Some Types of Compactness in Bitopological Spaces*

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Abstract

In this paper, we give the concept of N-open set in bitopological spaces, where N is the first letter of the name of one of the authors, then we used this concept to define a new kind of compactness, namely N-compactness and we define the N-continuous function in bitopological spaces.

We study some properties of N-compact spaces, and the relationships between this kind and two other known kinds which are S-compactness and pair-wise compactness.

1- Introduction

In 1963, the concept of "bitopological space" was introduced by Kelly[1]. A set equipped with two topologies is called a 'bitopological space" and denoted by (X,τ,τ'), where (X,τ), $((X,\tau')$ are two topological spaces. From that time many authors used the concept of bitopological space to define new concepts like seperation axioms, some types of connectedness and covering properties, for more details see [2] and [3].

In this paper, we introduce the concept of N-compactness, we study some properties of this kind with many examples, we also give some new properties about the S-compactness and pair-wise compactness which was introduced by Mrsevic and Reilly [4], where we give for example propositions 2.21, 2.23, 2.24, 2.27, 2.28, 2.40 and theorem 2.41. We also study the relationships between the three kinds of compactness, where we proved the valid directions and give counter examples for the invalid ones, and we put certain conditions to make the invalid direction true.

2.2 Definition

A subset A of a bitopological space (X,τ,τ') is called an "N-open set"if and only if it is open in the space $(X,\tau\vee\tau')$, where $\tau\vee\tau'$ is the supremum topology on X contains τ and τ'.

2.3 Definition

The complement of an N-open set in a bitopological space (X,τ,τ') is called "N-closed set".

2.4 Remark

Let (X,τ,τ') be a bitopological space, then:
(i) Every open set in (X,τ) or in (X,τ') is an N-open set in (X,τ,τ').
(ii) Every closed set in (X,τ) or in (X,τ') is an N-closed set in (X,τ,τ').

2.5 Note

The opposite direction of this remark 2.4 may be untrue as the following example shows:

Example

Let $X=\{1,2,3\}$, $\tau=\{\phi,\{1\},X\}$ and $\tau'=\{\phi,\{2\},X\}$ then $\tau\vee\tau'=\{\phi,\{1\},\{2\},\{1,2\},X\}$ is the family of all N-open subsets of (X,τ,τ'). $\{1,2\}$ is an N-open set in (X,τ,τ') but it is not open in both (X,τ) and (X,τ'). So $\{3\}$ is an N-closed set in (X,τ,τ') which is not closed in both (X,τ) and (X,τ').

2.6 Definition

Let (X,τ,τ') be a bitopological space, let A be a subset of X. A subcollection of the family $\tau\vee\tau'$ is called an "N-open cover of A" if the union of members of this collection contains A.

2.7 Definition

A bitopological space (X,τ,τ') is said to be an "N-compact space" if and only if every N-open cover of X has a finite subcover.

2.8 Proposition

If (X,τ,τ') is an N-compact space, then both (X,τ) and (X,τ') are compact spaces.

Proof:

Follows from remark (2.4).   ■

2.9 Note

The implication in proposition (2.8) is not reversible, as the following example shows:

Example

Let $\square$ be the set of all natural numbers, $\tau=\{\square\} \cup P(O^+)$ and $\tau'=\{\square\}\cup P(E^+)$. Then $\tau\vee\tau'$ is the discrete topology on N, where $P(O^+)$ and $P(E^+)$ are the power sets of O^+ and E^+ respectively.

Now, both $(\square,\tau)$ and $(\square,\tau')$ are compact spaces, but $(\square,\tau,\tau')$ is not N-compact. Since the N-open cover $\{\{n\}\,|\,n\in\square\}$ of $\square$ has no finite subcover.

The opposite direction of proposition (2.8) becomes valid in a special case, when τ is a subfamily of τ', as the following proposition shows:

2.10 Proposition

If τ is a subfamily of τ', then (X,τ,τ') is an N-compact space if and only if (X,τ') and (X,τ) are compact.

Proof:

Necessity, follows from proposition (2.8).

Sufficiency, in view of τ is a subfamily of τ', then $\tau \vee \tau' = \tau'$. So (X,τ,τ') is N-compact. ∎

2.11 Proposition

The N-closed subset of an N-compact space is N-compact.

Proof:

Let (X,τ,τ') be an N-compact space and let A be an N-closed subset of X To show that A is an N-compact set. Let $\{U_i : i \in \Lambda\}$ be an N-open cover of A. Since A is N-closed subset of X, then X–A is N-open subset of X, so $\{X-A\} \cup \{U_i : i \in \Lambda\}$ is an N-open cover of X, which is an N-compact space.

Therefore, there exists $i_1, i_2,\ldots,i_n \in \Lambda$, such that $\{ X-A, U_{i_1}, U_{i_2}, \ldots, U_{i_n} \}$ is a finite subcover of X. As $A \subseteq X$ and X–A covers no part of A, then $\{ U_{i_1}, U_{i_2},\ldots, U_{i_n} \}$ is a finite subcover of A. So A is N-compact set. ∎

2.12 Definition

A function f: $(X,\tau,\tau') \longrightarrow (Y,T,T')$ is said to be an "N-continuous function" if and only if the inverse image of each N-open subset of Y is an N-open subset of X.

2.13 Proposition

The N-continuous image of an N-compact space is an N-compact space.

Proof:

Let (X,τ,τ') be an N-compact space, and let f: $(X,\tau,\tau') \longrightarrow (Y,T,T')$ be an N-continuous, onto function.

To show that (Y,T,T') is an N-compact space. Let $\{U_i : i \in \Lambda\}$ be an N-open cover of Y, then $\{f^{-1}(U_i): i \in \Lambda\}$ is an N-open cover of X, which is N-compact space. So, there exists $i_1, i_2,\ldots,i_n \in \Lambda$, such that the family $\{f^{-1}(U_{ij}): j=1, 2, \ldots,n\}$ covers X and since f is onto, then $\{U_{ij}: j=1, 2, \ldots,n\}$ is a finite subcover of Y. ∎

2.14 Proposition

If A and B are two N-compact subsets of a bitopological space (X,τ,τ'), then $A \cup B$ is an N-compact subset of X.

Proof:

Clear. ∎

2.15 Remark

If A and B are two N-compact subsets of a bitopology space (X,τ,τ'), then $A \cap B$ need not be N-compact.

For example, let $X = \square \cup \{0,-1\}$ and let $\tau = P(\square) \cup \{H \subseteq X \mid -1,0 \in H \wedge (X-H) \text{ is finite}\}$. Let $\tau' = \tau \cup \{H \subseteq X \mid (-1 \in H \text{ or } 0 \in H) \wedge (X - H) \text{ finite}\}$.

Now, let $A = \square \cup \{0\}$ and $B = \square \cup \{-1\}$, then both A and B are N-compact subsets of the bitopologycal space (X,τ,τ'), but $A \cap B = \square$ is not N-compact set.

In the following definition, we study another kind of open sets in bitopological spaces, namely "S-open set".

2.16 Definition [4]

A subset A of a topological space (X,τ,τ') is said to be "S-open set" if it is τ-open or τ'-open.

The complement of the S-open set is called "S-closed set".

2.17 Remark
(i) Every S-open set in a bitopological space (X,τ,τ') is an N-open set.
(ii) Every S-closed set in a bitopological space (X,τ,τ') is an N-closed set.

2.18 Note
    The implication in remark (2.17) is not reversible. See the example of note (2.5), where the set {1,2} is N-open set which is not S-open set. So the set {3} is   N-closed set which is not S-closed set.

2.19 Definition [4]
    Let (X,τ,τ') be a bitopological space, let A be a subset of X. A subcollection of the family $\tau \cup \tau'$ is called an "S-open cover" of A if the union of members of this collection contains A.

    In the definitions (2.16) and (2.19), we use the concept of S-open sets in bitopological spaces inorder to expose another type of compactness in bitopological spaces, called S-compactness, which was introduced in the first time by Mrsevic and Reilly, (4).

2.20 Definition [4]
    A bitopological space (X,τ,τ') is called an "S-compact space" if and only if every S-open cover of X has a finite subcover.

2.21 Proposition
    If (X,τ,τ') is an S-compact space, then both (X,τ) and (X,τ') are compact.
Proof:
    Clear. ■

2.22 Note
    The opposite direction of proposition (2.21) may be false.
For example:

    Let X =[0,1] and let $\tau = \{\phi, X, \{0\}\}$ and $\tau' = \{\phi, X, (0,1]\} \cup \{(\frac{1}{n}, 1] \mid n \in \square \}$. Then both (X,τ) and (X,τ') are compact spaces, but (X,τ,τ') is not S-compact, since the S-open cover $\{\{0\}\} \cup \{(\frac{1}{n}, 1] \mid n \in \square \}$ of X has no finite subcover.

    The opposite direction of proposition (2.21) becomes valid in a special case, where τ is a subfamily of τ', as the following proposition shows:

2.23 Proposition
    If τ is a subfamily of τ', then (X,τ,τ') is an S-compact space if and only if (X,τ') and (X,τ) are compact spaces.
Proof:
    Clear. ■

2.24 Proposition
    An S-closed subset of an S-compact space is S-compact.
Proof:
    Let (X,τ,τ') be an S-compact space, let A be an S-closed subset of X. To show that A is an S-compact set.

Let $\{U_i : i \in \Lambda\}$ be an S-open cover of A. Since A is an S-closed subset of X, then X–A is an S-open subset of X. Then $\{U_i : i \in \Lambda\} \cup \{X–A\}$ is an S-open cover of X, which is S-compact space. Therefore, there exists $i_1, i_2,…,i_n \in \Lambda$, such that $\{U_{ij}: j=1, 2, …,n\} \cup \{X–A\}$ is a finite subcover of X. Since $A \subseteq X$ and X–A covers no part of A, then $\{U_{ij}: j=1, 2, …,n\}$ is a finite subcover of A. So A is an S-compact. ■

2.25 Definition [5]

Let f: $(X,\tau,\tau') \longrightarrow (Y,T,T')$ be a function, then f is said to be a "bicontinuous function" if and only if $f^{-1}(U) \in \tau$, for each $U \in T$, and $f^{-1}(v) \in \tau'$, for each $v \in T'$.

2.26 Example

Let $X=\{1,2,3\}$, $\tau=\{\phi,X,\{1\},\{2\},\{1,2\}\}$ and $\tau' = \tau_D$. And let $Y=\{a,b,c\}$, $T=\{\phi,Y,\{a\}\}$ and $T'= \tau_I$.

Define f: $(X,\tau,\tau') \longrightarrow (Y,T,T')$, such that f(1) = a, f(2) = b and f(3) = c. Then f is bicontinuous function.

Where τ_D and τ_I are the discrete and indiscrete topologies on X and Y respectively.

2.27 Proposition

A bicontinuous image of an S-compact space is an S-compact space.

Proof:

Let f: $(X,\tau,\tau') \longrightarrow (Y,T,T')$ be a bicontinuous, onto function and let (X,τ,τ') be an S-compact space.

To prove that (Y,T,T') is an S-compact.

Let $\{U_i : i \in \Lambda\}$ be an S-open cover of Y, then $\{f^{-1}(U_i) : i \in \Lambda\}$ is an S-open cover of X, which is an S-compact space. Therefore, there exists $i_1, i_2,…,i_n \in \Lambda$, such that $\{f^{-1}(U_{ij}): j=1, 2, …,n\}$ is a finite subcover of X and since f is onto, then we get $\{U_{ij}: j=1, 2, …,n\}$ is a finite subcover of Y. So Y is an S-compact space. ■

2.28 Proposition

If A and B are two S-compact subsets of a topological space (X,τ,τ'), then $A \cup B$ is an S-compact subset of X.

Proof:

Clear. ■

2.29 Remark

If A and B are two S-compact subsets of a bitopological space (X,τ,τ'), then $A \cap B$ need not be S-compact set. For example:

See the example of remark (2.15), both A and B are S-compact subsets of the bitopological space (X,τ,τ'), $A \cap B = \square$ is not S-compact set.

2.30 Proposition

Every N-compact space is an S-compact.
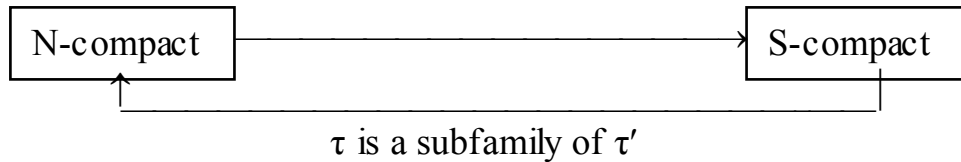
Proof:

Follows from remark (2.17). ■

2.31 Proposition

Let (X,τ,τ') be a topological space. If τ is a subfamily of τ', then the concepts of S-compactness and N-compactness are coincident.

Proof:

Follows from propositions (2.10) and (2.23). ■

The following diagram shows the relationships between N-compact and   S-compact spaces:

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┌──────────────┐                              ┌──────────────┐
│  N-compact   │─────────────────────────────>│  S-compact   │
└──────────────┘                              └──────────────┘
        ▲                                              │
        └──────────────────────────────────────────────┘
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τ is a subfamily of τ'

Now, we shall recall another kind of compactness on bitopological spaces called "pair-wise compact" to study this kind and compare it with the two above kinds of compactness in bitopological spaces.

2.32 Definition [4]
Let (X,τ,τ') be a bitopological space, $A \subseteq X$, an S-open cover of A is called a "pair-wise open cover" if it contains at least one non-empty element from τ, and at least one non-empty element from τ'.

2.33 Example
Let $X = \{1,2,3\}$, $\tau = \{\varnothing,X,\{1\}\}$ and $\tau' = \{\varnothing,X,\{2\},\{3\},\{2,3\}\}$. Then the cover $C = \{\{1\},\{2\},\{3\}\}$ is a pair-wise open cover of X.

2.34 Remark
Every pair-wise open cover of the bitopological space (X,τ,τ') is an S-open cover.

2.35 Note
The implication in remark 2.34 is not reversible.
For example:
Let $X = \{1,2,3\}$, $\tau = \{\varnothing,\{1\},X\}$ and $\tau' = \{\varnothing,\{2\},\{3\},\{2,3\},\{1,2\},X\}$. Then the cover $C = \{\{1,2\},\{3\}\}$ is an S-open cover of X, but it is not pair-wise open cover..

2.36 Definition [4]
A bitopological space (X,τ,τ') is called a "pair-wise compact space" if every pair-wise open cover of X has a finite subcover.

2.37 Remark
Let (X,τ,τ') be a bitopological space. If $\tau = \tau_I$ or $\tau' = \tau_I$, then X is a pair-wise compact space.

2.38 Proposition
Every S-compact space isa pair-wise compact space.
Proof:
Follows from remark 2.34.

2.39 Note
The converse of proposition 2.38 may be false.
For example:
$(\square,\tau_u, \tau_I)$ is pair-wise compact space, but not S-compact.

The example of note 2.39 shows that, if (X,τ,τ') is a pair-wise compact space, then (X,τ') need not to be compact and the example of note 2.9 shows that, if (X,τ) and (X,τ') are compact space, then (X,τ,τ') need not to be a pair-wise compact.

2.40 Proposition

If τ is a subfamily of τ' and (X,τ') is a compact space, then (X,τ,τ') is a pair-wise compact space.

Proof:

Follows from proposition 2.10 and proposition 2.38.

2.41 Theorem

If (X,τ) and (X,τ') are compact spaces, then (X,τ,τ') is S-compact if and only if it is a pair-wise compact.

Proof:

Necessity, follows from proposition 2.38.

Sufficiency, suppose (X,τ,τ') is a pair-wise compact space, to prove it, is an S-compact space.

Let W be an S-open cover of X, then there are three probabilities

i) If W is a τ-open cover, since (X,τ) is compact, then W has a finite subcover of X, so the proof is over.

ii) If W is a τ'-open cover, since (X,τ') is compact space, then W has a finite subcover of X, so the proof is over.

iii) If W is a pair-wise open cover, since (X,τ,τ') is a pair-wise compact space, then W has a finite subcover.

iv) Therefore, (X,τ,τ') is an S-compact space.

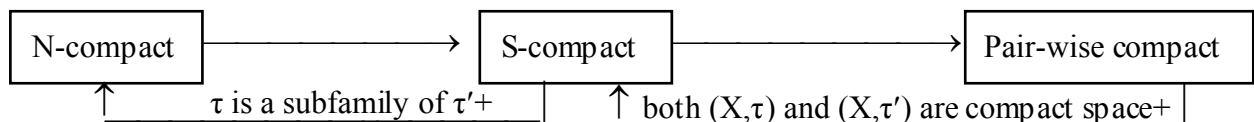From proposition 2.38 and theorem 2.41, we get the following diagram::

$$\boxed{\text{S-compact}} \xrightarrow{\hspace{4cm}} \boxed{\text{Pair-wise compact}}$$
both (X,τ) and (X,τ') are compact space +
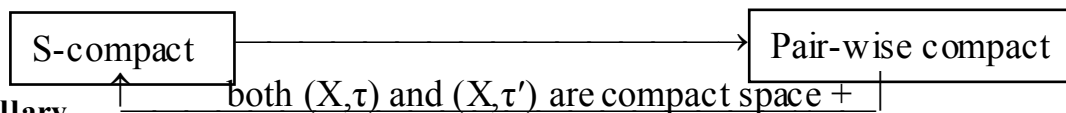
2.42 Corollary

If τ is a subfamily of τ', and (X,τ') is a compact space, then (X,τ,τ') is a pair-wise compact if and only if it is an S-compact space.

2.43 Remark

The following diagram shows the relations among the different types of compactness that are studied in this section:

In a bitopological space (X,τ,τ')

$$\boxed{\text{N-compact}} \xrightarrow{\hspace{2cm}} \boxed{\text{S-compact}} \xrightarrow{\hspace{2cm}} \boxed{\text{Pair-wise compact}}$$
τ is a subfamily of τ'+   both (X,τ) and (X,τ') are compact space+

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بعض انواع الفضاءات ثنائية الرص

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الخلاصة

قمنـا فـي هـذا البحـث بتعريـف نـوع جديـد مـن المجموعـات المفتوحـة فـي الفضـاءات التبولوجيـة الثنائيـة اسـميناها المجموعـات المفتوحـة مـن نـوع – N إذ ان N هـو الحـرف الاول لاسـم أحـد البـاحثين و مـن ثـم أسـتعملنا هـذا المفهـوم فـي تعريـف نـوع جديـد مـن التـراص وهـو التـراص مـن نـوع – N وكـذلك عرفنـا الدالـة المسـتمرة مـن نـوع – N فـي الفضـاءات الثنائيـة.

ولقـد درسـنا بعـض الخـواص للتـراص مـن نـوع – N كمـا درسـنا علاقـة هـذا النـوع بنـوعين آخـرين معـروفين همـا التـراص مـن نـوع – S والتراص الثنائي.