TIME SERIES AND THE ALGEBRAIC MATRIX RICCATI EQUATION

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One way of modelling certain kinds of time series is via the Yule-Walker equations. These are a set of (over-determined) linear equations for estimating the parameters in the models. The coefficients in these equations are estimates, obtained from the data, of the autocorrelations. In this paper two ways of solving the Yule-Walker equations are considered. The first is the well known method using the pseudo-inverse and the second uses the algebraic matrix Riccati equation. A number of numerical examples are used to illustrate and compare the two different approaches.

Keywords: autoregressive time series, Yule-Walker equations, algebraic matrix Riccati equations, total least squares

Introduction

Determination of the autoregressive parameters

$$a_i, i = 0,1,\cdots,n, \tag{1}$$

in the stochastic process

$$A(q^{-1})y(t) = D(q^{-1})w(t) \tag{2}$$

where

$$A(q^{-1}) = \sum_{i=o}^{n} a_i q^{-i} \tag{3}$$

is investigated.

In *Eq.(2)*, $y(t)$ denotes noisy data, $w(t)$ is white noise and $D(q^{-1})$ is a polynomial containing the moving average parameters of the process. The expression q^{-1} in *Eq.(2)* is the unit delay operator defined by

$$q^{-1}y(t) = y(t-1). \tag{4}$$

More extensive details and background material can be found in Ref. [1]. Here the Yule-Walker method is used to estimate the parameters a_i by two different techniques. Firstly the direct method involving the inverse or pseudo-inverse of the autocorrelation matrix, depending on the invertability or otherwise of the latter, is used.

Secondly an algorithm is used which is based on the result that a total least squares solution to the Yule-Walker equations can be found by solving an appropriate matrix Riccati equation.

A number of numerical examples from Ref. [1] serve to illustrate the two techniques and to compare the results with a more sophisticated stochastic analysis given in Ref. [1]. A similar investigation has been carried out in Ref. [5] but the TSL solutions to the Yule-Walker equation are obtained by singular value decomposition and simulated time series are used.

The Yule-Walker Method

For a pure (i.e., $D(q^{-1}) \equiv 1$) AR process the Yule-Walker equations are (Ref.[1], Ref. [2]),

$$\begin{bmatrix} r_0 & r_1 \cdots r_{n-1} \\ r_1 & r_0 \cdots r_{n-2} \\ \vdots & \ddots & \vdots \\ r_m & r_{m-1} \cdots r_{m-(n-1)} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \tag{5}$$

where,

$$r_k = \frac{c_k}{c_0}, \text{ so that } r_0 = 1;$$

Table 1 Numerical results for Series C

n x m	1x1	1x10	1x50	1x100	TENT	FINAL
LS	-.805	-.814	-.847	-.881	-.81	-.82
TLS	-.805	-.815	-.854	-.892	-.81	-.82

Table 2 Numerical results for Series D

n x m	1x1	1x10	1x50	1x100	TENT	FINAL
LS	-.861	-.860	-.908	-.908	-.86	-.87
TLS	-.861	-.860	-.916	-.912	-.86	-.87

c_k is an estimate of the relevant unknown autocovariance given by

$$c_k = \frac{1}{N}\sum_{t=1}^{N-k}\left(y_t - \bar{y}\right)\left(y_{t+k} - \bar{y}\right), \quad k = 0,1,2,\cdots$$

and N is the number of observations in the time series. In case $m > n$ then the linear equations *(Eq.(5))* are overdetermined and the pseudo-inverse has to be used.

Numerical Investigation

Four different autoregressive time series taken from Ref. [1] have been used as numerical illustrations.

Example 1 Series C

This is a time series, with $N = 226$ observations, of temperatures in a chemical process taken at one minute intervals. A model with two possible sets of parameters, one tentative (tent) the other more finalized (final) is given in Ref. [1]. These are the following:

$$\text{Tent: } \nabla y(t) - 0.81\nabla y(t-1) = w(t) ,$$

$$\text{Final: } \nabla y(t) - 0.82\nabla y(t-1) = w(t) .$$

The results of the numerical investigation are given in *Table 1*.

The table entries are solutions (rounded to 3 decimal places) of n x m Yule-Walker equations by the least squares method (LS) and by the total least squares method (TLS). The former is found using the pseudo-inverse matrix and the latter using Algorithm 1 of [3]. (A convenient check on the LS solution is also obtained by using A1 as explained in [4]). For this example a closer analysis shows that $m = 22$ is about the best number of equations to use and TLS is slightly better than LS.

Example 2 Series D

The tentative and final models for this example are:

$$\text{Tent: } y(t) - 0.86y(t-1) = w(t)$$

Table 3 Numerical results for Series E. Model 1

n x m	2x2	2x10	2x50	2x99	TENT	FINAL
LS	-1.318	-1.398	-1.475	-1.515	-1.32	-1.42
	.634	.704	.724	.770	.63	.73
TLS	-1.318	-1.432	-1.549	-1.586	-1.32	-1.42
	.634	.733	.790	.834	.63	.73

Table 4 Numerical results for Series E. Model 2

n x m	3x3	3x10	3x50	3x99	TENT	FINAL
LS	-1.369	-1.821	-2.104	-2.155	-1.37	-1.57
	0.740	1.378	1.775	1.849	0.74	1.02
	-0.0805	-0.367	-0.571	-0.599	-0.08	-0.21
TLS	-1.369	-1.975	-2.363	-2.412	-1.37	-1.57
	0.740	1.609	2.188	2.277	0.74	1.02
	-0.0805	-0.485	-0.783	-0.825	-0.08	-0.21

$$\text{Final: } y(t) - 0.87y(t-1) = w(t) .$$

There are $N = 310$ observations and the data is a set of chemical process viscosity readings taken at hourly intervals. The numerical results are shown in *Table 2*.

For this example $m = 19$ appears to be the best number of equations to take with TLS getting nearer to -0.87 than LS with the same number of equations.

Example 3 Series E

For this, much studied example, the data shows the number of sunspots that occurred in each year from 1770 to 1869; so there are 100 observations. Two different sets of parameter values are given for each of two different suggested models:

Model 1

$$\text{Tent: } y(t) - 1.32y(t-1) + 0.63y(t-2) = w(t) ,$$

$$\text{Final: } y(t) - 1.42y(t-1) + 0.73y(t-2) = w(t) ,$$

Model 2

$$\text{Tent: } \begin{aligned} & y(t) - 1.37y(t-1) + \\ & + 0.72y(t-2) - 0.08y(t-3) = w(t) \end{aligned} ,$$

$$\text{Final: } \begin{aligned} & y(t) - 1.57y(t-1) + \\ & + 1.02y(t-2) - 0.21Y(t-3) = w(t) \end{aligned} .$$

Tables 3 and *4* give the numerical results in this case. For the first model $m = 9$ or 10 give good approximations to the parameter values given in [1] and for the second model $m = 7$ or 8 seem to be reasonable values.

Table 5 Numerical results for Series F

n x m	2x2	2x10	2x50	2x69	TENT	FINAL
LS	0.32	0.319	0.331	0.365	0.32	0.34
	-0.18	-0.170	-0.178	-0.149	-0.18	-0.19
TLS	0.32	0.323	0.385	0.451	0.32	0.34
	-0.18	-0.168	-0.157	-0.104	-0.18	-0.19

Example 4 Series F

Here the time series F consists of 70 readings of the yields from a batch chemical process and the suggested model is:

Tent: $y(t) + 0.32y(t-1) - 0.18y(t-2) = w(t)$

Final: $y(t) + 0.34y(t-1) - 0.19y(T-2) = w(t)$

The results of the calculations are shown in *Table 5*. For this example the second parameter increases with m so the best m for this parameter is 2! For the first parameter, however, m = 25 gives a value of -0.338 with 0.169 for the second parameter.

"Portmanteau" lack-of-fit test

A test for model adequacy, which depends on the first K auto-correlations of the residuals $r_k(\hat{a})$, is proposed in [1]. The statistic

$$\hat{Q} = n(n+2) \sum_{k=1}^{K} r_k^2(\hat{a})/(n-k) \qquad (6)$$

is computed and referred to a table of the chi-squared distribution. In *Eq.(6)* $n = N - d$ where d is the degree of differencing required to approximate to stationarity in the time series being analysed.

For series D, with $K = 25$, there are 24 degrees of freedom and *Eq.(6)* becomes

$$\hat{Q} = 310 \times 312 \sum_{k=1}^{25} r_k^2(\hat{a})/(310-k) \ ,$$

which is readily computed by MATHEMATICA, for example, as follows:

Subtract the mean of series D (9.13258) from each of its elements and call the resulting series, augmented by a back-forecasted value as its initial value, $s(D)$.

Compute the series of residuals.

$s(D) = Table\ [Part\ [s(D),i] - \Phi\ Part[s(D),i-1],\{i,2,310\}]$

with Φ the estimate of the model parameter. Compute $\hat{\Omega}$ as follows:

$s(D) = Table[Part[s(D),i] - 0.00243941,\{i,1,310\}]$

(0.00243941 is the mean of the previous $s(D)$).

$r_o(\hat{a}) = Sum[Part[s(D),i] * Part[s(D),i],\{i,1,310\}]$

$g = Table[Sum[Part[s(D),i] *$

$* Part[s(D),i+j],\{i,1,310-j\}],\{j,1,l\}]/r_o(\hat{a})$

(This produces a table of l auto-correlations of residuals)

$\hat{\Omega} = 310 * 312 Sum[Part[g,i]\hat{\ }2/(310-i),\{i,1,25\}]$.

Taking $\Phi = -0.87$ (the final value from [1]) gives $\hat{\Omega} = 11.5$

Now when m x n = 19x1 the parameter Φ obtained by LS is 0.868034 and by TLS it is 0.868979. The corresponding values of $\hat{\Omega}$ are 11.44 and 11.47. Similarly when m x n =20x1 the values for Φ are 0.870544 and 0.87177 with corresponding values of 11.51 and 11.55 for $\hat{\Omega}$.

Thus TLS gets closer to the value of $\hat{\Omega}$ given in [1] with fewer linear equations.

Conclusion

A numerical study has shown that the TLS method is slightly superior to the LS method for finding the unknown parameters in models of auto-regressive time series.

The techniques used can be extended to estimation of moving average parameters [2]. Special purpose methods for solving the Yule-Walker equation are also discussed in [2].

Finally, when Algorithm 1 is used on its own, and not in comparison with other methods as it is here, the computations should be stopped as soon as some test of lack of fit (such as the portmanteau test from [1]) is adequately satisfied.

SYMBOLS

a_i	autoregressive parameters
A, D	polynomials in q^{-1}
q^{-1}	unit delay operator
$y(t)$	noisy data
$w(t)$	white noise
n	number of autoregressive parameters
m	number of Yule-Walker equations
$\bar{y}$	mean of y(t)
N	number of observations in time series
d	degree of differencing for stationarity
$\hat{\Omega}$	statistic used in Portmanteau test
i, j, k	running incides
l	number of autocorrelations taken in Portmanteau test.

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