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NEUMANN BOUNDARY VALUE PROBLEMS WITH BEM AND COLLOCATION

N. HERRMANN[1] AND V. KARNJANATAWEE[2]

[1]Institut für Angewandte Mathematik, Universität Hannover, Hannover, Germany
[2]Dept. of Mathematics, King Mongkut University of Technology Thonburi, Bangkok, Thailand

This paper is an introduction into the Boundary Element Method (BEM) with collocation to find the numerical solution of two different types of Neumann problems. At first we start with the Laplace equation and continue later with the heat equation on a bounded convex domain with smooth boundary in two dimensions. We will show how to transform the governing problem into a boundary integral equation which can be solved by dividing the boundary into a finite number of segments and applying the collocation method. We finish presenting an example of the heat equation.

Keywords: Boundary element method, Laplace equation, heat equation, Neumann boundary value problem, Collocation

Introduction

Let Ω be a convex open domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$. Consider the Laplace equation:

$$\Delta u(x,y) = 0 \qquad \text{in } \Omega,$$

$$\frac{\partial u(x,y)}{\partial n} = g(x,y) \qquad \text{on } \partial\Omega,$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

and the heat equation

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2},$$

where $(x,y) \in \Omega, \quad t \in (0,T]$

with initial and boundary conditions,

$$u(x,y,0) = f(x,y), \qquad (x,y) \in \overline{\Omega}$$

$$\frac{\partial u(x,y,t)}{\partial n} = g(x,y,t), \quad (x,y) \in \partial\Omega, t \in (0,T]$$

By using the fundamental solution we will develop the Boundary Integral Equation (BIE) from the governing boundary value problem and will create a Fredholm integral equation of the second kind. According to the fact that the solution of a Neumann value problem is not unique the same is true for the approximate solution which we get by using BEM and collocation. Piecewise constant and piecewise linear functions will be used to form the ansatz functions which lead to a simple linear system of equations.

Neumann Boundary Value Problem for the Laplace Equation

Let Ω be a convex open domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$. Consider the Neumann boundary value problem (NBVP):

$$\Delta u(x,y) = 0 \qquad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u(x,y)}{\partial n} = g(x,y) \qquad \text{on } \partial\Omega. \tag{2}$$

It is well known that for two dimensions

$$E(x,\xi) = -\frac{log|x-\xi|}{2\pi} \tag{3}$$

is the Green's function or the fundamental solution for the operator Δ, that means it is the solution of the problem

$$\Delta_\xi E(x,\xi) = -\delta(x-\xi) \quad \forall x, \xi \in \mathbf{R}^2. \tag{4}$$

It can be seen that E is not defined at $x = \xi$, where E is singular.

Boundary Integral Equation

For $x \in \Omega$ we follow the definition of the distribution δ, and with (1) and (4)

$$u(x) = \delta(x-\xi)(u)$$

$$= \int_{\Omega} \left(\Delta u(\xi) E(x,\xi) - \left(\Delta_{\xi} E(x,\xi)\right) u(\xi)\right) d\xi$$

The second Green-Gauß formula gives

$$u(x) = \int_{\partial\Omega} \left(\frac{\partial u(\xi)}{\partial n} E(x,\xi) - \frac{\partial E(x,\xi)}{\partial n} u(\xi) \right) d\sigma_{\xi} \quad \forall x \in \Omega . (5)$$

There is no singularity for $x \in \Omega$. Therefore for $x \in \Omega$ we can compute $u(x)$ by knowing the boundary data $u(x)$ and $\frac{\partial u}{\partial n}$ on $\partial\Omega$. We will refer to this formula as the representation of the solution $u(x)$ for $x \in \Omega$.

Our Neumann problem from above gives $\frac{\partial u}{\partial n}$ on $\partial\Omega$. We have to find $u(x) = g(x)$ on $\partial\Omega$.

Both integrals in (5)

$$\int_{\partial\Omega} \frac{\partial u(\xi)}{\partial n} E(x,\xi) d\sigma_{\xi} \quad \text{and} \quad \int_{\partial\Omega} \frac{\partial E(x,\xi)}{\partial n_{\xi}} u(\xi) d\sigma_{\xi} \qquad (6)$$

are well defined for $x \in \Omega$. But for $x \in \partial\Omega$ we have a jump of magnitude $\frac{u(x)}{2}$ for the second integral :

$$\lim_{x \to x_0 \in \partial\Omega} \int_{\partial\Omega} \frac{\partial E(x,\xi)}{\partial n_{\xi}} u(\xi) d\sigma_{\xi} = \frac{u(x_0)}{2} + \int_{\partial\Omega} \frac{\partial E(x_0,\xi)}{\partial n_{\xi}} u(\xi) d\sigma_{\xi}$$

and so we obtain the boundary integral equation

$$u(x_0) = \frac{u(x_0)}{2} + \int_{\partial\Omega} \left(E(x_0,\xi) \frac{\partial u(\xi)}{\partial n} - \frac{\partial E(x_0,\xi)}{\partial n_{\xi}} u(\xi) \right) d\sigma_{\xi} \quad (7)$$

$\forall x_0 \in \partial\Omega$

Then we get from (7) a Fredholm integral equation of the second kind

$$\frac{u(x)}{2} + \int_{\partial\Omega} \frac{\partial E(x,\xi)}{\partial n_{\xi}} u(\xi) d\sigma_{\xi} = \int_{\partial\Omega} E(x,\xi) \frac{\partial u(\xi)}{\partial n} d\sigma_{\xi} \qquad (8)$$

$\forall x \in \partial\Omega$ where u on the left hand side is unknown and $\frac{\partial u(x)}{\partial n} = g(x)$ on $\partial\Omega$ is given.

Collocation Method

The integral equation (9) does generally not admit a solution in closed form. We will show how to solve such a problem numerically using the collocation method.

Given $\frac{\partial u(x)}{\partial n} = g(x), \quad x \in \partial\Omega,$ solve for the unknown Dirichlet data $u(x)$, $x \in \partial\Omega$ from

$$\frac{u(x)}{2} + \int_{\partial\Omega} \frac{\partial E(x,\xi)}{\partial n_{\xi}} u(\xi) d\sigma_{\xi} = \int_{\partial\Omega} E(x,\xi) g(\xi) d\sigma_{\xi}, \qquad (9)$$

$\forall x \in \partial\Omega$

The solution of (9) is not unique since if we replace $u(x)$ by $\bar{u}(x) = u(x) + c$ where c is constant then $\bar{u}$ is a solution, too. We need a compatibility condition so that the Neumann boundary value problem is well posed: $g(x)$ should satisfy

$$0 = \int_{\Omega} \Delta u(x) dx = \int_{\partial\Omega} \frac{\partial u(x)}{\partial n} d\sigma = \int_{\partial\Omega} g(x) d\sigma \qquad (10)$$

We use now the collocation method to discretize the problem. We subdivide the boundary $\partial\Omega$ into n arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ and call the midpoints of the arcs $x_1, x_2, \ldots, x_n$ with $x_n = x_1$. We take the ansatz:

$$\tilde{u}(x) = \sum_{i=1}^{n} a_i \chi_i(x) \qquad (11)$$

where $\chi_i(x)$ is the characteristic function on Γ_i and a_i are unknown parameters. That means $\tilde{u}(x)$ is a piecewise constant approximation to $u(x)$ on $\partial\Omega$. As collocation points we use the midpoints. This leads to

$$\frac{a_j}{2} + \sum_{i=1}^{n} a_i \int_{\Gamma_i} \frac{\partial E(x_j,\xi)}{\partial n_{\xi}} d\sigma_{\xi} = \int_{\partial\Omega} E(x_j,\xi) g(\xi) d\sigma_{\xi}, \quad (12)$$

$1 \le j \le n$

These are n linear equations with the unknowns $a_1, \ldots, a_n$, so that we have $n \times n$ linear system of equations

Neumann Boundary Value Problem for the Heat Equation

Let Ω be a bounded convex domain with smooth boundary $\partial\Omega = \Gamma$ in $\mathbf{R}^2$. Consider the heat equation

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}, \qquad (13)$$

$(x,y) \in \Omega, \quad t \in (0,T]$

with initial and boundary conditions,

$$u(x,y,0) = f(x,y), \qquad (x,y) \in \overline{\Omega} \qquad (14)$$

$$\frac{\partial u(x,y,t)}{\partial n} = g(x,y,t), \quad (x,y) \in \Gamma, t \in (0,T]. \qquad (15)$$

The problem will be transformed into an integral equation by using the fundamental solution and will be solved by applying the collocation method in the same manner as for the Laplace equation (1) and (2).

Boundary integral equation

The well-known Green's function or fundamental solution for the heat equation

$$E(\xi,\tau;x,t)=\begin{cases}\dfrac{1}{4\pi(t-\tau)}e^{\frac{-|\xi-x|^2}{4(t-\tau)}} & \text{if } t>\tau\\ 0 & \text{if } t\le\tau\end{cases}, \qquad (16)$$

is used as weight function to generate the integral equation

$$\int_T\int_\Omega\left(\Delta_\xi u(\boldsymbol{\xi},\tau)-\frac{\partial u(\boldsymbol{\xi},t)}{\partial\tau}\right)E(\boldsymbol{\xi},\tau;\mathbf{x},t)\,d\boldsymbol{\xi}\,d\tau=0$$

where

$u(\mathbf{x},t)$ is the unknown solution of the heat equation,

$\boldsymbol{\xi}=(\xi_1,\xi_2)$, $\mathbf{x}=(x,y)$,

$\Delta_\xi=\dfrac{\partial^2}{\partial\xi_1^2}+\dfrac{\partial^2}{\partial\xi_2^2}$.

Let Ω be a bounded convex domain in $\mathbf{R}^2$ with smooth boundary $\Gamma=\partial\Omega$.

By using the Gauss-Green formula the integral becomes:

$$u(x,t)=\int_0^t\int_\Gamma\frac{\partial u(\xi,\tau)}{\partial n_\xi}\cdot E(\xi,\tau;x,t)\,d\sigma_\xi d\tau-\int_0^t\int_\Gamma u(\xi,\tau)\frac{\partial E(\xi,\tau;x,t)}{\partial n_\xi}\,d\sigma_\xi d\tau+\int_\Omega f(\xi)E(\xi,0;x,t)d\xi$$

for $(x,t)\in\Omega\times(0,T]$

The left hand side of the formula has to be replaced by the famous jump relation if *x* is on the boundary. So we are led to a boundary integral equation for $x\in\Gamma$ and $0<t\le T$.

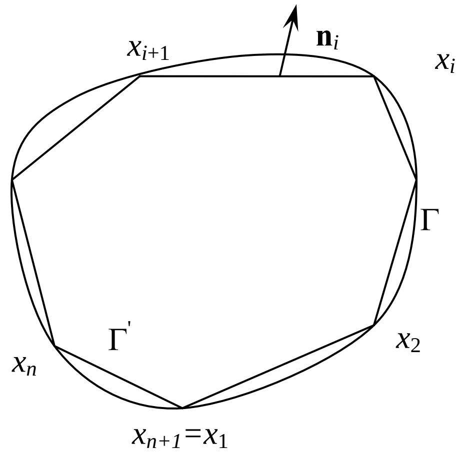
$$\frac{1}{2}u(x,t)=\int_0^t\int_\Gamma\frac{\partial u(\xi,\tau)}{\partial n_\xi}\cdot E(\xi,\tau;x,t)\,d\sigma_\xi d\tau-\int_0^t\int_\Gamma u(\xi,\tau)\frac{\partial E(\xi,\tau;x,t)}{\partial n_\xi}\,d\sigma_\xi d\tau+\int_\Omega f(\xi)E(\xi,0;x,t)d\xi$$

Substitute the given boundary condition into $\dfrac{\partial u(\xi,\tau)}{\partial n_\xi}$ the boundary integral equation becomes

$$\frac{1}{2}u(x,t)+\int_0^t\int_\Gamma u(\xi,\tau)\frac{\partial E(\xi,\tau;x,t)}{\partial n_\xi}\,d\sigma_\xi d\tau=\int_0^t\int_\Gamma g(\xi,\tau)\cdot E(\xi,\tau;x,t)\,d\sigma_\xi d\tau+\int_\Omega f(\xi)E(\xi,0;x,t)d\xi$$

Collocation Method

We divide the boundary $\Gamma=\partial\Omega$ into *n* segments as shown.



Let $x_1,x_2,\ldots,x_n,x_{n+1}=x_1\in\Gamma$.

Define $\Gamma_i'=\{x\in\mathbf{R}^2 : x=\lambda x_i+(1-\lambda)x_{i+1},\ 0\le\lambda\le1\}$ where $i=1,\ldots,n$. Then $\Gamma_1'\cup\ldots\cup\Gamma_n'$ is a polygon in Ω since Ω is convex.

Let $\vec{n}_j'$ be the outer unit normal vector on Γ_i',

$h=\max\limits_{i=0,\ldots,n}|\Gamma_i'|$ be the step width in space and for $m\in\mathbf{N}$

$k=\dfrac{T}{m}$ be the step width in time

and we get the time steps $t_j=jk$ ($j=0,1,\ldots,m$).

Using collocation method the piecewise linear spline functions

$$\varphi_i(x')=\begin{cases}\dfrac{x'-x_{i-1}}{x_i-x_{i-1}}, & x'\in\Gamma_{i-1}'\\[2mm] \dfrac{x_{i+1}-x'}{x_{i+1}-x_i}, & x'\in\Gamma_i' \quad,\quad i=1,\ldots,n\\[2mm] 0 & \textit{otherwise}\end{cases}$$

are applied to space and the piecewise constant characteristic functions

$$\chi_j(t)=\begin{cases}0 & t\notin(t_{j-1},t_j]\\ 1 & t\in(t_{j-1},t_j]\end{cases},\quad j=1,\ldots,m$$

are applied to time.

With both we form the ansatz function

$$\tilde{u}(x',t)=\sum_{j=1}^m\sum_{i=1}^n u_i^j\varphi_i(x')\chi_j(t)$$

where u_i^j are the unknown coefficients which we have to find.

At the time step *p* of segment *i* the boundary integral equation becomes

$$\frac{1}{2}u_i^p + \int_0^{t_p}\int_{\Gamma'} \tilde{u}(\xi',\tau)\frac{\partial E(\xi',\tau;x_i,t_p)}{\partial n_{\xi'}}d\sigma'_{\xi'}d\tau$$

$$= \int_{t_0}^{t_p}\int_{\Gamma'} g(\xi',\tau)\cdot E(\xi',\tau;x_i,t_p)\,d\sigma'_{\xi'}d\tau$$

$$+ \int_{\Omega'} f(\xi')E(\xi',0;x_i,t_p)d\xi'$$

This is a system of linear equations. The coefficients can be written in short form as

$$b_{ij}^{pq} = \int_{t_{q-1}}^{t_q}\int_{\Gamma'} \varphi_j(\xi')\frac{\partial E(\xi',\tau;x_i,t_q)}{\partial n_{\xi'}}d\sigma'_{\xi'}d\tau$$

$$c_i^p = \int_{t_0}^{t_p}\int_{\Gamma'} g(\xi',\tau)\cdot E(\xi',\tau;x_i,t_p)\,d\sigma'_{\xi'}d\tau$$

$$+ \int_{\Omega'} f(\xi')E(\xi',0;x_i,t_p)d\xi'$$

The boundary integral equation at time step p of segment i is rewritten as:

$$\left(\begin{bmatrix}\frac{1}{2} & & \\ & \ddots & \\ & & \frac{1}{2}\end{bmatrix} + \left[b_{ij}^{pp}\right]\right)\begin{bmatrix}u_1^p\\ \vdots \\ u_n^p\end{bmatrix} = \left(-\sum_{q=1}^{p-1}\left[b_{ij}^{pq}\right]\right)\begin{bmatrix}u_1^q\\ \vdots \\ u_n^q\end{bmatrix} + \begin{bmatrix}c_1^p\\ \vdots \\ c_n^p\end{bmatrix}$$

which should be solved to get the solution.

Example

As example we consider the heat equation

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2},(x,y)\in\Omega, t\in(0,1]$$

with the following initial and Neumann boundary conditions,

$$u(x,y,0) = 1, \quad (x,y)\in\overline{\Omega} \tag{17}$$

$$\frac{\partial u(x,y,t)}{\partial n} = 0, \quad (x,y)\in\Gamma, t\in(0,1] \tag{18}$$

where Ω is the unit circle.

Forming the boundary integral equation and substituting the data given in (17) and (18) we get

$$\frac{1}{2}u(\mathbf{x},t) + \int_0^t\int_\Gamma u(\xi,\tau)\frac{\partial E(\xi,\tau;\mathbf{x},t)}{\partial n_\xi}d\sigma_\xi d\tau = \int_\Omega E(\xi,0;\mathbf{x},t)d\xi\,,$$

$$\mathbf{x} = (x,y)\in\Gamma,\ t\in(0,1]$$

To apply the collocation method we divide the perimeter of the unit circle into 6 segments and discretize the time into 10 time steps. With piecewise linear functions φ_i, $(i = 1,\ldots,6)$ used in space and piecewise constant functions χ_j, $(j = 1,\ldots,10)$ used in time, the ansatz function is

$$\tilde{u}(x',t) = \sum_{j=1}^{10}\sum_{i=1}^{6} u_i^j\varphi_i(x')\chi_j(t)$$

where u_i^j are the unknown constant collocation points. Then the boundary integral equation at time step p becomes

$$\frac{1}{2}u_i^p + \int_0^t\int_{\Gamma'} \tilde{u}(\xi',\tau)\frac{\partial E(\xi',\tau;\mathbf{x}_i,t_p)}{\partial n_{\xi'}}d\sigma_{\xi'}d\tau$$

$$= \int_\Omega E(\xi',0;\mathbf{x}_i,t_p)d\xi',\quad i = 1,\ldots,6$$

So at the first time step we have the equation

$$\frac{1}{2}u_i^1 + \int_0^t\int_{\Gamma'}\left(u_1^1\varphi_1 + \ldots + u_6^1\varphi_6\right)\frac{\partial E_i^1}{\partial n_{\xi'}}d\sigma_{\xi'}d\tau$$

$$= \int_\Omega E(\xi',0,x_i,t_1)d\xi' \quad ,$$

where $i = 1, \ldots, 6$ and $\dfrac{\partial E_i^1}{\partial n_{\xi'}} = \dfrac{\partial E(\xi',\tau;x_i,t_1)}{\partial n_{\xi'}}$.

This is a system of linear equations

$$\begin{bmatrix}\frac{1}{2} & 0 & \cdots & 0\\ 0 & \frac{1}{2} & \cdots & 0\\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \frac{1}{2}\end{bmatrix} + \begin{bmatrix}\int_0^{t_1}\int_{\Gamma'_6\cup\Gamma'_1}\frac{\partial E_1^1}{\partial n_{\xi'}}\varphi_1 d\sigma d\tau & \cdots & \int_0^{t_1}\int_{\Gamma'_5\cup\Gamma'_6}\frac{\partial E_1^1}{\partial n_{\xi'}}\varphi_6 d\sigma d\tau\\ \int_0^{t_1}\int_{\Gamma'_6\cup\Gamma'_1}\frac{\partial E_2^1}{\partial n_{\xi'}}\varphi_1 d\sigma d\tau & \cdots & \int_0^{t_1}\int_{\Gamma'_1\cup\Gamma'_2}\frac{\partial E_2^1}{\partial n_{\xi'}}\varphi_2 d\sigma d\tau\\ \vdots & & \vdots\\ \int_0^{t_1}\int_{\Gamma'_6\cup\Gamma'_1}\frac{\partial E_6^1}{\partial n_{\xi'}}\varphi_1 d\sigma d\tau & \cdots & \int_0^{t_1}\int_{\Gamma'_1\cup\Gamma'_2}\frac{\partial E_6^1}{\partial n_{\xi'}}\varphi_2 d\sigma d\tau\end{bmatrix}\begin{bmatrix}u_1^1\\ u_2^1\\ \vdots\\ u_6^1\end{bmatrix}$$

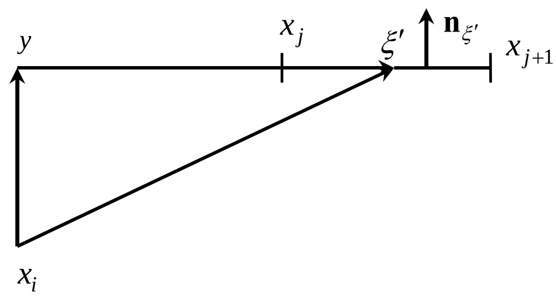
$$= \begin{bmatrix}\int_{\Omega'}E(\xi',0;\mathbf{x}_1,t_1)\\ \int_{\Omega'}E(\xi',0;\mathbf{x}_2,t_1)\\ \vdots\\ \int_{\Omega'}E(\xi',0;\mathbf{x}_6,t_1)\end{bmatrix}.$$

The quantities $\dfrac{\partial E_i^1}{\partial n_{\xi'}}$ with

$$E(\xi',\tau;x_i,t_1) = \begin{cases}\dfrac{1}{4\pi(t_1-\tau)}e^{\frac{-|\xi'-x_i|^2}{4(t_1-\tau)}} & \text{if } t_1 > \tau\\ 0 & \text{if } t_1 \le \tau\end{cases}$$

are $\quad \dfrac{\partial E_i^1}{\partial n_{\xi'}} = \dfrac{\partial E_i^1}{\partial \xi'}\cdot\mathbf{n}_{\xi'} = -\dfrac{(\xi'-\mathbf{x}_i)\cdot\mathbf{n}_{\xi'}}{8\pi(t_1-\tau)^2}e^{-\frac{|\xi'-\mathbf{x}_i|^2}{4(t_1-\tau)}}$.

The term $(\xi'-\mathbf{x}_i)\cdot\mathbf{n}_{\xi'}$ can be shown in the picture below



It is clear that

$$(\xi' - x_i)\cdot \mathbf{n}_{\xi'} = ((y - x_i) + (\xi' - y))\cdot \mathbf{n}_{\xi'}$$

$$= |y - x_i| = d_j \,.$$

d_j are constant. Then the elements of the coefficient matrix of above system are

$$\int_0^{t_1}\int_{\Gamma'_{j-1}\cup\Gamma'_j} \frac{\partial E_i^1}{\partial n_{\xi'}}\varphi_j d\sigma d\tau = -\int_{\Gamma'_{j-1}}\int_0^{t_1} \frac{d_{j-1}}{8\pi(t_1-\tau)} e^{-\frac{|\xi'-x_i|^2}{4(t_1-\tau)}}\varphi_j d\tau d\sigma_{\xi'}$$

$$-\int_{\Gamma'_j}\int_0^{t_1} \frac{d_j}{8\pi(t_1-\tau)} e^{-\frac{|\xi'-x_i|^2}{4(t_1-\tau)}}\varphi_j d\tau d\sigma_{\xi'} \,.$$

We integrate analytically with respect to the time variable and get

$$\int_0^{t_1}\int_{\Gamma'_{j-1}\cup\Gamma'_j} \frac{\partial E_i^1}{\partial n_{\xi'}}\varphi_j d\sigma d\tau = -\int_{\Gamma'_{j-1}} \frac{d_{j-1}}{2\pi|\xi'-x_i|^2} e^{-\frac{|\xi'-x_i|^2}{4t_1}}\varphi_j d\sigma_{\xi'}$$

$$-\int_{\Gamma'_j} \frac{d_j}{2\pi|\xi'-x_i|^2} e^{-\frac{|\xi'-x_i|^2}{4t_1}}\varphi_j d\sigma_{\xi'} \,.$$

We calculate each term numerically and obtain the system of linear equations

$$\mathbf{A}\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \\ u_6^1 \end{bmatrix} = \begin{bmatrix} 0.328298 \\ 0.328298 \\ 0.328298 \\ 0.328298 \\ 0.328298 \\ 0.328298 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.00202 & 0.00049 & 0.00002 & 0.00049 & 0.00202 \\ 0.00202 & 0.5 & 0.00202 & 0.00049 & 0.00002 & 0.00049 \\ 0.00049 & 0.00202 & 0.5 & 0.00202 & 0.00049 & 0.00002 \\ 0.00002 & 0.00049 & 0.00202 & 0.5 & 0.00202 & 0.00049 \\ 0.00049 & 0.00002 & 0.00049 & 0.00202 & 0.5 & 0.00202 \\ 0.00202 & 0.00049 & 0.00002 & 0.00049 & 0.00202 & 0.5 \end{bmatrix}$$

The solution of the first time step is

$$\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \\ u_4^1 \\ u_5^1 \\ u_6^1 \end{bmatrix} = \begin{bmatrix} 0.65005 \\ 0.65005 \\ 0.65005 \\ 0.65005 \\ 0.65005 \\ 0.65005 \end{bmatrix}$$

Conclusions

As shown in this paper the Boundary Element Method (BEM) can be applied to elliptic and parabolic differential equations. The benefit of the method compared to conventional methods such as finite different and finite element method is the reduction of dimension of the problem by one since the BEM deals only with the corresponding boundary integral equation. Moreover for applying the collocation method the unknown trial or ansatz function can be formed by simple functions i.e. piecewise constant and piecewise linear functions. However the disadvantage of the method is the knowledge of the fundamental solution. So its application is limited to linear differential equations

SYMBOLS

$a_{ij}^{pq}, b_{ij}^{pq}, c_{ij}^{pq}$	parameters
h	space step
k	time step
i, j	index parameters
m	number of time intervals
n	number elements
$\vec{n}_j$	unit normal vector
p, q	time parameters
t	time variable
$u(x,y,t)$	solution
u_i^n	collocation parameters
x, y	space variable
$\mathbf{x}$	space vector

Greek letters

α	a constant
χ, φ	trial function
Γ	boundary of the domain
Γ'	polygonal boundary
ξ, τ	integral variable
Ω	domain
Ω'	polygonal domain

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