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ENERGY ANALYSIS OF FREE TRANSVERSE VIBRATIONS OF THE VISCO-ELASTICALLY CONNECTED DOUBLE-MEMBRANE SYSTEM

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Abstract. *The presented paper deals with the analysis of energy transfer in the visco-elastically connected circular double-membrane system for free transverse vibration of the membranes. The system motion is described by a set of two coupled non-homogeneous partial differential equations. The solutions are obtained by using the method of separation of variables. Once the problem is solved, natural frequencies and mode shape functions are found, and then the form of solution for small transverse deflections of membranes is derived. Using the obtained solutions, forms of reduced kinetic, potential and total energies, as functions of dissipation of the whole system and subsystems, are determined. The numerical examples are given as an illustration of the presented theoretical analysis as well as the possibilities to investigate the influence of different parameters and different initial conditions on the energies transfer in the system.*

Key Words: *Double-membrane system, Visco-elastic layer, Dissipation function, Energy transfer, Multi-frequency vibration*

1. INTRODUCTION

Membrane structures and compound systems are widely used in many industrial applications. In addition, in the micro world the system of compound membranes may represent a biological model of a cell membrane. The application involves several disciplines and industrial contexts as, for example, microfiltration systems in biological, medical, food, dairy and beverage products and application in aeronautics, cosmonautics, civil and mechanical engineering (see Refs. [1] and [2]). Membrane, string, beam, plate or cable [3] structures can perform linear and nonlinear vibrations. Despite the fact that the membrane systems are, in general, strongly nonlinear systems, under certain assumptions they can be considered as linear. By assuming that the amplitude of a membrane deflection is

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smaller than its thickness and considering the initial tension on the membrane [4], the dynamic behavior of such structures can be observed within the linear vibration theory. Opposite to this, when no or small initial tension is applied and the membrane deflection amplitude is close to the value of the membrane thickness, the system is performing nonlinear vibration [4]. The linear as well as the nonlinear vibration analysis of membrane systems is important from the theoretical and practical point of view. Several papers have been published on the vibration analysis of elastically coupled double-membrane or plate-membrane systems, continuously connected by an elastic layer. Oniszczuk [5, 6], Noga [7, 8] and Karličić [9] present analytical expressions for the undamped free and forced vibrations of double-membrane and plate-membrane systems connected with the Winkler type of layers. Knowing the principles of phenomenological mapping and mathematical analogy, [10], it is clear that the composite membrane systems can be studied like coupled plate, beam or belt systems. Hedrih and Simonović [11-13] have studied transverse vibrations of the rectangular and circular plates connected with elastic or visco-elastic layers for both linear and nonlinear dynamic problems. Kelly [14] examines vibration of the elastically connected multiple beams system. Energy transfer problem is studied in [15-17] for the vibrations of double-membrane and double-plate systems, connected with elastic or visco-elastic type of layers.

In spite of many vibration studies on various types of hybrid systems, where different structures are coupled with different types of layers, according to the best of authors' knowledge, an energy transfer analysis of free transverse vibration of the visco-elastically connected circular double-membrane system has not been considered in the literature yet. Under the assumptions of small transverse vibration of the membranes and constant initial tension for both the membranes, we analyze our system within the linear vibration theory. Governing equations of the system are derived and the solution is proposed by using the method of separation of variables. Natural frequencies and mode shape functions are determined by solving the boundary and initial value problem. The expressions for time functions are used in order to find analytical forms of reduced kinetic, potential and total energy of the system.

2. FORMULATION OF GOVERNING EQUATIONS

As a model problem, we consider two circular membranes connected through visco-elastic layer, modeled by continuously distributed elements of the Kelvin-Voigt type. The scheme of such a mechanical model is depicted in Fig. 1. Both the membranes are assumed to be thin with mass densities ρ_i, *for* i=1,2 corresponding to the lower and upper membrane, respectively, ideally elastic and with neglected thickness.

The membranes are stretched and fixed along their entire boundaries in *xy* plane. The tensions per unit length σ_i [*N/m*], caused by stretching, are the same at all points in all directions and do not change during the motion. Small transverse deflections of membranes $w_i(r,\varphi,t)$, i=1,2 are considered, where b[*Ns/m*] is denoting the damping coefficient and $\tilde{c}$[*N/m*] is the constant stiffness coefficient per surface unit area of the distributed visco-elastic layer between the membranes. Using the D`Alambert principle, the governing system of non-homogenous coupled partial differential equations for free vibration of the double-membrane system is expressed in the following form:

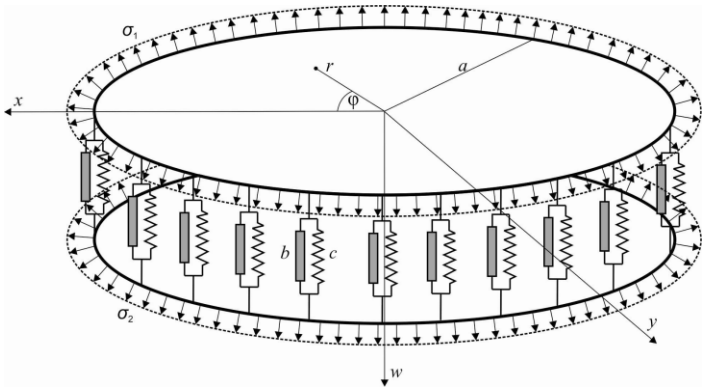

Fig. 1 The physical model of circular double-membrane compound system

$$\frac{\partial^2 w_i(r,\varphi,t)}{\partial t^2} = c_i^2 \Delta w_i(r,\varphi,t) + 2\delta_i \left(\frac{\partial w_{i+1}(r,\varphi,t)}{\partial t} - \frac{\partial w_i(r,\varphi,t)}{\partial t} \right) + a_i^2 \left(w_{i+1}(r,\varphi,t) - w_i(r,\varphi,t) \right) \quad (1)$$

where $i = 1,2$; $c_i = (\sigma_i / \rho_i)^{1/2}$[$m/s$] are velocities of the transverse wave propagation for both the membranes, Δ is Laplacian operator, and $a_i^2 = \tilde{c} / \rho_i$, $2\delta_i = b/\rho_i$ are notations for reduction coefficients.

Analytical solutions of the system of coupled partial differential equations are obtained by using the method of separation of variables. For the system of two coupled partial equations Eq. (1), for free vibration, we separate variables in $w_i(r,\varphi,t)$, $i=1,2$ and considering the eigenamplitude functions as $W_{i(nm)}(r,\varphi)$; $n,m=1,2,...,\infty$ and the time expansion with coefficients in the form of unknown time function as $T_{i(nm)}(t)$ we describe their time evolutions in the form:

$$w_i(r,\varphi) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{i(nm)} T_{i(nm)}(t), \quad i = 1,2 \quad (2)$$

wherein $m = 1,2,...,\infty$ denotes an infinite number of possible vibration modes.
Here, eigenamplitude functions $W_{i(nm)}(r,\varphi)$ are the same for both the membranes and written in the following form:

$$W_{(i)nm}(r,\varphi) = J_n(k_{nm} r)\cos(n\varphi + \varphi_{(i)0n}), \quad (3)$$

which are obtained for the decoupled system and for the same boundary conditions of membranes. In Eq. (3) $J_n(k_{nm}r)$ are Bessel`s functions of the first kind of $n$-th order, and $k_{nm}=x_{nm}/a$ are characteristic numbers for every $m$-th root of $n$-th Bessel`s functions over membrane radius a.

After introducing Eq. (2) into the governing system of Eq. (1), we obtain the following system of the homogeneous second order ordinary differential equations with respect to unknown time functions $T_{i(nm)}(t)$ for nm-family mode, in the following form:

$$\begin{aligned} &\ddot{T}_{1(nm)}(t) + 2\delta_1 \dot{T}_{1(nm)}(t) + \omega_{1(nm)}^2 T_{1(nm)}(t) - 2\delta_1 \dot{T}_{2(nm)}(t) - a_1^2 T_{2(nm)}(t) = 0 \\ &\ddot{T}_{2(nm)}(t) + 2\delta_2 \dot{T}_{2(nm)}(t) + \omega_{2(nm)}^2 T_{2(nm)}(t) - 2\delta_2 \dot{T}_{1(nm)}(t) - a_2^2 T_{1(nm)}(t) = 0, \end{aligned} \quad (4)$$

where $\omega_i^2{}_{(nm)} = \omega_{0i}^2{}_{(nm)} + a_i^2 = k_i^2{}_{(nm)} c_i^2 + a_i^2$; $i=1,2$; $n,m = 1,2,...,\infty$, are natural frequencies for the first and second membrane for the *nm*-th mode of vibration. To solve the system of two coupled ordinary differential Eqs. (4), it is necessary to form the frequent determinant and determine the eigenvalues of the system. We introduce matrices corresponding to the *nm*-th mode of the observed dynamical system: inertia $\mathbf{A}_{nm}$ matrix, stiffness coefficients matrix $\mathbf{C}_{nm}$ and damping coefficients matrix $\mathbf{B}_{nm}$ in the following forms:

$$\mathbf{A}_{nm} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{B}_{nm} = \begin{bmatrix} 2\delta_1 & -2\delta_1 \\ -2\delta_2 & 2\delta_2 \end{bmatrix}; \quad \mathbf{C}_{nm} = \begin{bmatrix} \omega_{1(nm)}^2 & -a_1^2 \\ -a_2^2 & \omega_{2(nm)}^2 \end{bmatrix} \tag{5}$$

The coupled system of Eqs. (4) for time domain may be treated as a system of equations corresponding to a two-degree-of-freedom discrete system with defined matrices in Eq. (5). The characteristic equation of the linearized coupled system is given in the form:

$$\left|\lambda_{nm}^2 \mathbf{A}_{nm} + \lambda_{nm} \mathbf{B}_{nm} + \mathbf{C}_{nm}\right| = \begin{vmatrix} \lambda_{nm}^2 + 2\delta_1\lambda_{nm} + \omega_{1(nm)}^2 & -2\delta_1\lambda_{nm} - a_1^2 \\ -2\delta_2\lambda_{nm} - a_2^2 & \lambda_{nm}^2 + 2\delta_2\lambda_{nm} + \omega_{2(nm)}^2 \end{vmatrix} = 0 . \tag{6}$$

We obtain eigenvalues of the system in *nm* modes as two pairs of complex conjugate roots in the form:

$$\lambda_{1,2(s)(nm)} = -\tilde{\delta}_{(s)(nm)} \mp i \quad \tilde{\omega}_{(s)(nm)}, \ s = 1, 2 \tag{7}$$

where $\tilde{\delta}_{(s)(nm)}$ and $\tilde{\omega}_{(s)(nm)}$ are real and imaginary parts of the corresponding pair of roots of the characteristic equation.

Now, the solution of the system of ordinary differential Eqs. (4) can be expressed as:

$$T_{i(nm)} = \sum_s C_{is} \left(A_{i(nm)}^{(s)} e^{\lambda_{1(s)(nm)} t} + \overline{A}_{i(nm)}^{(s)} e^{\lambda_{2(s)(nm)} t} \right). \tag{8}$$

We obtain eigenamplitude numbers $A^{(s)}{}_{i(nm)}$ and their conjugates $\overline{A}^{(s)}{}_{i(nm)}$ from:

$$\frac{A_{1(nm)}^{(s)}}{K_{21(nm)}^{(s)}} = \frac{A_{2(nm)}^{(s)}}{K_{22(nm)}^{(s)}} = C_{s(nm)} \quad \text{or} \quad \frac{A_{1(nm)}^{(s)}}{a_1^2 + 2\delta_1\lambda_{(s)(nm)}} = \frac{A_{2(nm)}^{(s)}}{\lambda_{(s)(nm)}^2 + 2\delta_1\lambda_{(s)(nm)} + \omega_{1(nm)}^2} = \tilde{C}_s \tag{8a}$$

where $K^{(s)}{}_{2i(nm)}$ are cofactors of the determinant in Eq. (6) and $C_{s(nm)}$ are known constants determined from the corresponding characteristic equation.

Considering the time functions in Eqs.(8) corresponding to frequencies $\tilde{\omega}_{(s)(nm)}$ of the damped coupling and taking into account conjugate complex roots (Eq. (7)) yields:

$$T_{i(nm)} = \sum_{s=1}^{2} e^{-\tilde{\delta}_{s(nm)} t} \left[u_{is(nm)} \cdot \cos(\tilde{\omega}_{(s)(nm)} t) + v_{is(nm)} \cdot \sin(\tilde{\omega}_{(s)(nm)} t) \right]. \tag{9}$$

The constants of integration $u_{is(nm)}$ and $v_{is(nm)}$ are defined as follows:

$$\begin{aligned} u_{is(nm)} &= A_{s(nm)} \operatorname{Re}(K_{2i}^{(s)}) + B_{s(nm)} \operatorname{Im}(K_{2i}^{(s)}), \\ v_{is(nm)} &= A_{s(nm)} \operatorname{Im}(K_{2i}^{(s)}) - B_{s(nm)} \operatorname{Re}(K_{2i}^{(s)}). \end{aligned} \tag{10}$$

$A_{s(nm)}$ and $B_{s(nm)}$ can be obtained through application of the initial conditions. The initial conditions are assumed as known functions:

$$w_i(r,\varphi,0) = g_i(r,\varphi), \tag{11}$$

$$\frac{\partial w_i(r,\varphi,t)}{\partial t}\Big|_{t=0} = \tilde{g}_i(r,\varphi). \tag{12}$$

Now, the particular solutions for a non-homogenous system of the coupled partial differential equations, for free vibrations, as membrane deflections, are:

$$\begin{aligned} w_i(r,\varphi,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{i(nm)}(r,\varphi)\Big\{ & e^{-\tilde{\delta}_{1(nm)}t}[u_{i1(nm)}\cdot\cos(\tilde{\omega}_1 t) + v_{i1(nm)}\cdot\sin(\tilde{\omega}_1 t)] + \\ & + e^{-\tilde{\delta}_{2(nm)}t}[u_{i2(nm)}\cdot\cos(\tilde{\omega}_2 t) + v_{i2(nm)}\cdot\sin(\tilde{\omega}_2 t)]\Big\}. \end{aligned} \tag{13}$$

The solutions from Eq. (13) are analytical results of our research on transversal vibrations of the visco-elastically connected double circular membrane system.

On the basis of the orthogonality properties of the mode shape functions, unknown constants $A_{s(nm)}$ and $B_{s(nm)}$ can be determined from the assumed initial conditions Eq. (11) and Eq. (12). Introducing the known functions of membranes' point displacements and velocities Eq. (11) and Eq. (12) into the solutions Eq. (2) and applying the classical orthonormality conditions of eigenamplitude functions $W_{i(nm)}(r,\varphi)$ and $W_{i(sr)}(r,\varphi)$, in the form:

$$M_{i(nmsr)} = \int_0^a\int_0^{2\pi} W_{i(nm)}(r,\varphi)W_{i(sr)}(r,\varphi)rdrd\varphi = \begin{cases} 0 & sr \neq nm, \\ M_{i(nm)} = \int_0^a\int_0^{2\pi}[W_{i(nm)}(r,\varphi)]^2 rdrd\varphi, \ i=1,\ 2. & sr = nm \end{cases}, \tag{14}$$

we obtain the values of the initial time functions:

$$T_{i0(nm)} = \frac{\int_0^a\int_0^{2\pi} g_i(r,\varphi)W_{(nm)}(r,\varphi)rdrd\varphi}{M_{i(nm)}} \quad \text{and} \quad \dot{T}_{i0(nm)} = \frac{\int_0^a\int_0^{2\pi} \tilde{g}_i(r,\varphi)W_{(nm)}(r,\varphi)rdrd\varphi}{M_{i(nm)}}. \tag{15}$$

In order to find the final forms of the transverse vibrations, the initial-value problem has to be solved:

$$\mathbf{T}_{0(nm)} = \mathbf{K}_{(nm)}\boldsymbol{\alpha}_{(nm)}, \tag{16}$$

where $\mathbf{T}_{0(nm)}=[T_{10(nm)}\ T_{20(nm)}\ \dot{T}_{10(nm)}\ \dot{T}_{20(nm)}]$ is the vector of the known functions Eq. (15), $\boldsymbol{\alpha}_{(nm)}=[A_{1(nm)}\ B_{1(nm)}\ A_{2(nm)}\ B_{2(nm)}]^T$ is the vector of unknown constants needed for solutions and $\mathbf{K}_{(nm)}$ is the functional matrix of the fourth order depending on the system's properties, given as:

$$\mathbf{K}_{(nm)} = \begin{bmatrix} \mathrm{Re}\,K_{21}^1 & \mathrm{Im}\,K_{21}^1 & \mathrm{Re}\,K_{21}^2 & \mathrm{Im}\,K_{21}^2 \\ \mathrm{Re}\,K_{22}^1 & \mathrm{Im}\,K_{22}^1 & \mathrm{Re}\,K_{22}^2 & \mathrm{Im}\,K_{22}^2 \\ -\tilde{\delta}_{(1)}\mathrm{Re}\,K_{21}^1 + \tilde{\omega}_{(1)}\mathrm{Im}\,K_{21}^1 & -\tilde{\delta}_{(1)}\mathrm{Im}\,K_{21}^1 - \tilde{\omega}_{(1)}\mathrm{Re}\,K_{21}^1 & -\tilde{\delta}_{(2)}\mathrm{Re}\,K_{21}^2 + \tilde{\omega}_{(2)}\mathrm{Im}\,K_{21}^2 & -\tilde{\delta}_{(2)}\mathrm{Im}\,K_{21}^2 - \tilde{\omega}_{(2)}\mathrm{Re}\,K_{21}^2 \\ -\tilde{\delta}_{(1)}\mathrm{Re}\,K_{22}^1 + \tilde{\omega}_{(1)}\mathrm{Im}\,K_{22}^1 & -\tilde{\delta}_{(1)}\mathrm{Im}\,K_{22}^1 - \tilde{\omega}_{(1)}\mathrm{Re}\,K_{22}^1 & -\tilde{\delta}_{(2)}\mathrm{Re}\,K_{22}^2 + \tilde{\omega}_{(2)}\mathrm{Im}\,K_{22}^2 & -\tilde{\delta}_{(2)}\mathrm{Im}\,K_{22}^2 - \tilde{\omega}_{(2)}\mathrm{Re}\,K_{22}^2 \end{bmatrix} \tag{16a}$$

From analytical solutions, Eq. (13), and corresponding solutions of constant system, Eq. (16), we can conclude that in one mode of vibration, two circular frequencies of coupling and a two-frequency time function $T_{i(nm)}(t)$ correspond to one eigenamplitude function. The first time mode is with lower damped frequency $\tilde{\omega}_{(1)(nm)}$, and the second one is with higher damped frequency $\tilde{\omega}_{(2)(nm)}$. Hence, the visco-elastic layer introduces into the system duplication of the number of circular frequencies which correspond to the one eigenamplitude function of the *nm*-family mode $n,m=1,2,...,\infty$.
We can rewrite the solutions for time functions in the form:

$$T_{i(nm)} = \sum_i D_{i(nm)} \mathbf{Y}_{i(nm)} e^{\lambda_{i(nm)} t} , \tag{17}$$

where $\mathbf{Y}_{i(nm)}$ are eigenvectors corresponding to following system matrix:

$$\mathbf{Q}_{nm} = \begin{bmatrix} \mathbf{0} & \vdots & \mathbf{I} \\ \cdots & \vdots & \cdots \\ -\mathbf{A}_{nm}^{-1}\mathbf{C}_{nm} & \vdots & -\mathbf{A}_{nm}^{-1}\mathbf{B}_{nm} \end{bmatrix}. \tag{18}$$

The constants of integration $D_{i(nm)}$ can be calculated by solving a system of simultaneous algebraic equations formulated in the matrix form as:

$$\mathbf{T}_{0(nm)} = \sum_i D_{i(nm)} \mathbf{Y}_{i(nm)}, \tag{19}$$

where $\mathbf{T}_{0(nm)}$ is the vector of inital condition constants given by Eq. (15). The system matrix Eq. (18) has two pairs of complex conjugate eigenvalues the same as values Eq. (7). Therefore, we conclude that the solutions given in Eq.(17), Eq. (8) and Eq. (9) are actually the same.

3. ENERGY ANALYSIS

We can analyze energy transfer in the double-membrane system by using reduced components of kinetic and potential energy of membranes, potential energy of the light distributed visco-elastic layer and the reduced Rayleigh function of dissipation for corresponding *nm*-family mode. Also, we can use the system of two ordinary differential Eqs. (4) for corresponding *nm*-th mode, which is considered as a system with two degrees of freedom. For the system of Eqs. (4) it is possible to write forms of kinetic and potential energies by using the matrices Eqs. (5). Reduced forms of energies corresponding to the *nm*-th mode are given as:

a) Reduced kinetic energy $\tilde{\tilde{E}}_{k(nm)}$, $n,m=1,2,...,\infty$:

$$\tilde{\tilde{E}}_{k(nm)} = \frac{1}{2}(\dot{T}_{1(nm)} \quad \dot{T}_{2(nm)})\,\mathbf{A}\begin{Bmatrix} \dot{T}_{1(nm)} \\ \dot{T}_{2(nm)} \end{Bmatrix} = \frac{1}{2}[\rho_1(\dot{T}_{1(nm)})^2 + \rho_2(\dot{T}_{2(nm)})^2] = \frac{E_{k(nm)}}{M_{i(nm)}}, \tag{20}$$

where $E_{k(nm)}$ and $M_{i(nm)}$ are kinetic energy and orthogonality function Eq. (14), respectively.

b) Reduced potential energy $\tilde{\tilde{E}}_{p(nm)}$:

$$\tilde{\tilde{E}}_{p(nm)} = \frac{1}{2}(T_{1(nm)} \quad T_{2(nm)})\,\mathbf{C}\begin{Bmatrix} T_{1(nm)} \\ T_{2(nm)} \end{Bmatrix} = \\ = \frac{1}{2}\tilde{c}[T_{2(nm)} - T_{1(nm)}]^2 + \left[\rho_1\omega_{01(nm)}^2(T_{1(nm)})^2 + \rho_2\omega_{02(nm)}^2(T_{2(nm)})^2\right] = \frac{E_{p(nm)}}{M_{i(nm)}} \quad , \quad (21)$$

where $E_{p(nm)}$ is potential energy.

c) Reduced Rayleigh function of dissipation $\tilde{\tilde{\Phi}}_{(nm)lay}$, which is related to the layer, is of the form:

$$\tilde{\tilde{\Phi}}_{(nm)lay} = \frac{1}{2}(\dot{T}_{1(nm)} \quad \dot{T}_{2(nm)})\begin{bmatrix} \rho_1 2\delta_1 & -\rho_1 2\delta_1 \\ -\rho_2 2\delta_2 & \rho_2 2\delta_2 \end{bmatrix}\begin{Bmatrix} \dot{T}_{1(nm)} \\ \dot{T}_{2(nm)} \end{Bmatrix} = \frac{1}{2}b[\dot{T}_{1(nm)} - \dot{T}_{2(nm)}]^2 = \frac{\Phi_{p(nm)\text{lay}}}{M_{i(nm)}}. \quad (22)$$

Also, we can separate terms for the kinetic and potential energies which correspond to the first and the second membrane, [8, 14, 15]:

a.1) Reduced kinetic energy of the membranes:

$$\tilde{\tilde{E}}_{k(nm)(i)} = \frac{1}{2}\rho_i(\dot{T}_{i(nm)})^2 = \frac{E_{k(nm)(i)}}{M_{i(nm)}}, \quad i = 1,2\,. \quad (23)$$

b.1) Reduced potential energy of the membranes and reduced potential energy of a visco-elastic layer for the corresponding membranes:

$$\tilde{\tilde{E}}_{p(nm)(i)} = \frac{1}{2}\rho_i\omega_{i(nm)}^2(T_{i(nm)})^2 = \frac{E_{p(nm)(i)}}{M_{i(nm)}} \quad (24)$$

b.2) Reduced potential energy of pure interaction between membranes induced by a visco-elastic layer:

$$\tilde{\tilde{E}}_{p(nm)\,\text{int}} = -\frac{1}{2}(\rho_1 a_1^2 + \rho_2 a_2^2)T_{2(nm)}T_{1(nm)} = \frac{E_{p(nm)\text{int}}}{M_{i(nm)}}\,. \quad (25)$$

b.3) Reduced potential energy of the membranes without reduced part of the potential energy of layer for the corresponding membranes:

$$\tilde{\tilde{E}}_{p(nm)(i)b} = \frac{1}{2}\rho_i\omega_{0i(nm)}^2(T_{i(nm)})^2 = \frac{E_{p(nm)(i)b}}{M_{i(nm)}} \quad . \quad (26)$$

b.4) Full reduced potential energy of a visco-elastic layer interaction between the membranes:

$$\tilde{\tilde{E}}_{p(1,2)(n,m)layer} = \frac{1}{2}\tilde{c}[T_{2(n,m)}(t) - T_{1(nm)}(t)]^2 = \frac{E_{p(1,2)(nm)layer}}{M_{i(nm)}}. \quad (27)$$

From the previous equations, we can see that the reduced potential energy is obtained by considering membranes' vibration on the elastic foundation of the Winkler type. Both

the membranes share potential energy of the visco-elastic layer, and only one part is interaction between the membranes depending on layers rigidity and on both time functions of the membranes. In the following, we will introduce:

c.1) Rayleigh function of dissipation-reduced part of a visco-elastic layer for the corresponding membranes:

$$\tilde{\tilde{\Phi}}_{p(nm)\text{layer}(i)} = \rho_i \delta_i (\dot{T}_{i(nm)})^2 = \frac{\Phi_{p(nm)\text{layer}(i)}}{M_{i(nm)}} . \tag{28}$$

c.2) Part of the Rayleigh dissipation function – pure interaction between the membranes induced by visco-elastic layer

$$\tilde{\tilde{\Phi}}_{p(nm)\text{layer(int)}} = -(\rho_1 \delta_1 + \rho_2 \delta_2)\dot{T}_{1(nm)}\dot{T}_{2(nm)} = \frac{\Phi_{p(nm)\text{layer(int)}}}{M_{i(nm)}} . \tag{29}$$

If we use the solutions from Eq. (9) of free vibrations and their derivatives with respect to time, the reduced total membranes energy for the *nm*-th mode can be expressed as follows.

Reduced total energy of the first, *i=1*, and second, *i=2*, membranes in the *nm*-mode are:

$$\tilde{\tilde{E}}_{(nm)(mi)} = \tilde{\tilde{E}}_{k(nm)(i)} + \tilde{\tilde{E}}_{p(nm)(i)} = \frac{1}{2}\rho_i \left[(\dot{T}_{i(nm)})^2 + \omega_{i(nm)}^2 (T_{i(nm)})^2 \right] = \frac{E_{k(nm)(i)} + E_{p(nm)(i)}}{M_{i(nm)}} . \tag{30}$$

Therefore, the reduced total energy of both membranes is:

$$\tilde{\tilde{E}}_{(nm)(m1,\, m2)} = \tilde{\tilde{E}}_{k(nm)(1)} + \tilde{\tilde{E}}_{p(nm)(1)} + \tilde{\tilde{E}}_{k(nm)(2)} + \tilde{\tilde{E}}_{p(nm)(2)} = \tilde{\tilde{E}}_{(nm)(m1)} + \tilde{\tilde{E}}_{(nm)(m2)} . \tag{31}$$

Reduced total energy of the system in the *nm*-th mode is equal to the sum of the reduced total energy of both the membranes and the reduced potential energy of pure interaction between them (Eq. (25)):

$$\tilde{\tilde{E}}_{(nm)system} = \tilde{\tilde{E}}_{(nm)(m1,\, m2)} + \tilde{\tilde{E}}_{p(nm)\text{int}} . \tag{32}$$

As concluded in [17] for transverse vibrations of the double membrane system with elastic layer, the total energy of the system remains constant and equal to the energy in the initial moment during the whole dynamic process which is indeed a property of conservative systems. Here, for a non-conservative system, we have energy dissipation which is equal to:

$$\frac{\tilde{\tilde{E}}_{(nm)system}}{dt} = -2\Phi_{(nm)\text{system}} = -2\Phi_{(nm)lay} , \tag{33}$$

where $\Phi_{(nm)lay}$ is the Rayleigh function.

This means that the dissipation function is a measure of the degradation of the mechanical energy of system, notwithstanding the choice of system parameters and the initial conditions that is also confirmed in the numerical results section.

4. NUMERICAL RESULTS FOR THE SYSTEM ENERGY

For the numerical calculation and representation of Eqs. (32) and (33), the following parameter values are considered: tension per unit length σ_1=600[*N/m*], density ρ_1=200[kg/m^3] and radius a=1*m* for the upper membrane and for the lower membrane we consider the same radius but different tension and density. For the discussion of the results their values are presented in Table 1. We introduce the following coefficients $\xi=\rho_2/\rho_1$ and $\eta=\sigma_2/\sigma_1$ with values presented in Table 1. All the simulations are performed in the first shape mode *nm*=01 when k_{01}=2.40483.

When we have $\xi=\eta=1$, it could be noticed that the damped higher frequency of the second time mode is constant $\tilde{\omega}_2$=4.16528[s^{-1}]=$k_{01}(\sigma_1/\rho_1)^{1/2}$=*const* and that mode is not damped $\tilde{\delta}_2$=0 for different values of layer parameters. The lower frequency aptly corresponds to the changes of stiffness coefficient $\tilde{c}$ and damping coefficient *b*. Lower natural frequency $\tilde{\omega}_1$ increases for an increase of the layers stiffness and it decreases for an increase of the damping coefficient.

Table 1 The eigenvalues of the system in Eq. (7) for proper 01-mode, for different system parameters

a=1[m]			$\xi=0.5$		$\xi=1$		$\xi=2$	
$\tilde{c}$ [N/m]	b [Ns/m]		$\eta=0.5$	$\eta=1$	$\eta=0.5$	$\eta=1$	$\eta=0.5$	$\eta=1$
100	100	$\lambda_{1,2(1)}$	-0.75 ± 4.28i	-0.53 ± 5.87i	-0.27 ± 4.16i	-0.5 ± 4.25i	-0.26 ± 4.2i	-0.26 ± 4.19i
		$\lambda_{1,2(2)}$	0 ± 4.17i	-0.22 ± 4.27i	-0.23 ± 3.06i	0 ± 4.16i	-0.11 ± 2.14i	-0.11 ± 3.0i
	200	$\lambda_{1,2(1)}$	-1.50 ± 4.07i	-1.10 ± 5.52i	-0.5 ± 3.89i	-1.0 ± 4.16i	-0.51 ± 4.11i	-0.53 ± 4.07i
		$\lambda_{1,2(2)}$	0 ± 4.16i	-0.39 ± 4.46i	-0.5 ± 3.22i	0 ± 4.16i	-0.24 ± 2.17i	-0.21 ± 3.06i
	300	$\lambda_{1,2(1)}$	-2.25 ± 3.71i	-1.9 ± 5.02i	-0.3 ± 3.61i	-1.5 ± 4.01i	-0.76 ± 3.97i	-0.84 ± 3.86i
		$\lambda_{1,2(2)}$	0 ± 4.16i	-0.35 ± 4.69i	-1.2 ± 3.31i	0 ± 4.16i	-0.36 ± 2.21i	-0.28 ± 3.18i
250	100	$\lambda_{1,2(1)}$	-0.75 ± 4.53i	-0.57 ± 6.02i	-0.32 ± 4.26i	-0.5 ± 4.43i	-0.27 ± 4.29i	-0.28 ± 4.29i
		$\lambda_{1,2(2)}$	0± 4.16i	-0.18 ± 4.34i	-0.18 ± 3.16i	0 ± 4.16i	-0.1 ± 2.22i	-0.09 ± 3.05i
	200	$\lambda_{1,2(1)}$	-1.50 ± 4.34i	-1.18 ± 5.72i	-0.64 ± 4.01i	-1.0 ± 4.34i	-0.54 ± 4.21i	-0.57 ± 4.18i
		$\lambda_{1,2(2)}$	0 ± 4.16i	-0.32 ± 4.48i	-0.35 ± 3.31i	0 ± 4.16i	-0.21 ± 2.24i	-0.17 ± 3.1i
	300	$\lambda_{1,2(1)}$	-2.25 ± 4.00i	-1.94 ± 5.28i	-1.21 ± 3.6i	-1.5 ± 4.19i	-0.81 ± 4.07i	-0.89 ± 4i
		$\lambda_{1,2(2)}$	0 ± 4.16i	-0.31 ± 4.66i	-0.29± 3.55i	0 ± 4.16i	-0.32 ± 2.28i	-0.23 ± 3.18i
500	100	$\lambda_{1,2(1)}$	-0.75 ± 4.93i	-0.62 ± 6.27i	-0.38 ± 4.47i	-0.5 ± 4.7i	-0.29 ± 4.45i	-0.30 ± 4.46i
		$\lambda_{1,2(2)}$	0 ± 4.16i	-0.13 ± 4.42i	-0.12 ± 3.27i	0 ± 4.16i	-0.08 ± 2.32i	-0.07 ± 3.11i
	200	$\lambda_{1,2(1)}$	-1.50 ± 4.75i	-1.27 ± 6.03i	-0.78 ± 4.29i	-1.0 ± 4.62i	-0.58 ± 4.38i	-0.62 ± 4.37i
		$\lambda_{1,2(2)}$	0 ± 4.16i	-0.23 ± 4.51i	-0.22 ± 3.36i	0 ± 4.16i	-0.17 ± 2.34i	-0.12 ± 3.15i
	300	$\lambda_{1,2(1)}$	-2.25 ± 4.45i	-2 ± 5.66i	-1.27 ± 4.0i	-1.5 ± 4.48i	-0.87 ± 4.25i	-0.96± 4.23i
		$\lambda_{1,2(2)}$	0 ± 4.16i	-0.25 ± 4.63i	-0.22 ± 3.48i	0 ± 4.16i	-0.25 ± 2.37i	-0.17 ± 3.2i

It is interesting to note that the same is true for $\xi=\eta=const$, which is marked with gray cells in Table 1. This mechanism of changes is also the same for other arbitrary values of σ_1 and ρ_1 when $\xi=\eta=const$, where we have obtained $\tilde{\omega}_2=k_{01}(\sigma_1/\rho_1)^{1/2}=const$ and $\tilde{\delta}_2$=0 for all values of layers parameters.

Comparing to the other values in Table 1, it is obvious that the damped natural frequencies increase for an increase of the layers stiffness, whereas damped lower frequency $\tilde{\omega}_1$ decreases and higher frequency $\tilde{\omega}_2$ increases for an increase of damping coefficient. The final forms of time functions, Eqs.(9) rely on the values of vector $\boldsymbol{\alpha}_{(nm)}$. The relations for time functions and their time derivatives determine the values of reduced energies, Eqs. (30-32), and dissipative function, Eq. (33), of subsystems and the

system itself. It is possible to obtain the diagrams for the system and subsystems energies for every particular value of parameters. For the qualitative analyses several different parameters for membranes and layers are selected for the diagrams of energies and presented in the following Figures.

Fig. 2a shows the reduced total energy of the system, Eq. (32), in appropriate 01-mode for different values of visco-elastic layer stiffness and the same initial conditions. The parameters used in simulation for this Figure are $\sigma_1=2\sigma_2=600$ [*N/m*], $\rho_1=5\rho_2=200$ [kg/m^3], $b=200$ [*Ns/m*] for three different values of $\tilde{c}=\{100, 250 \text{ and } 500\}$[*N/m*], and initial conditions $w_{10}=0.0025W_{01}$, $w_{20}=0.001W_{01}$, $\dot{w}_{10}=\dot{w}_{20}=0$. As can be seen from Fig. 2a, the total energy of the system is larger and more slowly dissipated for higher values of the visco-elastic layer stiffness coefficient than for the lower values. At initial time, total energy depends on the energy given to the system by appropriate initial conditions, as they are the same for those three lines starting from the same appropriate point.

Fig. 2b shows the double reduced Rayleigh function of the dissipation, Eq. (29), in the upper part of diagrams, and time exchange of reduced total energy of the system $\tilde{\tilde{E}}_{(nm)system}/dt$ in the lower part of diagrams. Those diagrams are obtained for the same values of parameters and initial conditions as in Fig. 2a. Also, as it stems from Eq.(33), time exchange of total energy is equal to the negative double value of dissipation function and diagrams from Fig. 2b are completely symmetric.

Figs. 3a and 3b show similar diagrams as Figs. 2a and 2b, respectively, with the same initial conditions but for different system parameters: $\sigma_1=5\sigma_2=600$ [*N/m*], $\rho_1=\rho_2/2=200$ [kg/m^3], $\tilde{c}=250$[*N/m*] and for varied values of damping coefficient $b=\{100, 200 \text{ and } 300\}$[*Ns/m*]. In this case, the total system energy dissipates faster as the damping constant increases.

In order to investigate the effect of change of the initial conditions, numerical simulations are performed for the same stiffness and damping coefficients. Diagrams in Figs. 4a and 4b are similar to those in Figs. 2a and 2b, respectively. However, in this case calculations are performed for different system parameters: $\sigma_1=5$, $\sigma_2=600$ [*N/m*], $\rho_1=\rho_2/2=200$ [kg/m^3], $\tilde{c}=250$[*N/m*], $b=300$ [*Ns/m*]. In addition, three different cases of initial displacements and velocities on the first and the second membranes are employed: I) $w_{10}=0.002W_{01}$, $w_{20}=0$, $\dot{w}_{10}=\dot{w}_{20}=0$, the lower membrane is at rest in the initial moment; II) $w_{10}=0.002W_{01}$, $w_{20}=-0.002W_{01}$, $\dot{w}_{10}=\dot{w}_{20}=0$, the membrane points are having the opposite initial positions corresponding to the first amplitude shape function, so-called initially anti-phased membranes, and III)$w_{10}=w_{20}=0.002W_{01}$, $\dot{w}_{10}=\dot{w}_{20}=0$, the membrane points are having the same initial positions corresponding to the first amplitude shape function, so-called initially phased membranes. The shapes of membranes at the initial moment for the three cases are presented in Figs. 5a, 5b and 5c, respectively.

The forms of membrane deflection and velocity at the initial moment determine the mechanical energy given to the double membrane system. In Fig. 4a one can see different starting points of the total energy function for different initial conditions. The largest values obtained for the case of initially anti-phase membranes, since the system has received the greatest potential energy in the initial moment when the membrane points are having the opposite positions corresponding to the first amplitude shape function.

Since the system is damped, after some period of time the total energy of the system is dissipated to the zero value. As can be seen from Fig. 4 for all the three simulated conditions the period of time needed for dissipation of the total energy does not change and we can say that the system's damped properties depend only on the system parameters.

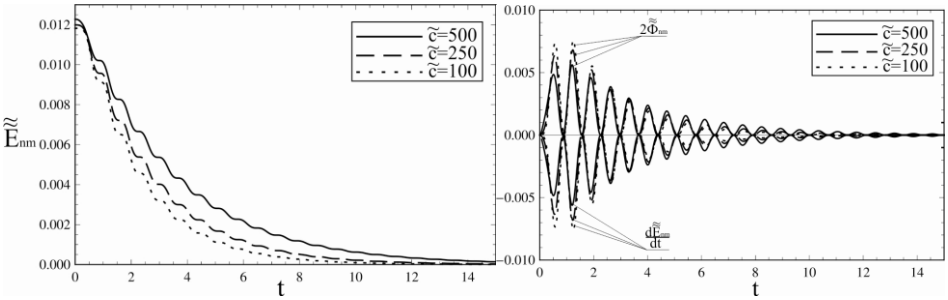

Fig. 2 a) The reduced total energy of the system and b) the double reduced Rayleigh function of dissipation and time exchange of the system's reduced total energy in appropriate 01-mode for system parameters and different values of stiffness coefficient c̃

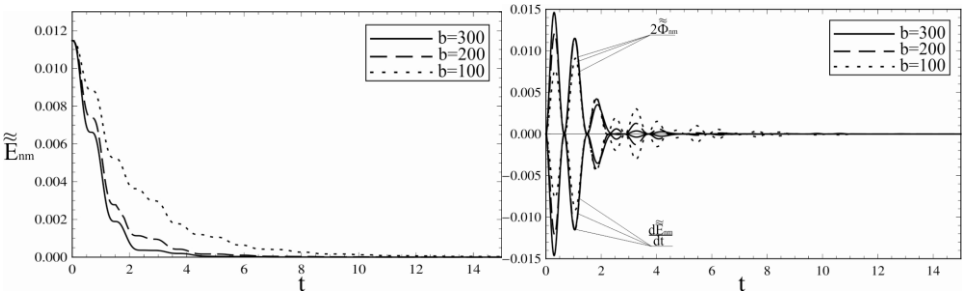

Fig. 3 a) The system's reduced total energy and b) the double reduced Rayleigh function of dissipation and time exchange of the system's reduced total energy in appropriate 01-mode for system parameters and different values of damping coefficient b

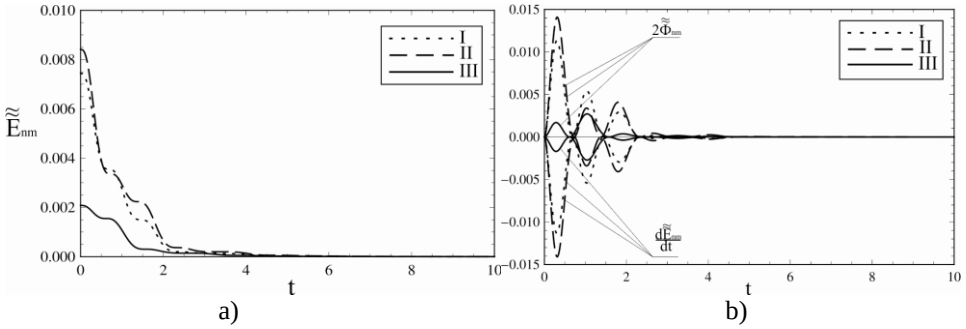

Fig. 4 a) The system's reduced total energy and b) the double reduced Rayleigh function of dissipation and time exchange of the system's reduced total energy in appropriate 01-mode for system parameters and different three values of initial displacement and velocity on the first and the second membranes (I, II, III)

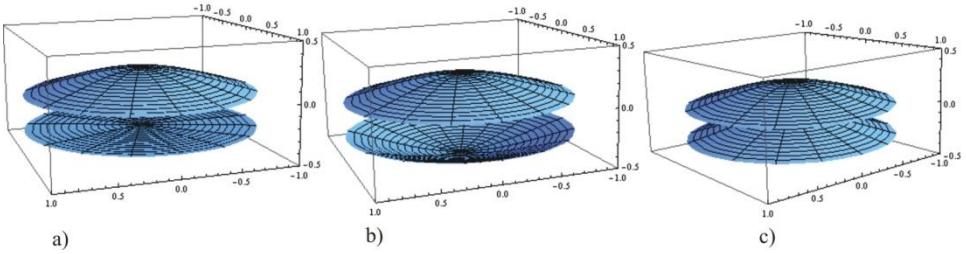

Fig. 5 The shape of membranes at different initial moment: a) the lower membrane is at rest at the initial moment; b) so-called initially anti-phased membranes, and c) so-called initially phased membranes

5. CONCLUSIONS

In this paper, the free vibration and energy analysis of a double-membrane system joined by the Kelvin-Voigt type layer is performed analytically and numerically. The obtained forms of solutions for time functions Eq.(9) in proper mode shape are rather down-to-earth solutions since they are applicable in numerical experiment. Furthermore, those forms of solutions could be used in the analysis of forced and nonlinear system oscillations. The analytical analysis showed that the Kelvin-Voigt type layer is responsible for the appearance of two-frequency regimes, which corresponds to one eigenamplitude function. Time functions for different modes of vibrations are uncoupled and energy transfer appears in a single mode. The system is non-conservative and damped by a visco-elastic layer, so the dissipation function is a measure of the degradation of system's mechanical energy, notwithstanding the choice of system parameters and the change of the initial conditions. The mechanical energy given to the system at the initial time decreases until the whole energy of the system is dissipated. It can be concluded that influence of the system parameters is more significant for the mechanical energy analysis than the change of the initial conditions, which is, in general, a property of the linear systems.

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ENERGIJSKA ANALIZA SLOBODNIH TRANSVERZALNIH VIBRACIJA SISTEMA VISKO-ELASTIČNO SPREGNUTIH MEMBRANA

Predstavljeni rad je posvećen analizi prenosa energije kod slobodnih transverzalnih oscilacija sistema visko-elastično spregnute dve membrane. Kretanje sistema je opisano sistemom dve spregnute nehomogene parcijalne diferencijalne jednačine. Rešenja su dobijena primenom metode razdvajanja promenljivih. Rešavajući problem dobijaju se sopstvene kružne frekvencije i osnovni amplitudni oblici oscilovanja sistema, a potom i oblici rešenja za male transverzalne pomeraje membrana. Koristeći dobijena rešenja određene su redukovane vrednosti kinetičke, potencijalne energije i funkcije rasipanja kako celog sistema tako i podsistema. Primeri numeričkog proračuna su dati kao ilustracija prikazane teorijske analize i kao mogućnost da se prouče uticaji različitih parametara sistema i različitih početnih uslova na prenos energije u sistemu.

Ključne reči: *sistem dve membrane, visko-elastični sloj, funkcija rasipanja, prenos energije, više-frekventne oscilacije.*