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On Cumulative Residual Renyi's Entropy

Sobre la entropía residual acumulada de Renyi

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Abstract

At the entropy measures and their generalization path, in the direction of statistics and information science, recently, Sunoj & Linu (2012) proposed the cumulative residual Renyi's entropy of order α and its dynamic version and studied its main properties. In this paper, we introduce an alternative measure of cumulative residual Renyi's entropy (CRRE) of order α which, unlike the mentioned one, is positive for all distributions and all values of α. We also consider its dynamic version and study their main properties in the context of reliability theory and stochastic orders. We give an estimator of the proposed CRRE and investigate its exact and asymptotic distribution. Numerous examples illustrating the theory are also given.

***Key words*:** Aging classes; Cumulative residual entropy, Mean residual lifetime, Stochastic orders, Shannon entropy, Tsallis entropy.

Resumen

En las medidas de entropía y su camino de generalización, en la dirección de las estadísticas y la ciencia de la información, recientemente, Sunoj & Linu (2012) propuso el residual acumulativo la entropía de Renyi de orden α y su versión dinámica y se estudiaron sus principales propiedades. En este artículo presentamos una medida alternativa de la entropía residual acumulada de Renyi (CRRE) de orden α que, a diferencia de la mencionada, es positiva para todas las distribuciones y todos los valores de α. También consideramos su versión dinámica y estudiamos sus principales propiedades en el contexto de la teoría de la confiabilidad y los órdenes estocásticos. Damos un estimador del CRRE propuesto e investigamos su distribución exacta y asintótica. También se dan numerosos ejemplos que ilustran la teoría.

***Palabras clave*:** Clases de envejecimiento; Entropía residual acumulada; Entropía de Shannon, Entropía de Tsallis; Vida útil residual media; Órdenes estocásticas.

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1. Introduction

It is well-known that the approach in Shannon (1948) was one of the first works for mathematically quantifying of the information entropy that employed the probability and chance notion and linked between these two notions. Considering different point of views, various generalizations of Shannon's entropy have been given by many researchers. For a comprehensive entropy-related works review and the history of the derivation of Shannon's entropy and its different generalizations, we refer the reader to Nanda & Chowdhury (2019). It is worth to mention that Shannon's entropy and its different versions were firstly introduced for the discrete probability spaces. The continuous versions of the entropy measures have been usually given by replacing the sum notation with the integral, straightforwardly. Among the several generalizations of Shannon's entropy, Rényi (1961) has introduced an important one which for a non-negative continuous random variable X with density function $f(x)$ is given by

$$\mathcal{E}_\beta(X) = \frac{1}{1-\beta}\log(\int_0^\infty f^\beta(x)dx), \quad \beta \neq 1, \quad \beta > 0,$$

where log is the natural logarithm. The dynamic version of $\mathcal{E}_\beta(X)$ has been studied by Abraham & Sankaran (2005). However, the density function is not necessarily exist for all random variables. On the other hand, the obtained entropy measures by the density function may does not have the required main properties of an information measure. For example, they may take negative values (see, Figure 1 for a plot of $\mathcal{E}_\beta(X)$). Regarding these and other limitations, alternative generalizations of entropy measures have been introduced by researchers through replacing the density function with the survival and distribution functions. This kind of generalization has been started by Rao et al. (2004) when they proposed the generalized version of Shannon's entropy as

$$\mathcal{E}(X) = -\int_0^\infty \bar{F}(x) log \bar{F}(x)dx,$$

where $\bar{F}(x) = 1 - F(x)$ is the survival function of random variable X with distribution function F. Recently, motivated by the usefulness of Renyi's entropy and Rao et al.'s cumulative residual entropy (CRE) measure, Sunoj & Linu (2012) have introduced cumulative residual Renyi's entropy (CRRE) of order β as

$$\gamma(\beta) = \frac{1}{1-\beta}\log(\int_0^\infty \bar{F}^\beta(x)dx), \quad \beta \neq 1, \quad \beta > 0.$$

They also considered the dynamic version of the CRRE (DCRRE, by extending it to the residual lifetime variable) and studied its main properties useful in reliability modelling.

As Figure 1 depicts, $\gamma(\beta)$ has still the drawback that it may take negative values for some distributions. It is also worth to recall that the Tsallis entropy is one of the well-known entropy measures which its cumulative version of order α (CTE) has been introduced by Rajesh & Sunoj (2019) as

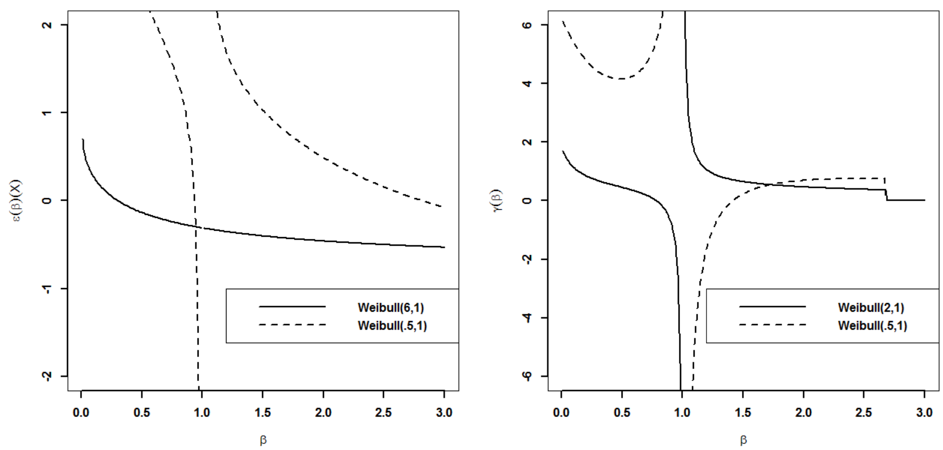

FIGURE 1: Plots of $\mathcal{E}_\beta(X)$ and $\gamma(\beta)$.

$$\mathcal{T}_\alpha(X) = \frac{1}{\alpha - 1}\int_0^\infty [\bar{F}(x) - \bar{F}^\alpha(x)]dx, \quad \alpha \neq 1, \quad \alpha > 0. \tag{1}$$

Here, $\mathcal{T}_\alpha(X)$ is always positive.

In this paper, by re-parametrization of $\gamma(\beta)$ (replacing β by $\alpha + 1$) and by normalizing it, we propose an alternative measure of the CRRE of order α by the following:

$$\gamma_\alpha(X) = -\frac{1}{\alpha}\log\left(\frac{\int_0^\infty \bar{F}^{\alpha+1}(x)dx}{\int_0^\infty \bar{F}(x)dx}\right), \quad \alpha > 0. \tag{2}$$

It is clear that $\gamma_\alpha(X)$ is always positive. The rest of the paper is organized as follows. In Section 2, we first give the main properties of $\gamma_\alpha(X)$. Comparing values of the CRRE under various stochastic orders between random variables are also studied in this section. Section 3 is devoted to the dynamic CRRE and its properties. The estimation of the proposed CRRE and its properties are investigated in Section 4. Finally, some conclusions are given in Section 5.

Before proceeding to give the main results of the paper, we overview some preliminary concepts of ageing and stochastic orders (For more details of these concepts see, for example, Shaked & Shanthikumar, 2007).

Let X and Y be non-negative random variables with the distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, hazard functions

$$\lambda_X(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\log\bar{F}(t) \text{ and } \quad \lambda_Y(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\log\bar{G}(t),$$

and mean residual life functions

$$m_X(t) = \frac{1}{\bar{F}(t)}\int_t^\infty \bar{F}(x)\mathrm{d}x \text{ and } \; m_Y(t) = \frac{1}{\bar{G}(t)}\int_t^\infty \bar{G}(x)\mathrm{d}x,$$

respectively. Throughout this paper we assume that these functions all exist and increasing (decreasing) means non-decreasing (non-increasing).

Definition 1. The random variable X is said to be:

(i) increasing (decreasing) failure rate in average, IFRA (DFRA), if $-\frac{1}{t}\log \bar{F}(t)$ is increasing (decreasing) in t,

(ii) new better (worse) than used, NBU(NWU), if $\bar{F}(x+t) \leq (\geq) \bar{F}(x)\bar{F}(x)$, for all $x, t > 0$,

(iii) new better (worse) than used in expectation, NBUE(NWUE), if $m_X(t) \leq (\geq)\ m_X(0)$, for all $t > 0$,

(iv) increasing (decreasing) mean residual life (IMRL(DMRL)) if $m_X(t)$ is increasing (decreasing) in t,

(v) smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all t,

(vi) smaller than Y in the hazard rate ordering (denoted by $X \leq_{hr} Y$) if $\lambda_X(t) \geq \lambda_Y(t)$ for all t,

(vii) smaller than Y in the mean residual lifetime ordering (denoted by $X \leq_{mrl} Y$) if $m_X(t) \leq m_Y(t)$ for all t,

(viii) smaller than Y in the DMRL order (denoted by $X \leq_{dmrl} Y$) if $\frac{m_Y(G^{-1}(u))}{m_X(F^{-1}(u))}$ is increasing in $u \in [0,1]$,

(ix) smaller than Y in the NBUE order (denoted by $X \leq_{nbue} Y$) if $\frac{m_X(F^{-1}(u))}{m_Y(G^{-1}(u))} \leq \frac{E[X]}{E[Y]}$ for all $u \in [0,1]$,

(x) smaller than Y in the increasing convex order (denoted by $X \leq_{icx} Y$) if $E[\phi(X)] \leq E[\phi(Y)]$, for all increasing convex functions ϕ,

(xi) smaller than Y in the dispersive order (denoted by $X \leq_{disp} Y$) if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$, whenever $0 < \alpha \leq \beta < 1$,

(xii) smaller than Y in the Lorenz order (denoted by $X \leq_L Y$) if $\frac{1}{E(X)}\int_0^u F^{-1}(v)\mathrm{d}v \geq \frac{1}{E(Y)}\int_0^u G^{-1}(v)\mathrm{d}v$ for all $u \in [0,1]$.

2. Some Properties of $\gamma_\alpha(X)$

Note that $\gamma_\alpha(X)$ can also be written as

$$\gamma_\alpha(X) = -\frac{1}{\alpha}\log(E[\bar{F}^\alpha(X_e)]), \tag{3}$$

where, X_e is the equilibrium random variable corresponding to X with density function $f_e(x) = \frac{\bar{F}(x)}{\mu}$, $\mu = E[X] = \int_0^\infty \bar{F}(x)dx$. The following theorem gives the main properties of the CRRE $\gamma_\alpha(X)$ for $\alpha > 0$.

Theorem 1. **(a)** $\gamma_\alpha(X) \geq 0$.

(b) $\gamma_\alpha(X)$ *is decreasing in* α.

(c) $\lim_{\alpha\to 0}\gamma_\alpha(X) = \frac{\mathcal{E}(X)}{\mu}$.

(d) *If* $Y = aX$, $a > 0$, *then* $\gamma_\alpha(Y) = \gamma_\alpha(X)$.

(e) $\frac{\mathcal{T}_{\alpha+1}(X)}{\mu} \leq \gamma_\alpha(X) \leq \frac{\mathcal{E}(X)}{\mu}$.

(f) $\gamma_\alpha(X) \geq -\frac{1}{\alpha}\log\left(\frac{\alpha+1}{2^\alpha\mu}E^{1-\alpha}[X^{\frac{1}{1-\alpha}}]\right)$, *for* $0 < \alpha < 1$.

***Proof*.** We only give the proof of parts (b), (e) and (f). For a proof of (b), use (3) and the Lyapounov inequality. To prove (e), first note that $\gamma_\alpha(X)$ can also be expressed as

$$\gamma_\alpha(X) = -\frac{1}{\alpha}\log\left(1-\frac{\alpha}{\mu}\int_0^\infty m_X(x)\bar{F}^\alpha(x)dF(x)\right) \tag{4}$$

$$= -\frac{1}{\alpha}\log(1-\frac{\alpha}{\mu}\mathcal{T}_{\alpha+1}(X)), \tag{5}$$

where the last equation follows from equation (7) in Rajesh & Sunoj (2019). Now, applying the inequality $-\log(1-x) \geq x$, $0 \leq x \leq 1$, implies that $\gamma_\alpha(X) \geq \frac{\mathcal{T}_{\alpha+1}(X)}{\mu}$. On the other hand, using the log-sum inequality we have

$$\gamma_\alpha(X) = -\frac{1}{\alpha}\log\left(\frac{\int_0^\infty \bar{F}^{\alpha+1}(x)dx}{\int_0^\infty \bar{F}(x)dx}\right) = \frac{1}{\alpha}\log\left(\frac{\int_0^\infty \bar{F}(x)dx}{\int_0^\infty \bar{F}^{\alpha+1}(x)dx}\right) \leq \frac{\mathcal{E}(X)}{\mu}.$$

This completes the proof. □

To prove part (f), the integration by parts gives that

$$\gamma_\alpha(X) = -\frac{1}{\alpha}\log\left(\frac{\alpha+1}{\mu}\int_0^\infty x\bar{F}^\alpha(x)dF(x)\right). \tag{6}$$

The result now follows from Holder's inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$.

Example 1. Let X be a random variable with a Weibull distribution and survival function $\bar{F}(x) = e^{-(\lambda x)^\beta}$. Then

$$\gamma_\alpha(X) = \frac{1}{\alpha\beta}\log(\alpha+1).$$

Example 2. Let X be distributed as Pareto with survival function $\bar{F}(x) = \left(\frac{b}{b+x}\right)^a$, $x, b > 0, a > 1$. Then

$$\gamma_\alpha(X) = -\frac{1}{\alpha}\log\left(\frac{a-1}{a\alpha+a-1}\right).$$

The following theorem compares the CRREs of two random variables when one is smaller than the other in some stochastic orders.

Theorem 2. **(i)** *If $X \leq_{st} Y$, then $\gamma_\alpha(X) \geq \gamma_\alpha(Y) - \frac{1}{\alpha}\log\left(\frac{E[Y]}{E[X]}\right)$, for all $\alpha > 0$.*

(ii) *If $X \leq_{icx} Y$ and $E[X] = E[Y]$, then $\gamma_\alpha(X) \leq \gamma_\alpha(Y)$, for all $\alpha > 0$.*

(iii) *If $X \leq_{nbue} Y$, then $\gamma_\alpha(X) \leq \gamma_\alpha(Y)$, for all $\alpha > 0$.*

(iv) *If $X \leq_L Y$, then $\gamma_\alpha(X) \leq \gamma_\alpha(Y)$, for all $\alpha > 0$.*

***Proof*.** □

For part (ii), first note that the CTE given in (1) can also be rewritten as

$$\mathcal{T}_\alpha(X) = \frac{1}{\alpha - 1}\int_0^\infty F^{-1}(u)[1 - \alpha(1-u)^{\alpha-1}]du. \tag{7}$$

This along with Theorem 4.A.4 in Shaked & Shanthikumar (2007, p. 183) implies that if $X \leq_{icx} Y$, then $\mathcal{T}_{\alpha+1}(X) \leq \mathcal{T}_{\alpha+1}(Y)$. Part (ii) now follows from (5) and the hypothesis $E[X] = E[Y]$.

To prove part (iii), one can see that equation (4) can also be given as

$$\gamma_\alpha(X) = -\frac{1}{\alpha}\log\left(1 - \frac{\alpha}{E[X]}\int_0^1 m_X(F^{-1}(u))(1-u)^\alpha du\right).$$

The result now follows from the fact that $X \leq_{nbue} Y$ is equivalent to that

$$\frac{m_X(F^{-1}(u))}{E[X]} \leq \frac{m_Y(G^{-1}(u))}{E[Y]}, \quad 0 \leq u \leq 1,$$

(see Shaked & Shanthikumar, 2007).

Finally, for the proof of (iv), using the integration by parts, we obtain from equation (6) that

$$\begin{aligned}
\gamma_\alpha(X) &= -\frac{1}{\alpha}\log((\alpha+1)\int_0^1 \frac{F^{-1}(u)}{E(X)}(1-u)^\alpha du) \\
&= -\frac{1}{\alpha}\log(\alpha(\alpha+1)\int_0^1 (1-u)^{\alpha-1}L_X(u)du),
\end{aligned} \tag{8}$$

where $L_X(u) = \frac{1}{E(X)}\int_0^u F^{-1}(v)dv$ is the Lorenz curve corresponding to X. The result now follows from the fact that under the hypothesis, $L_X(u) \geq L_Y(u)$, for all $u \in [0,1]$.

It is readily seen from (8) that, for any integer $k \geq 1$, $\gamma_k(X) = -\frac{1}{k}\log(1 - G_k)$, where G_k is the kth Gini index (see Farris, 2010).

Remark 1. Note that the CTE can also be given by

$$\mathcal{T}_\alpha(X) = \int_0^\infty \varphi_F(x;\alpha)dF(x),$$

where $\varphi_F(x;\alpha) = \frac{1}{\alpha-1}\int_0^x [1-\bar{F}^{\alpha-1}(y)]dy$ is an increasing function of x for all $\alpha > 0$. Additionally, $X \leq_{st} Y$ if and only if $E[\phi(X)] \leq E[\phi(Y)]$ for all increasing function ϕ Shaked & Shanthikumar (2007, p. 4). Thus, $X \leq_{st} Y$ implies that $\mathcal{T}_\alpha(X) \leq \int_0^\infty \varphi_F(x;\alpha)dG(x) \leq \int_0^\infty \varphi_G(x;\alpha)dG(x) = \mathcal{T}_\alpha(Y)$, for $\alpha > 0$. Since $\lim_{\alpha\to 1}\mathcal{T}_\alpha(X) = \mathcal{E}(X)$, we get that if $X \leq_{st} Y$, then $\mathcal{E}(X) \leq \mathcal{E}(Y)$. This improves the inequality given in Proposition 2.1 in Navarro et al. (2010).

Remark 2. One can see from the proof of part (ii) in the above theorem that if $X \leq_{icx} Y$, then $\mathcal{T}_{\alpha+1}(X) \leq \mathcal{T}_{\alpha+1}(Y)$. Letting α goes to zero implies that if $X \leq_{icx} Y$, then $\mathcal{E}(X) \leq \mathcal{E}(Y)$. This relation has been used in Zardasht (2015) to test the increasing convex order hypothesis.

3. Dynamic Cumulative Residual Renyi's Entropy

If random variable X has survival function $\bar{F}$, then, the corresponding residual lifetime variable $X_t = X - t \mid X > t$ has survival function $\bar{F}_t(x) = \frac{\bar{F}(x)}{\bar{F}(t)}$, $x > t$. Indeed, if X is the failure time of a new product or an engineering system, X_t is that of the product or system at its age t. The probability distribution of X_t and its properties play an important role in reliability and life testing studies. By replacing $\bar{F}(x)$ with $\bar{F}_t(x)$ in (2), the dynamic CRRE can be defined as

$$\begin{aligned}\gamma_\alpha(X;t) &= -\frac{1}{\alpha}\log\left(\frac{\int_t^\infty \left(\bar{F}(x)/\bar{F}(t)\right)^{\alpha+1}dx}{\int_t^\infty \left(\bar{F}(x)/\bar{F}(t)\right)dx}\right)\\ &= -\frac{1}{\alpha}\log\left(\frac{\int_t^\infty \bar{F}^{\alpha+1}(x)dx}{\bar{F}^\alpha(t)\int_t^\infty \bar{F}(x)dx}\right), \quad \alpha > 0. \end{aligned} \tag{9}$$

Example 3. Let X be a random variable with a Weibull distribution and survival function $\bar{F}(x) = e^{-(\lambda x)^\beta}$. Then

$$\gamma_\alpha(X;t) = -\frac{1}{\alpha}\log\left(\frac{\bar{F}_g((\alpha+1)(\lambda t)^\beta)e^{\alpha(\lambda t)^\beta}}{(\alpha+1)^{\frac{1}{\beta}}\bar{F}_g((\lambda t)^\beta)}\right),$$

where $\bar{F}_g(.)$ is the survival function of the gamma distribution with shape and scale parameters $\frac{1}{\beta}$ and 1, respectively.

Example 4. Let X be distributed as power with distribution function $F(x) = x^\beta$, $0 \leq x \leq 1, \beta > 0$. Then

$$\gamma_\alpha(X;t) = -\frac{1}{\alpha}\log\left(\frac{F_b(1-t^\beta,\alpha+2,\frac{1}{\beta})B(\alpha+2,\frac{1}{\beta})}{(1-t^\beta)^\alpha F_b(1-t^\beta,2,\frac{1}{\beta})B(2,\frac{1}{\beta})}\right),$$

where, $F_b(x,a,b)$ is the distribution function of a beta random variable with parameters a and b, and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

Figure 2 shows the plot of $\gamma_\alpha(X;t)$ for the above two examples.

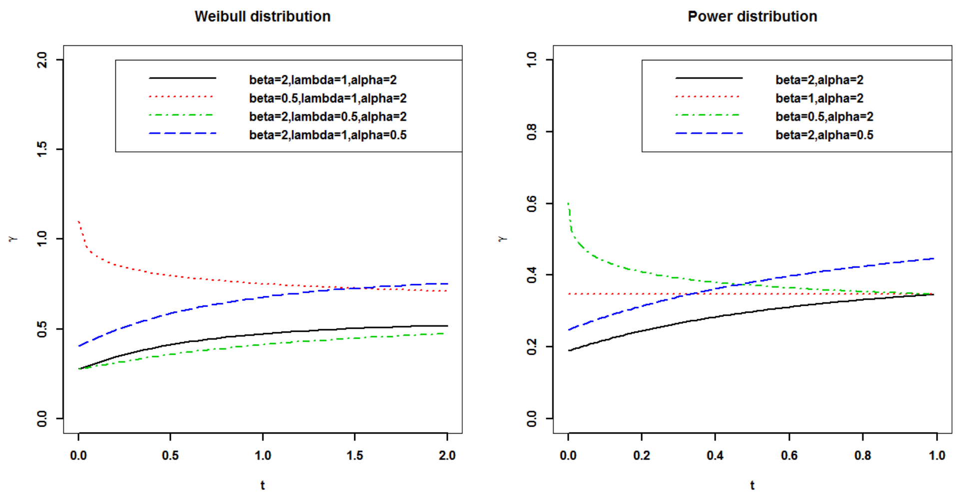

FIGURE 2: Plot of $\gamma_\alpha(X;t)$ for Power and Weibull distribution.

It is clear that $\gamma_\alpha(X;0) = \gamma_\alpha(X)$. Analog to equations (4), (5) and (6), the DCRRE can also be rewritten as

$$\begin{aligned}\gamma_\alpha(X;t) &= -\frac{1}{\alpha}\log\left(1-\frac{\alpha\int_t^\infty m_X(x)\bar{F}^\alpha(x)dF(x)}{\bar{F}^\alpha(t)\int_t^\infty \bar{F}(x)dx}\right) && (10)\\ &= -\frac{1}{\alpha}\log\left(1-\frac{\alpha\mathcal{T}_{\alpha+1}(X;t)}{m_X(t)}\right), && (11)\\ &= -\frac{1}{\alpha}\log\left(\frac{(\alpha+1)\int_t^\infty (x-t)\bar{F}^\alpha(x)dF(x)}{\bar{F}^{\alpha+1}(t)m_X(t)}\right), && (12)\end{aligned}$$

where, $m_X(t) = \frac{1}{\bar{F}(t)}\int_t^\infty \bar{F}(x)dx$ is the mean residual lifetime function of X and equation (11) is obtained using equation (12) in Rajesh & Sunoj (2019), an equivalent equation for the CRTE which is given by

$$\mathcal{T}_\alpha(X;t) = \frac{1}{\alpha-1}\left(m_X(t) - \frac{\int_t^\infty \bar{F}^\alpha(x)dx}{\bar{F}^\alpha(t)}\right) = \frac{\int_t^\infty m_X(x)\bar{F}^{\alpha-1}(x)dF(x)}{\bar{F}^\alpha(t)}.$$

The following theorem gives some results for $\gamma_\alpha(X;t)$ when X belongs to some reliability aging classes.

Theorem 3. **(a)** *If $X \in IFRA(DFRA)$, then*

$$\gamma_\alpha(X;t) \le (\ge) -\frac{1}{\alpha}\log\left(\frac{m_X((\alpha+1)t)\bar{F}((\alpha+1)t)}{(\alpha+1)m_X(t)\bar{F}^{\alpha+1}(t)}\right).$$

(b) *If $X \in NBU(NWU)$, then $\gamma_\alpha(X;t) \ge (\le)\gamma_\alpha(X) + \frac{1}{\alpha}\log(\frac{m_X(t)}{\mu})$.*

(c) *If* $X \in NBUE(NWUE)$, *then* $\gamma_\alpha(X;t) \leq (\geq) -\frac{1}{\alpha}\log\left(1-\frac{\alpha\mu}{(\alpha+1)m_X(t)}\right)$.

(d) *If* $X \in DMRL(IMRL)$, *then* $\gamma_\alpha(X;t) \leq (\geq)\frac{1}{\alpha}\log(\alpha+1)$.

***Proof*.** To prove part (a), it is not hard to show that $X \in IFRA(DFRA)$ is equivalent to $\bar{F}^a(t) \geq (\leq)\bar{F}(at)$, for $a \geq 1$. Thus, for $\alpha > 0$,

$$\begin{aligned}\int_t^\infty \bar{F}^{\alpha+1}(x)dx \quad &\geq (\leq) \quad \int_t^\infty \bar{F}((\alpha+1)x)dx \\ &= \quad \frac{1}{\alpha+1}\int_{(\alpha+1)t}^\infty \bar{F}(x)dx \\ &= \quad \frac{1}{\alpha+1}m_X((\alpha+1)t)\bar{F}((\alpha+1)t).\end{aligned}$$

The result now follows from equation (9). For part (b), the hypothesis implies that

$$\begin{aligned}\gamma_\alpha(X;t) \quad &= \quad -\frac{1}{\alpha}\log\left(\int_0^\infty\left(\frac{\bar{F}(x+t)}{\bar{F}(t)}\right)^{\alpha+1}dx\right)+\frac{1}{\alpha}\log(m_X(t)) \\ &\geq (\leq) \quad -\frac{1}{\alpha}\log\left(\int_0^\infty \bar{F}^{\alpha+1}(x)dx\right)+\frac{1}{\alpha}\log(m_X(t)) \\ &= \quad \gamma_\alpha(X)+\frac{1}{\alpha}\log(\frac{m_X(t)}{\mu}).\end{aligned}$$

This completes the proof. Parts (c) and (d) similarly follow from equation (10). □

As an application in reliability theory, let $X_1, X_2, \ldots, X_n$ be the independent random lifetimes of the components of a series system which are copies of X. Then, the lifetime of the system is $X_{1:n} = \min\{X_1, X_2, \ldots, X_n\}$. It is not difficult to see from equation (9) that

$$\alpha\gamma_\alpha(X_{1:n};t) = (n(\alpha+1)-1)\gamma_{n(\alpha+1)-1}(X;t)-(n-1)\gamma_{n-1}(X;t). \tag{13}$$

This reveals that the DCRRE of series systems is straightly related to that of its components.

Consider also another series systems with $Y_1, Y_2, \ldots, Y_n$ being its components lifetime which are independent and are copies of Y. For these series systems, the following result gives that, if $\gamma_\alpha(X;t) \leq \gamma_\alpha(Y;t)$, for $t \geq 0$, then under a condition, the DCRREs of the systems are also ordered.

Theorem 4. *If* $\gamma_\alpha(X;t) \leq \gamma_\alpha(Y;t)$, *for all* $\alpha > 0, t \geq 0$, *and if* $X \in IMRL$ *and* $Y \in DMRL$, *then* $\gamma_\alpha(X_{1:n};t) \leq \gamma_\alpha(Y_{1:n};t)$, *for* $t \geq 0$.

***Proof*.** Under the first assumption, and using equation (13) we obtain that

$$\alpha[\gamma_\alpha(Y_{1:n};t)-\gamma_\alpha(X_{1:n};t)] \geq (n-1)[\gamma_{n-1}(X;t)-\gamma_{n-1}(Y;t)].$$

The second assumption along with Theorem 3(d) now gives the result.

Furthermore, using equation (9), for $n \geq 1$, we have

$$\gamma_n(X;t) = -\frac{1}{n}\log\left(\frac{m_{X_{1:n+1}}(t)}{m_X(t)}\right), \tag{14}$$

which ensures that $X_{1:n} \leq_{mrl} X$.

The following theorem shows that when the components of two series systems are ordered in the sense of the mean residual lifetime and the DCRREs, then their mean residual lifetime functions are also ordered. $\square$

Theorem 5. *If $X \leq_{mrl} Y$ and $\gamma_{n-1}(X;t) \geq \gamma_{n-1}(Y;t)$, for $t \geq 0$, then $X_{1:n} \leq_{mrl} Y_{1:n}$.*

Proof. Using equation (14) $\gamma_{n-1}(X;t) \geq \gamma_{n-1}(Y;t)$ is equivalent to that $\frac{m_{X_{1:n}}(t)}{m_X(t)} \leq \frac{m_{Y_{1:n}}(t)}{m_Y(t)}$. The result now follows from the fact that $X \leq_{mrl} Y$ means that $m_X(t) \leq m_Y(t)$, for $t \geq 0$. $\square$

In the sequel, we give some results comparing the DCRRE of two random variables which are stochastically ordered in some notions.

Theorem 6. *If $X \leq_{hr} Y$, then $\gamma_\alpha(X;t) \geq \gamma_\alpha(Y;t) - \frac{1}{\alpha}\log\left(\frac{m_Y(t)}{m_X(t)}\right)$, for $t > 0$.*

Proof. The hypothesis is equivalent to $\frac{\bar{F}(x)}{\bar{F}(t)} \leq \frac{\bar{G}(x)}{\bar{G}(t)}$, for all $t \leq x$ (cf. Shaked & Shanthikumar, 2007, p. 16). The result now follows from (9). $\square$

Theorem 7. *If $X \leq_{dmrl} Y$, then $\gamma_\alpha(X;F^{-1}(p)) \leq \gamma_\alpha(Y;G^{-1}(p))$, for $0 < p < 1$.*

Proof. First, using equation (10) we have

$$1 - e^{-\alpha\gamma_\alpha(X;F^{-1}(p))} = \frac{\alpha\int_p^1 m_X(F^{-1}(u))(1-u)^\alpha du}{(1-p)^{\alpha+1}m_X(F^{-1}(p))}. \tag{15}$$

On the other hand, the hypothesis implies that $\frac{m_X(F^{-1}(u))}{m_X(F^{-1}(p))} \leq \frac{m_Y(G^{-1}(u))}{m_Y(G^{-1}(p))}$, for $p \leq u$. The result now, follows from the above equation. $\square$

Theorem 8. *If X and Y are non-negative random variables with common left endpoint zero, and if $X \leq_{disp} Y$, $X \in DMRL$, then $\gamma_\alpha(X;F^{-1}(p)) \leq \gamma_\alpha(X;G^{-1}(p))$, for $0 < p < 1$.*

Proof. Using equation (15) and by applying inequalities 3.C.5 and 3.C.9, and Theorem 3.b.13(a) in Shaked & Shanthikumar (2007), pp. 165, 166 and 154, respectively, we obtain that

$$\begin{aligned}(1 - e^{-\alpha\gamma_\alpha(X;F^{-1}(p))})m_X(F^{-1}(p)) &\leq (1 - e^{-\alpha\gamma_\alpha(X;G^{-1}(p))})m_X(G^{-1}(p)) \\ &\leq (1 - e^{-\alpha\gamma_\alpha(X;G^{-1}(p))})m_X(F^{-1}(p)).\end{aligned}$$

Hence, we get that $\gamma_\alpha(X;F^{-1}(p)) \leq \gamma_\alpha(X;G^{-1}(p))$, for $0 < p < 1$. This completes the proof. $\square$

Differentiating equation (2) with respect to t, we have

$$\alpha m_X(t)\gamma'_\alpha(X;t) = e^{\alpha\gamma_\alpha(X;t)} - \alpha m'_X(t) - (\alpha+1). \tag{16}$$

A new class of distributions can be considered based on the mathematical behaviour of the DCRRE.

Definition 2. A random variable X is said to be increasing (decreasing) DCRRE, denoted by IDCRRE(DDCRRE), if $\gamma_\alpha(X;t)$ is an increasing (decreasing) function of t.

Note that, equation (16) implies that $X \in IDCRRE(DDCRRE)$ if

$$\alpha\gamma_\alpha(X;t) \geq (\leq) \log(\alpha m'_X(t) + \alpha + 1),$$

or equivalently, if

$$\alpha\gamma_\alpha(X;t) \geq (\leq) \log(\alpha m_X(t)\lambda_X(t) + 1).$$

Example 5. If X has a weibull distribution with survival function $\bar{F}(x) = e^{(\lambda x)^\beta}$, then $X \in IDCRRE(DDCRRE)$ if $\beta > (<)1$.

Example 6. Let X be distributed uniformly on $(0,\beta)$. Then, $\gamma_\alpha(X;t) = -\frac{1}{\alpha}\log\left(\frac{2}{\alpha+2}\right)$, which is a constant function of t.

Sunoj & Linu (2012) have characterized some distributions using the relationship between their own version of the DCRRE and the mean residual lifetime function. Rajesh & Sunoj (2019) have also obtained characterizations for some distributions using the DCTE and mean residual lifetime function. The following theorem gives a characterization of some distributions using the same relationship between the mean residual lifetime function and $\gamma_\alpha(X;t)$.

Theorem 9. *Let X be a non-negative random variable with continuous survival function $\bar{F}(x)$ and the mean residual lifetime function $m_X(x)$. Suppose that the relationship $\alpha\gamma_\alpha(X;t) = \log(c(t)m_X(t))$ holds, for a nonnegative function $c(t)$. Then, for $t \geq 0$*

$$m_X(t) = \frac{1}{\sqrt{c(t)}e^{\frac{C(t)}{2\alpha}}}[\int_0^t \frac{-(\alpha+1)}{2\alpha}\sqrt{c(x)}e^{\frac{C(x)}{2\alpha}}dx + k], \tag{17}$$

where, $C'(t) = c(t)$, and k is a constant.

***Proof*.** Under the given relationship, equation (16) implies that

$$m'_X(t) + \left(\frac{\alpha c'(t) - c^2(t)}{2\alpha c(t)}\right)m_X(t) = \frac{-(\alpha+1)}{2\alpha},$$

which is a differential equation and has a solution in the form of (17). $\square$

The following result gives a characterization result similar to that of Theorem 2.3 in Sunoj & Linu (2012).

Theorem 10. *Let* $\alpha\gamma_\alpha(X;t) = \log(C)$*. Then,* X *has*

(i) *Pareto distribution with survival function* $\bar{F}(x) = (1+bx)^{-a}$*,* $x, b > 0, a > 1$*,*

(ii) *exponential distribution with survival function* $\bar{F}(x) = e^{-\lambda x}$*,* $x, \lambda > 0$*,*

(iii) *finite range distribution with survival function* $\bar{F}(x) = (1-bx)^a$*,* $0 < x < \frac{1}{b}, a, b > 0$*,*

according as $\frac{C-(\alpha+1)}{\alpha} \overset{\geqq}{<} 0$*.*

Proof. Under the above relationship, equation (16) follows that $m'_X(t) = \frac{C-(\alpha+1)}{\alpha} = k$, a constant. This is equivalent to $m_X(t) = kt + d$, where d is also a constant, which characterizes the distributions (i)-(iii) according as $k \overset{\geqq}{<} 0$. The converse part is easy to prove. □

Note that the uniform distribution in Example 3 is just the distribution in part (iii) of the above theorem with $b = \frac{1}{\beta}$ and $a = 1$.

4. The estimation of CRRE

In this section, we propose an estimator of $\gamma_\alpha(X)$ and investigate its exact and asymptotic distribution under some conditions. Let $X_1, \ldots, X_n$ be independent positive random sample from the population of X with continuous distribution functions F and corresponding order statistics $X_{(1)}, \ldots, X_{(n)}$. Let also F_n be the empirical distribution function of X.

Regarding equation (6), $\gamma_\alpha(X)$ can also be given through an L-functional (cf. Shao, 2003, p. 343) by

$$\gamma_\alpha(X) = -\frac{1}{\alpha}\log\left(\frac{\int_0^\infty xJ_\alpha(F(x))dF(x)}{\int_0^\infty xdF(x)}\right), \tag{18}$$

where, $J_\alpha(u) = (\alpha+1)(1-u)^\alpha$. Now, by replacing F in (18) with F_n, an estimator of $\gamma_\alpha(X)$ can be given by the following:

$$\begin{aligned}\gamma_\alpha(F_n) =& -\frac{1}{\alpha}\log\left(\frac{1}{\bar{X}}\int_0^\infty xJ_\alpha(F_n(x))dF_n(x)\right)\\ =& -\frac{1}{\alpha}\log\left(\frac{1}{n\bar{X}}\sum_{i=1}^n J_\alpha(\frac{i}{n})X_{(i)}\right).\end{aligned} \tag{19}$$

The following theorem gives the exact distribution of the above estimator under a random sample from an exponential population.

Theorem 11. *Let $X_1, \dots, X_n$ be a random sample from an exponential distribution with an arbitrary hazard rate. Then, the survival function of $\gamma_\alpha(F_n)$ is given by*

$$P[\gamma_\alpha(F_n) > x] = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{d_{i,n}^{\alpha} - e^{-\alpha x}}{d_{i,n}^{\alpha} - d_{j,n}^{\alpha}} \right) I(d_{i,n}^{\alpha} > e^{-\alpha x}),$$

where, $d_{i,n}^{\alpha} = \frac{1}{n-i+1} \sum_{j=i}^{n} c_{j,n}^{\alpha}$, $c_{i,n}^{\alpha} = (\alpha+1)(1-\frac{i}{n})^{\alpha}$ and $I(.)$ is the indicator function.

Proof. First note that, (19) can also be expressed as

$$\gamma_\alpha(F_n) = -\frac{1}{\alpha} \log \left(\frac{\sum_{i=1}^{n} c_{i,n}^{\alpha} X_{(i)}}{\sum_{i=1}^{n} X_{(i)}} \right).$$

On the other hand, by using the normalized spacings $D_i = (n-i+1)(X_{(i)} - X_{(i-1)})$, $i = 1, \dots, n$ ($X_{(0)} \equiv 0$), one can see easily that

$$\begin{aligned} \sum_{i=1}^{n} d_{i,n}^{\alpha} D_i &= \sum_{i=1}^{n} (\sum_{j=i}^{n} c_{j,n}^{\alpha})(X_{(i)} - X_{(i-1)}) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{j} c_{j,n}^{\alpha} (X_{(i)} - X_{(i-1)}) = \sum_{j=1}^{n} c_{j,n}^{\alpha} X_{(j)}, \end{aligned}$$

which implies that $\gamma_\alpha(F_n)$ can also be given by

$$\gamma_\alpha(F_n) = -\frac{1}{\alpha} \log \left(\frac{\sum_{i=1}^{n} e_{i,n}^{\alpha} D_i}{\sum_{i=1}^{n} D_i} \right).$$

The result now follows by applying the Theorem 3.1 in Belzunce et al. (2005). □

It is clear from the almost sure convergence property of the L-estimators (see Example 1 and 2 in Wellner, 1977 and Helmers, 1977) and the continuous mapping theorem (cf. Theorem 1.10 in Shao, 2003, p. 59) that as $n \to \infty$,

$$\gamma_\alpha(F_n) \to \gamma_\alpha(X),$$

with probability one, provided that the population mean is finite. The following theorem gives the asymptotic distribution of $\gamma_\alpha(F_n)$ under some mild conditions.

Theorem 12. *Assume that $E(X^2) < \infty$ and*

$$\sigma_\alpha^2(F, J) = 2 \int_0^\infty \int_x^\infty F(x) \bar{F}(y) J_\alpha(F(x)) J_\alpha(F(y)) dy dx > 0. \tag{20}$$

Then, as $n \to \infty$,

$$\sqrt{n}[\gamma_\alpha(F_n) - \gamma_\alpha(X)] \xrightarrow{d} N(0, \sigma^2),$$

where, $\xrightarrow{d}$ *denotes convergence in distribution,* $N(0,\sigma^2)$ *stands for the normal random variable with mean zero and variance* $\sigma^2 = \frac{\sigma^2_\alpha(F,J)}{\alpha^2\gamma^2_{1\alpha}(X)}$,

$$\gamma_{1\alpha}(X) = (\alpha+1)\int_0^\infty x\bar{F}^\alpha(x)dF(x).$$

***Proof*.** We have $\gamma_\alpha(F_n) = -\frac{1}{\alpha}\log(\frac{\gamma_{1\alpha}(F_n)}{\bar{X}})$ where $\gamma_{1\alpha}(F_n) = \frac{1}{n}\sum_{i=1}^n J_\alpha(\frac{i}{n})X_{(i)}$. It follows from Theorem 2 and 3 in Stigler (1974) that as $n\to\infty$, $\sqrt{n}[\gamma_{1\alpha}(F_n) - \gamma_{1\alpha}(X)]$ converges in distribution to a normal random variable with mean zero and variance (20). On the other hand, $\bar{X}$ is a consistent estimator of the population mean, $\mu = \int_0^\infty \bar{F}(x)dx$. Applying the Slutsky and Delta-method theorems now gives the result. □

It is worth to mention that a consistent estimator of the asymptotic variance can be obtained by replacing F in (20) with F_n.

Crescenzo & Longobardi (2009) have used the following data sets to apply their cumulative entropy for analyzing the lifetime data. As an example, we use these data and compute the estimators of the CRRE $\gamma_\alpha(X)$ and $\gamma(\beta)$ in Sunoj & Linu (2012). Since, it has not been proposed any estimator of $\gamma(\beta)$ in Sunoj & Linu (2012), we rewrite it as

$$\gamma(\beta) = \frac{1}{1-\beta}\log(\beta\int_0^\infty x\bar{F}^{\beta-1}(x)dF(x)),$$

and consider the corresponding estimator by the following

$$\hat{\gamma}(\beta) = \frac{1}{1-\beta}\log(\frac{\beta}{n}\sum_{i=1}^n(1-\frac{i}{n})^{\beta-1}X_{(i)}).$$

Example 7. The data set analyzed in Crescenzo & Longobardi (2009) includes 43 sample lifetime data which are as follows.

$$7, 47, 58, 74, 177, 232, 273, 285, 317, 429, 440, 445, 455, 468, 495, 497, 532,$$

$$571, 579, 581, 650, 702, 715, 779, 881, 900, 930, 968, 1077, 1109, 1314, 1334,$$

$$1367, 1534, 1712, 1784, 1877, 1886, 2045, 2056, 2260, 2429, 2509.$$

For these data, Crescenzo & Longobardi (2009) obtained the estimate of their cumulative entropy as 572.3. Figure 3 depicts the plot of the estimators $\gamma_\alpha(F_n)$ and $\hat{\gamma}(\beta)$ for different values of α and β. One can see from the plot that the estimator $\hat{\gamma}(\beta)$ takes negative values for $\beta > 1$.

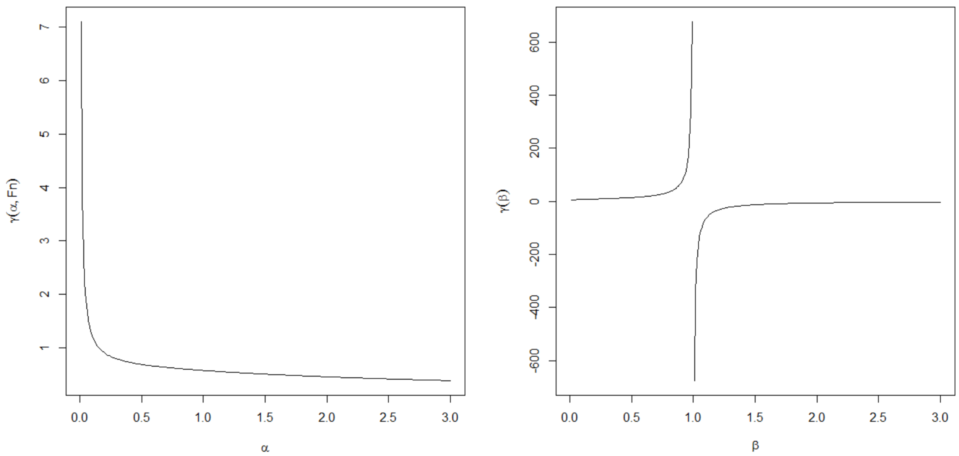

FIGURE 3: Plots of $\gamma_\alpha(F_n)$ (left) and $\hat{\gamma}(\beta)$(right).

5. Conclusion

In this paper, we have proposed an alternative measure of cumulative residual Renyi's entropy (CRRE) of order α which unlike the one by Sunoj & Linu (2012) preserves the main property of an information measure and is always positive. We have investigated the main properties of the proposed measure and studied its relation with other entropy measures. Assuming some well-known stochastic orders between two random variables, the imposed orders between their corresponding CRRE were revealed. The dynamic version of the CRRE was also considered and its main properties and its relation to Tsallis's Entropy were studied. The dynamic CRREs of the stochastically ordered random variables were also compared. The estimator of the proposed CRRE and its exact distribution under a random sample from an exponential distribution and also its asymptotic distribution were studied. Throughout the paper numerous examples and plots illustrating the theory were given.

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