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Evaluation of the Mean Control Chart Under a Bayesian Approach

Evaluación de la carta de control para la media de un proceso bajo un enfoque bayesiano

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Abstract

A previous study on the evaluation of control charts for the mean with a Bayesian approach, based on predictive limits, was performed in such a way that neither prior nor sample information was taken into account. This work was developed to make a more complete study to evaluate the influence of the combination of the prior distribution with the sample information. It is assumed that the quality characteristic to be controlled can be modeled by a Normal distribution and two cases are considered: known and unknown variance. A Bayesian conjugate model is established, therefore the prior distribution for the mean is Normal and, in the case where the variance is unknown, the prior distribution for the variance is defined as the Inverse-Gamma(ν, ν). The posterior predictive distribution, which is also Normal, is used to establish the control limits of the chart. Signal propability is used to measure the performance of the control chart in phase II, with the predictive limits calculated under different specifications of the prior distributions, and two different sizes of the calibration sample and the future sample. The simulation study evaluates three aspects: the effects of sample sizes, the distance of the prior mean to the mean of the calibration sample, and an indicator of how informative is the prior distribution of the population mean. In addition, in the case of unknown variance, we study what is the effect of changing values in the parameter ν. We found that the false alarm rate could be quite large if the prior distribution is very informative which in turn leads to an ARL (average run length) biased chart, that is, the maximum of the ARL is not given when the process is under control. Besides, we found great influence of the prior distribution on the control chart power when the size of the calibration and future samples are small, particulary when the

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prior is very informative. Finally, regarding the effect of the parameter ν, we found that the smaller the value, which means having a less informative prior distribution, the lower the power of the control chart.

***Key words*:** control charts; Bayesian approach; ARL; conjugate prior; informative prior.

Resumen

Un estudio previo sobre la evaluación de las gráficas de control para la media con un enfoque Bayesiano, basadas en límites predictivos, fue realizado de tal manera que no se tuvo en cuenta ni la información a priori ni la información muestral. En este trabajo hemos desarrollado un estudio más completo para evaluar la influencia de la combinación de la distribución a priori con la información muestral. Se asume que la característica de calidad a controlar puede modelarse mediante una distribución Normal y se consideran dos casos: varianza conocida y desconocida. Para la aproximación Bayesiana se establece un modelo conjugado, por lo tanto la distribución a priori para la media es Normal y, en el caso donde la varianza es desconocida, se define como distribución a priori para la varianza la Gamma-Inversa(ν, ν). La distribución predictiva posterior, que también es Normal, es utilizada para establecer los límites de control de la gráfica. Se utiliza la probabilidad de señal para medir el desempeño de la gráfica en la denominada phase II de control, con los límites predictivos calculados bajo diferentes especificaciones de las distribuciones a priori, del tamaño de la muestra de calibración y del tamaño de la muestra futura. El estudio de simulación evalúa tres aspectos: efectos del tamaño de muestra, de la distancia de la media a priori con relación a la media de la muestra de calibración, y un indicador de cuán informativa es la distribución a priori de la media poblacional. Adicionalmente, cuando la varianza es desconocida, se estudia el efecto de los valores del parámetro ν. Se encuentra que la tasa de falsas alarmas puede ser exageradamente grande si se especifica una a priori muy informativa, lo que a su vez puede conducir a una gráfica de control con una ARL (average run length) sesgada, es decir, que el máximo de la ARL no se dará cuando el proceso está en control. Además, cuando el tamaño de las muestras de calibración y de la muestra futura son pequeñas, hay gran influencia de la especificación de la a priori sobre la potencia de la gráfica de control, en especial cuando la a priori es muy informativa. Finalmente, en cuanto al efecto del parámetro ν, se encuentra que entre más pequeño es su valor, lo cual indica que la distribución a priori para la varianza es menos informativa, menor es la potencia de la gráfica de control, en especial si los tamaños de muestra son pequeños.

***Palabras clave*:** gráficas de control; enfoque Bayesiano; ARL; distribución a priori conjugada; distribución a priori informativa.

1. Introduction

Statistical Process Control (SPC) is a method of quality control which employs statistical methods to understand, monitor and improve a process. Frequently, monitoring consists of the detection of changes in the parameter(s) within the probability distribution of one variable or several variables of the process. Control charts are techniques of SPC, which aim to monitor the process over time in order to detect changes in the performance. Many types of statistical methods have been incorporated into SPC procedures, for example regression models, variance components, and time series, among others. In Woodall & Montgomery (1999) and Woodall & Montgomery (2014) the authors provide an overview of SPC methods and offer some directions for further research.

According to Woodall (2000), the understanding of the variation in values of a quality characteristic is of primary importance in SPC. The sources of variability are:

- Common: due to the inherent nature of the process.
- Assignable or special: unusual shocks or other disruptions in the process, the causes of which can and should be removed.

A control chart is a tool for monitoring process performance, which consists of a graphic designed to detect unusual variations due to assignable causes. This chart plots the values of a quality feature and compare those with the control limits. Control of the process mean quality level is usually done with control chart for means, or the $\bar{x}$ chart, introduced in Shewhart (1931). This chart plots, in a time-ordered sequence, the sample mean $\bar{x}$ of random samples collected generally with a regular frequency. Three sigma limits are used to set the upper and lower control limits, under the assumption of Normal distribution. We conclude that the process is under control when all values of $\bar{x}$ plot inside the control limits and no systematic behavior is evident; otherwise, it is produced an out of control signal. As Woodall (2000) explains, there are other rules for signaling an out-of-control situation based on "non-random" patterns in the chart. Figure 1 shows the typical appearance of the $\bar{x}$ chart.

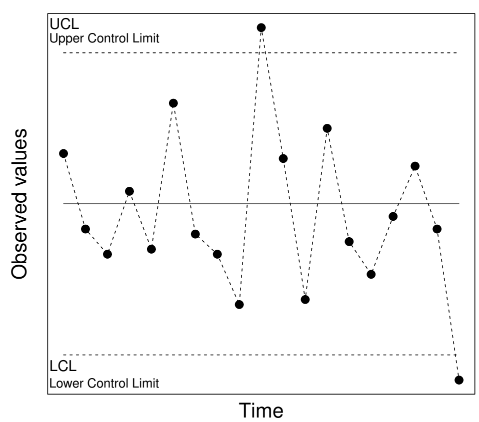

FIGURE 1: An example of $\bar{x}$ control chart.

Process monitoring involves two phases (Woodall & Montgomery 1999,Woodall & Montgomery 2014, Woodall 2000):

- Phase I or retrospective analysis. In this phase basically the following tasks are performed:
  - ◇ Collect data to understand the process and check statistical control.
  - ◇ Select an appropriate in-control model.
  - ◇ Estimate the parameter(s) of this model.
- Phase II or process monitoring phase. The monitoring method is implemented with data collected over time in order to detect changes within the process with respect to the assumed in-control. From a practical perspective, it is important to design monitoring methods in such a way that there would be a reasonable low number of false alarms.

Woodall & Montgomery (1999) comment that the implementation of these two phases can be a challenging problem because shifts in the underlying distribution distort parameter estimation which, in turn, masks the shifts.

The calculation of any statistical measure of performance requires an assumption about the probability distribution of the quality characteristic. The majority of the studies about the performance of control charts for variables have been based on the assumptions of an underlying Normal distribution and independence of samples over time. Also, the control limits are based on this assumption (Woodall & Montgomery 1999, Woodall & Montgomery 2014, Woodall 2000).

In phase II, performance is usually measured with probability of a signal or with some parameter of the run length (ARL). When evaluating performance in this phase, we usually assume that the parameters are known, ignoring that these values were estimated in phase I. However, it is necessary to recognize that the control limits in phase II are random variables and, therefore, this affects the performance of the control charts (Jensen et al. 2006, Psarakis et al. 2014). Consequently, studying the $\bar{x}$ chart with a Bayesian approach is clearly justified. In this regard Colosimo & del Castillo (2007) says "In the idealized situation that all process readings follow a common Normal distribution with known parameters and are independent, all the properties of the chart are well-known... However, increasing realization exists that process parameters are hardly ever known to an adequate precision for these theoretical calculations to be plausible. Furthermore, many process drift". Additionally Woodall (2000) comments: "... Deming stated that no process, except in artificial demonstrations by use of random numbers, is steady, unwavering...Deming's objection to measures of statistical performance of control charts because no process is stable can be overcome at least in part by modeling the instability of the process distribution. For example, one might consider a Normal distribution with constant variance, but with a mean that itself is normally distributed. This approach is useful in situations for which there is more than one component of common cause variability".

Woodall & Montgomery (2014) point out that a few Bayesian methods have been proposed for process monitoring and it would be useful to review this approach. The authors say: "These methods do not seem widely used, which is somewhat odd considering the success of Bayesian methods in other areas of applied statistics. Disadvantages of Bayesian methods include an added layer of complexity and the amount of computation required. A discussion of the general framework, advantages, disadvantages, and limitations of Bayesian surveillance approaches, however, could be very interesting and informative".

Some authors have studied and proposed alternatives to the $\bar{x}$ control chart with a Bayesian approach. Nenes (2013), Nikolaidis & Tagaras (2017), Tagaras & Nikolaidis (2002) evaluate the economic performance of various adaptive control schemes to derive conclusions about their relative effectiveness. The analysis concentrates on Bayesian control charts used for monitoring the process mean. Bhat & Gokhale (2014) and Saghir (2015) study $\bar{x}$ control chart using the posterior distribution. The control limits of the proposed chart are derived under the assumption that the process mean has a conjugate prior distribution. They analyze the power of the proposed control chart. On the other hand, Xiaosong et al. (2015) present a control chart based on conjugate Bayesian approach for multi-batch and low volume production. With a case study, they show that the conjugate Bayesian approach outperforms the traditional frequency approach when sample size is small.

Additionally, Chen (2016) proposes a Bayesian nonparametric control charts for individual measurements. Tsiamyrtzis & Hawkins (2005) propose a model for statistical process control in short production runs whose objetive is to detect online whether the mean of the process has exceeded a prespecified upper threshold value. Ali & Riaz (2020) consider different symmetric and asymmetric loss functions for designing Bayesian control charts. To get the desired in-control performance under different loss functions, they propose a corrected design of the Bayesian control charts. Monfared & Lak (2013) propose a Bayesian approach to estimate the change point when implementing an $\bar{x}$ control chart. They show how it is possible to use the information in an $\bar{x}$ control chart and construct an informative prior for the change point.

This article aims to contribute in the analysis of the Bayesian approach proposed in Menzefricke (2002), where a $\bar{x}$ control chart is designed when there is uncertainty in the parameters, using the posterior predictive distribution to set control limits. The objective is to evaluate the effect of the prior distribution for the unknown parameters of a process on signal probability, when the quality feature is assumed Normal, considering both the hypothetical cases in which variance is known and unknown.

The article is organized as follows. In Section 2, $\bar{x}$ control limits with predictive limits are given when variance is known and unknown. Also, how to evaluate these control limits. In Section 3, there is an explanation of the simulation study carried out to evaluate the $\bar{x}$ control chart and also shows the results of the simulation study and Section 4 concludes the article.

2. Background

2.1. $\bar{x}$ Control Chart With Predictive Limits, σ^2 Known

Following Menzefricke (2002), suppose we want to control a quality characteristic X whose distribution is Normal with known variance σ^2 and unknown mean μ. We assume in phase I that a random sample of size n_c is available from a stable process, where $\bar{x}_c$ is the sample mean. We denote the likelihood by

$$p(x_1.x_2,\ldots,x_{n_c}|\mu) = (2\pi\sigma^2)^{-n_c/2}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n_c}(x_i-\mu)^2\right]. \tag{1}$$

The prior distribution for μ is Normal(m_0,σ^2/n_0), where n_0 measures the degree of uncertainty about the parameter μ, therefore the larger its value the lower the uncertainty given that the prior variance is smaller. It can be shown that the posterior distribution is:

$$p(\mu|\bar{x}_c) = \text{Normal}\left(m_1,\frac{\sigma^2}{n_1}\right), \tag{2}$$

where $m_1 = (n_0 m_0 + n_c\bar{x}_c)/n_1$ and $n_1 = n_0 + n_c$ (measures the impact of the size of the training sample or the prior information). Under the assumption that the process remains stable, we can now derive the control chart limits for a future sample n, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, for which the sufficient statistic is the sample mean, $\bar{y}_n$. Given μ, the distribution of $\bar{y}_n$ is Normal(μ,σ^2/n) and the predictive distribution for $\bar{y}_n$ is:

$$p(\bar{y}_n|\bar{x}_c) = \text{Normal}\left(m_1,\sigma^2\left(\frac{1}{n}+\frac{1}{n_1}\right)\right). \tag{3}$$

From equation (3), Menzefricke (2002) deduces as lower and upper control limits, respectively,

$$\text{LCL} = m_1 - z_{\alpha/2}\sigma\sqrt{\frac{1}{n}+\frac{1}{n_1}}, \tag{4}$$

$$\text{UCL} = m_1 + z_{\alpha/2}\sigma\sqrt{\frac{1}{n}+\frac{1}{n_1}}, \tag{5}$$

where $z_{\alpha/2}$ is the quantile $(1-\alpha/2)\,100\%$ of the distribution Normal$(0,1)$.

2.2. $\bar{x}$ Control Chart with Predictive Limits, σ^2 Unknown

Following Menzefricke (2002), we suppose that the variable X whose mean we want to control, has a Normal distribution with unknown mean and variance. Suppose that in phase I there is a random sample of size n_c from the stable process, with sample mean $\bar{x}_c$ and sample variance s_x^2; the distribution of $(\bar{x}_c, s_x^2|\mu,\sigma^2)$ is

$f\left(\bar{x}_c, s_x^2|\mu,\sigma^2\right) = f\left(\bar{x}_c|\mu,\sigma^2\right) f\left(s_x^2|\sigma^2\right)$, where $f(\bar{x}_c|\mu,\sigma^2) = \text{Normal}\left(\mu, \sigma^2/n_c\right)$ and $f(s_x^2|\sigma^2) = \text{Gamma}\left(\frac{n_c-1}{2}, \frac{n_c-1}{2\sigma^2}\right)$. The prior distribution for (μ,σ^{-2}) is a Normal-Gamma distribution, that is, $\mu|\sigma^2 \sim \text{Normal}\left(m_0, \frac{\sigma^2}{n_0}\right)$ and $\sigma^{-2} \sim \text{Gamma}\left(\frac{v_0}{2}, \frac{v_0 s_0^2}{2}\right)$. Then, the posterior distribution of $(\mu,\sigma^{-2})|\bar{x}_c, s_x^2$ is,

$$p\left(\mu|\bar{x}_c, s_x^2, \sigma^2\right) = \text{Normal}\left(m_1, \frac{\sigma^2}{n_1}\right), \tag{6}$$

$$p\left(\sigma^{-2}|\bar{x}_c, s_x^2\right) = \text{Gamma}\left(\frac{\nu_1}{2}, \frac{\nu_1}{2} s_1^2\right), \tag{7}$$

where $n_1 = n_0 + n_c$, $m_1 = \left(n_0 m_0 + n_c \bar{x}_c\right)/n_1$, $v_1 = v_0 + n_c$ and $v_1 s_1^2 = v_0 s_0^2 + (n_c - 1)s_x^2 + n_c n_0 (m_0 - \bar{x}_c)^2 / \left(n_c + n_0\right)$.

Note that the assumed Bayesian model is a conjugate model: since the prior distribution for the mean is Normal and the distribution for σ^{-2} is Gamma, a Normal-Gamma distribution is obtained as posterior distribution. The latter is used to find the predictive distribution for $\bar{y}_n$, the mean of the future sample of size n, and with it, find the predictive control limits of phase II. Under the assumption that the process remains stable, we can now derive the control chart limits for the sample mean of a future sample $\mathbf{y} = (y_1, y_2, \dots, y_n)$, for which the sufficient statistic is the sample mean, $\bar{y}_n$. Given μ and σ^2, the distribution of $\bar{y}_n$ is Normal(μ,σ^2/n), therefore, the predictive distribution for $\bar{y}_n$ is

$$\bar{y}_n|\bar{x}_c, s_x^2 \sim \text{St}\left(v_1, m_1, \left(\frac{1}{n} + \frac{1}{n_1}\right) s_1^2\right), \tag{8}$$

where St() in noncentral t-distribution with v_1 degrees of freedom. From this distribution, Menzefricke (2002) determines as lower and upper control limits for the mean, respectively,

$$\text{LCL} = m_1 - t_{\alpha/2, v_1} s_1 \sqrt{\frac{1}{n} + \frac{1}{n_1}}, \tag{9}$$

$$\text{UCL} = m_1 + t_{\alpha/2, v_1} s_1 \sqrt{\frac{1}{n} + \frac{1}{n_1}}, \tag{10}$$

where $t_{\alpha/2,v_1}$ is the quantile $(1-\alpha/2)\,100\%$ of the t distribution with v_1 degrees of freedom.

2.3. Evaluation of Control Chart in Phase II

In order to assess the effectiveness of the control chart with predictive limits in phase II, Menzefricke (2002) used the probability of rejection or signal. We assume that μ_0 and σ_0^2 are the parameters of the stable process.

2.3.1. Known σ^2

Suppose that there is a shift in the process mean in such a way that the future random sample $\mathbf{y} = (y_1, y_2, \ldots y_n)$ has a Normal $\left(\mu_0 + a_1, \sigma_0^2\right)$, $a_1 \in \mathbb{R}$, and therefore $\bar{y}_n \sim$ Normal $\left(\mu_0 + a_1, \sigma_0^2/n\right)$. Assuming μ_0 and σ_0^2 known, the probability of rejection or signal, denoted by $\alpha\left(\bar{x}_c, \mu_0 + a_1\right)$, is

$$\begin{aligned}\alpha\left(\bar{x}_c, \mu_0 + a_1\right) &= 1 - \int_{\mathrm{LCL}}^{\mathrm{UCL}} \mathrm{Normal}(\bar{y}_n | \mu_0 + a_1, \sigma_0^2/n) d\bar{y}_n \\ &= 1 - \Phi\left(\frac{m_1 - (\mu_0 + a_1)}{\sigma_0/\sqrt{n}} + z_{\alpha/2}\sqrt{1+p}\right) \\ &+ \Phi\left(\frac{m_1 - (\mu_0 + a_1)}{\sigma_0/\sqrt{n}} - z_{\alpha/2}\sqrt{1+p}\right), \end{aligned} \tag{11}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of standard Normal distribution, $p = n/n_1$, and LCL, UCL given by (4) and (5), respectively.

2.3.2. Unknown σ^2

Suppose that the future data corresponds to a random sample $\mathbf{y} = (y_1, y_2, \cdots y_n)$ has a Normal $\left(\mu_0 + a_1, a_2^2\sigma_0^2\right)$, with $a_1 \in \mathbb{R}$, and $a_2 \in \mathbb{R}^+$, and therefore $\bar{y}_n \sim$ Normal $\left(\mu_0 + a_1, a_2^2\sigma_0^2/n\right)$. Assuming μ_0 and σ_0^2 known, the probability of rejection or signal, denoted by $\alpha\left(\bar{x}_c, \mu_0 + a_1, a_2^2\sigma_0^2\right)$, is

$$\begin{aligned}\alpha\left(\bar{x}_c, \mu_0 + a_1, a_2^2\sigma_0^2\right) &= 1 - \int_{\mathrm{LCL}}^{\mathrm{UCL}} \mathrm{Normal}(\bar{y}_n | \mu_0 + a_1, a_2^2\sigma^2/n) d\bar{y}_n \\ &= 1 - \Phi\left(\frac{m_1 - (\mu_0 + a_1)}{a_2\sigma_0/\sqrt{n}} + t_{\alpha/2, v_1}\frac{s_1}{a_2\sigma_0}\sqrt{1+p}\right) \\ &+ \Phi\left(\frac{m_1 - (\mu_0 + a_1)}{a_2\sigma_0/\sqrt{n}} - t_{\alpha/2, v_1}\frac{s_1}{a_2\sigma_0}\sqrt{1+p}\right), \end{aligned} \tag{12}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of standard Normal distribution, $p = n/n_1$, and LCL, UCL given by (9) and (10), respectively.

According to Menzefricke (2002), $p = n/n_1$ measures the degree of uncertainty of μ, with low values suggesting little uncertainty. However, Menzefricke (2002) did not evaluate the effect of the prior distributions in (11) and (12), since he took $m_1 = \mu_0$ and $s_1^2 = \sigma_0^2$, a simplification that does not allow to see the influence of the combination of the prior distribution with the sample information. His conclusions are only in terms of the effects of n_1, a_1 (only positive values) and ν_1.

3. Simulation Study

A simulation study was performed in order to analyze the effect of the prior distribution and the sample information in (11) and (12). Without loss of generality we may assume that $\mu_0 = 7$ and $\sigma_0 = 1$. As usual with control charts, we set the nominal false alarm rate at $\alpha = 0.0027$. In order to determine whether the prior distributions have effects in (11) and (12), we decided to control the following parameters.

- Sample sizes on phase I (n_c) and phase II (n): $n = n_c = 5,\ 30$.
- $p = n/n_1 = 0.01, 0.2, 1$. Consequently, n_0 was set as a function of p, n and n_c, as follows: $n_0 = \frac{n}{p} - n_c$; therefore, n_0 takes the following values,

TABLE 1: Values of n_0 taken as a function of p, n and n_c

	$n_0 = \frac{n}{p} - n_c$	
p	$n = n_c = 5$	$n = n_c = 30$
0.01	495	2970
0.2	20	120
1	0	0

- Mean of the prior distribution of μ:
  1. When σ^2 is known, the prior mean is $m_0 = \bar{x}_c + \frac{k\sigma_0}{\sqrt{n_c}}$, where $\bar{x}_c$ is the sample mean in phase I. $\frac{k\sigma_0}{\sqrt{n_c}}$ is the magnitude and direction of the deviation of m_0 with respect to $\bar{x}_c$.
  2. When σ^2 is unkown, we define $m_0 = \bar{x}_c + k\frac{s_x}{\sqrt{n_c}}$. In this case, the difference between m_0 and $\bar{x}_c$ will vary as a function of $k\frac{s_x}{\sqrt{n_c}}$.

  In the cases above, we take $k = -3.0, -0.5, 0.0, 0.5, 3.0$.
- Hyperparameters of the prior distribution of σ^{-2}: following Gelman (2006), we take $\frac{\nu_0}{2} = \frac{\nu_0 s_0^2}{2} = \nu$, with $\nu = 0.001$ and 1, where large values indicate little uncertainty regarding to σ^{-2}.
- Other parameters: we fix $a_2 = 1$ and a_1 from -3 to 3 with steps of 0.05.

3.1. Simulation procedure, known σ^2

With n, n_c, p and k fix, we generate 150000 random training samples of size n_c. We calcule the sample mean $\bar{x}_c$ and evaluate (11) vs. a_1, obtaining an approximate curve of the probability of rejection as a function of a_1. These values were summarized by calculating the first and third quartiles, mean and median. The results are shown in Figures 3 and 4.

3.2. Simulation Procedure, Unknown σ^2

With n, n_c, p, ν and k fix, we generate 150000 random training samples of size n_c. We calcule the sample mean $\bar{x}_c$ and the sample variance s_x^2 and evaluate (12) vs. a_1. This values were summarized by calculating the first and third quartiles, mean and median. The results are shown in the Figures from 6 to 9.

3.3. Results with Known σ^2

Figures 3 and 4 show that the greater the sample size (n, n_c) the control chart is more powerful in detecting deviations of the process mean, as expected. Furthermore, as $p \to 0$, the control chart is more sensitive to positive or negative a_1 deviations, also depending on the magnitude and direction in which the prior mean m_0 moves away from the training sample mean $\bar{x}_c$ (that is, according to k). Indeed, if $k < 0$, there is higher power when deviations are positive ($a_1 > 0$), consequently minor ARL in that direction; if $k > 0$ the power is higher when deviations are negative ($a_1 < 0$) and therefore the ARL is lower for those values of a_1. Figures 2 shows the ARL behavior when $p = 0.01$, known σ^2, $k = -3$, $k = 3$, and samples sizes $n_c = n = 5$ y $n_c = n = 30$. We see that with both samples sizes, when $k = -3$, if $a_1 > 0$ then $1 \leq \text{ARL} \leq 2$, and if $a_1 < 0$, then $1 \leq \text{ARL} \leq 30$ approximately; on the other hand, when $k = 3$, if $a_1 > 0$ then $1 \leq \text{ARL} \leq 30$, and if $a_1 < 0$, then $1 \leq \text{ARL} \leq 2$.

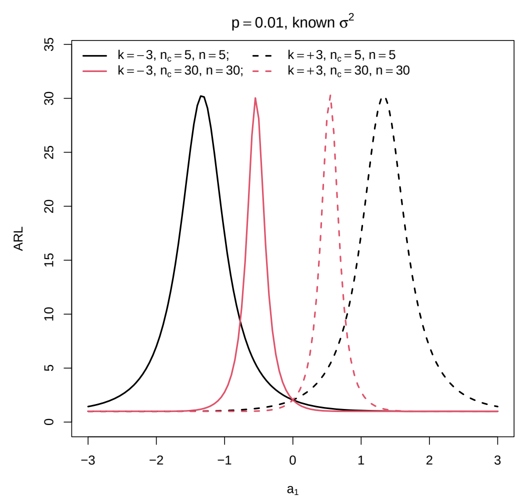

Figure 2: ARL curves with $k = -3$, 3, $p = 0.01$, when σ^2 is known.

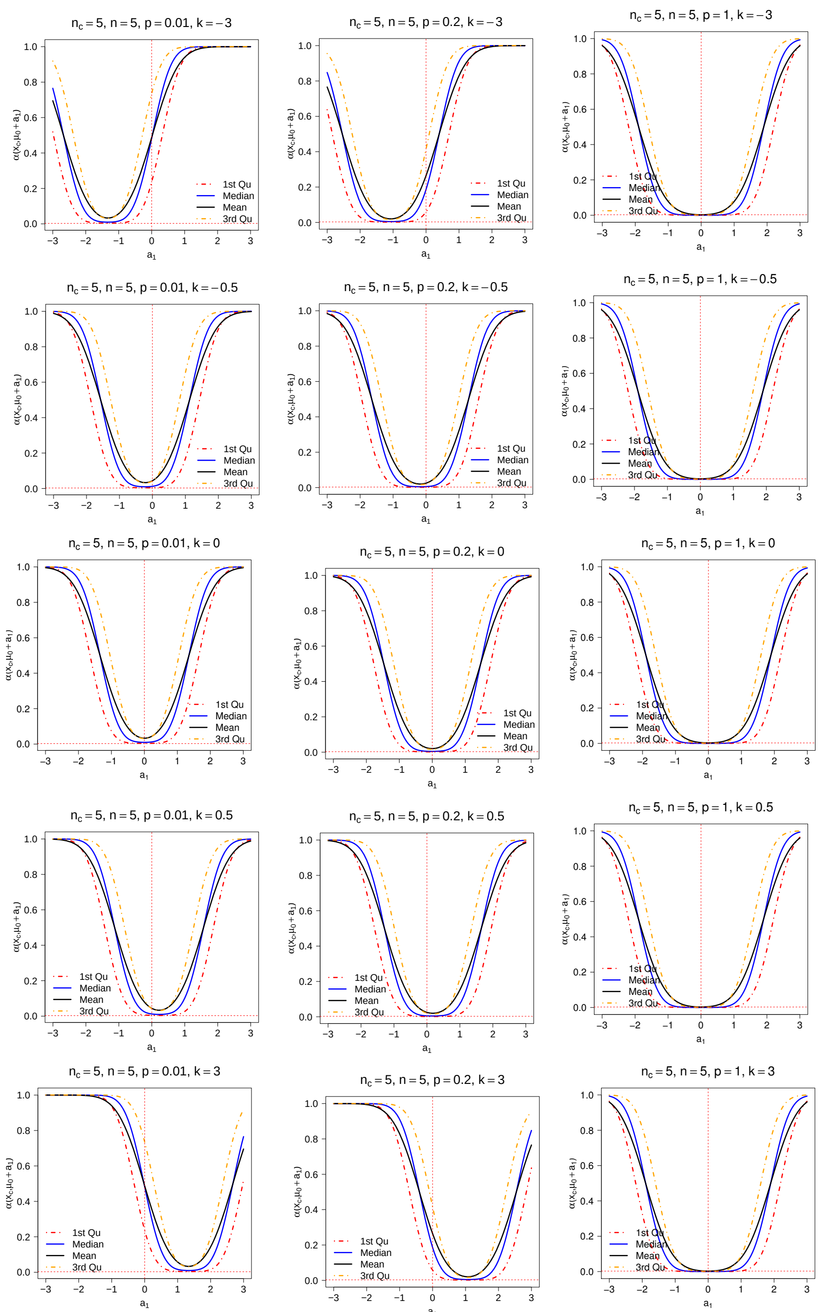

FIGURE 3: Curves of 1st quantile (red dash line), median (blue solid line), mean (black solid line) and 3rd quantile (orange dot line), of $\alpha\left(\bar{X}_c, \mu_0 + a_1\right)$, σ^2 known, with $n = n_c = 5$. From top to bottom varying $k = -3, -0.5, 0, 0.5, 3$ and from left to right varying $p = 0.01, 0.2, 1$.

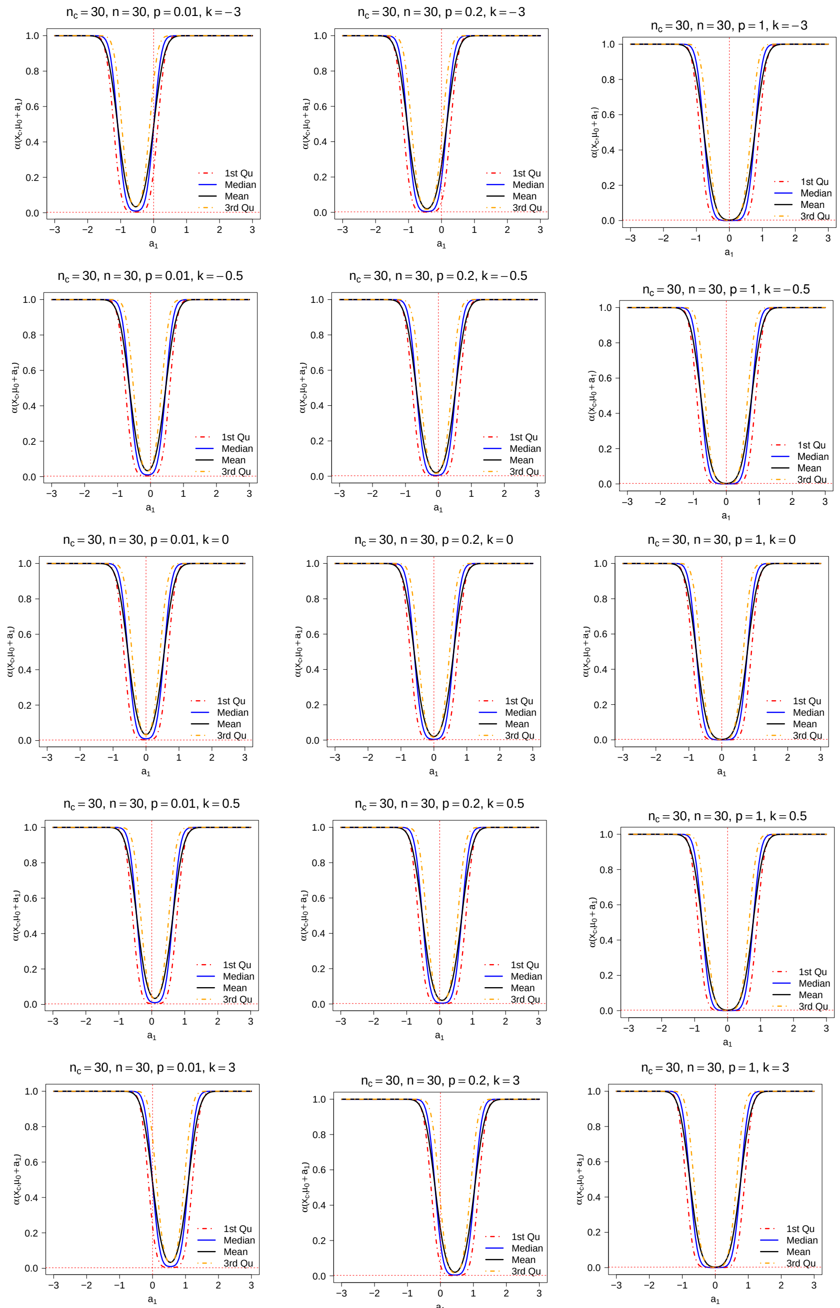

FIGURE 4: Curves of 1st quantile (red dash line), median (blue solid line), mean (black solid line) and 3rd quantile (orange dot line), of $\alpha\left(\bar{X}_c, \mu_0 + a_1\right)$, σ^2 known, with $n = n_c = 30$. From top to bottom varying $k = -3, -0.5, 0, 0.5, 3$ and from left to right varying $p = 0.01, 0.2, 1$.

It is also visible in Figures 3 and 4 an unfavorable feature, the probability of detection or signal when there is not a shift of the mean process (when $a_1 = 0$) can be greater than the nominal rate of 0.0027, therefore the ARL is biased, that is, the maximum is not when $a_1 = 0$. The maximum value of ARL is when $a_1 < 0$, $k < 0$ and $a_1 > 0$, $k > 0$, in both cases the maximum ARL is smaller than $1/0.0027 \approx 370$ (see Figure 2). The above patterns in the probability of detection are more marked if the sample sizes (n y n_c) are small, and consequently the observed problems may be more serious with these sample sizes.

Conversely, when the prior distribution for μ is less informative, ($p \to 1$), the false alarm rate is close to 0.0027. The signal probability is symmetric around $a_1 = 0$, similar to the ordinary $\bar{x}$ chart, but with less power in detecting deviations small of the process mean. For this last chart, under the normality assumption, the probability of detection or signal is symmetric around $a_1 = 0$, but with less power in detecting small deviations in the process mean. Therefore for this control chart the ARL is symmetrical with a maximum value around 370, Figure 5 shows the ARL curves for this situation, when $p = 1$, known σ^2, $k = -3$, $k = 3$, and samples sizes $n_c = n = 5$ and $n_c = n = 30$. We can see that the ARL curves overlap when $k = -3, 3$ for each sample size.

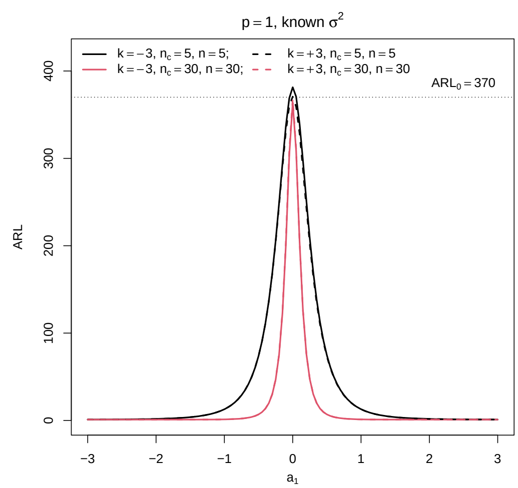

Figure 5: ARL curves with $k = -3$, 3, $p = 1$, when σ^2 is known.

3.4. Results with unknown σ^2

Figures from 6 to 9 reveal that there is a big influence of the prior distribution of the mean when the sample sizes (n, n_c) are small, particulary when this prior is very informative, that is, when p is small, and depending the magnitud and direction of the deviation of the prior mean (m_0) with respect to the sample mean of the calibration sample $(\bar{x}_c)$.

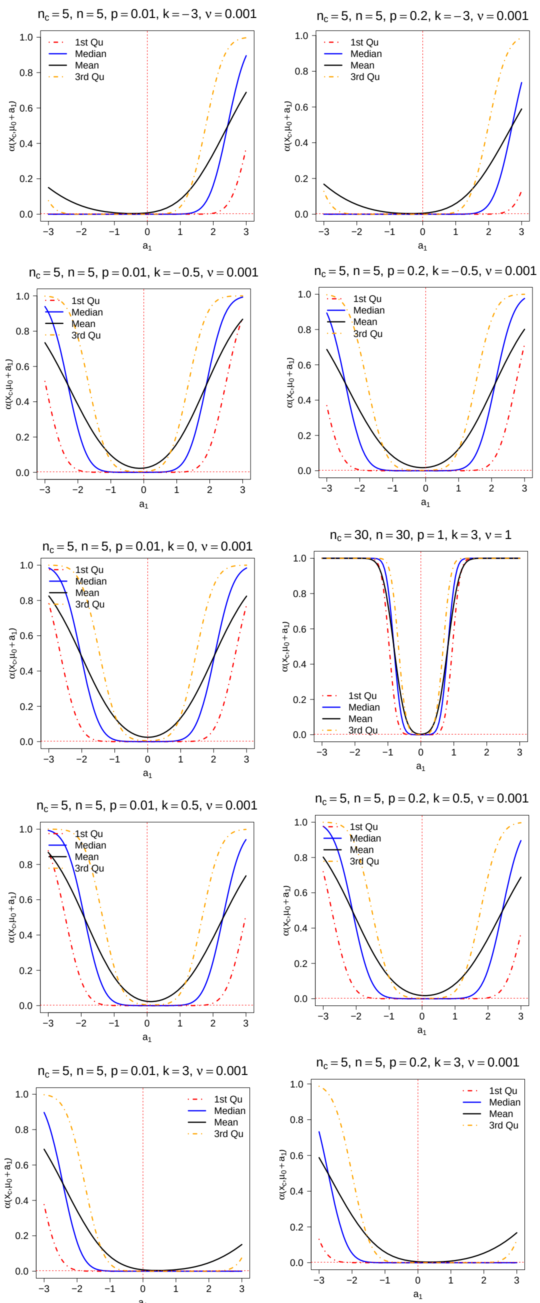

FIGURE 6: Curves of 1st quantile (red dash line), median (blue solid line), mean (black solid line) and 3rd quantile (orange dot line), of $\alpha\left(\bar{X}_c, \mu_0 + a_1, a_2\sigma_0^2\right)$, σ^2 unknown, with $n = n_c = 5$, $\nu = 0.001$. From top to bottom varying $k = -3, -0.5, 0, 0.5, 3$ and from left to right varying $p = 0.01, 0.2, 1$.

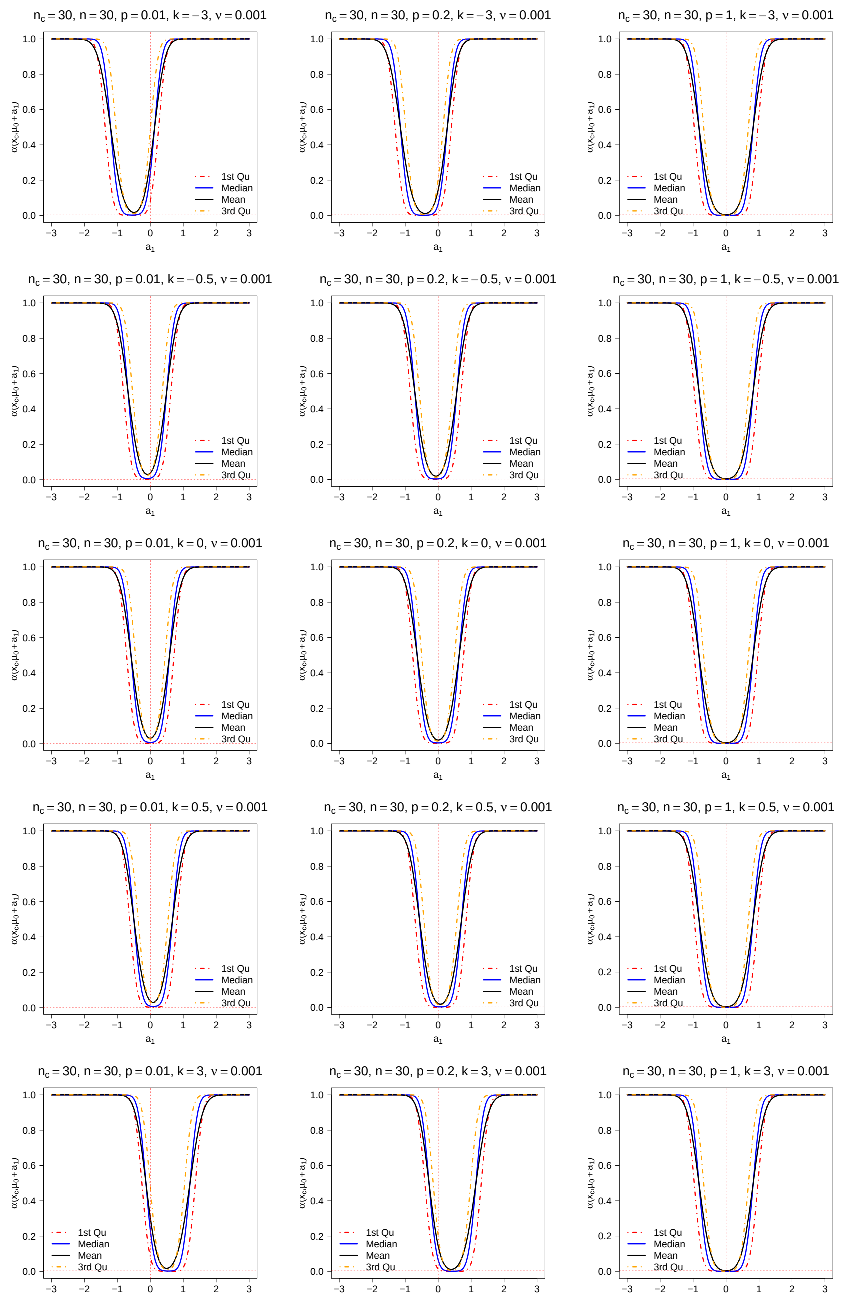

Figure 7: Curves of 1st quantile (red dash line), median (blue solid line), mean (black solid line) and 3rd quantile (orange dot line), of $\alpha\left(\bar{X}_c, \mu_0 + a_1, a_2\sigma_0^2\right)$, σ^2 unknown, with $n = n_c = 30$, $\nu = 0.001$. From top to bottom varying $k = -3, -0.5, 0, 0.5, 3$ and from left to right varying $p = 0.01, 0.2, 1$.

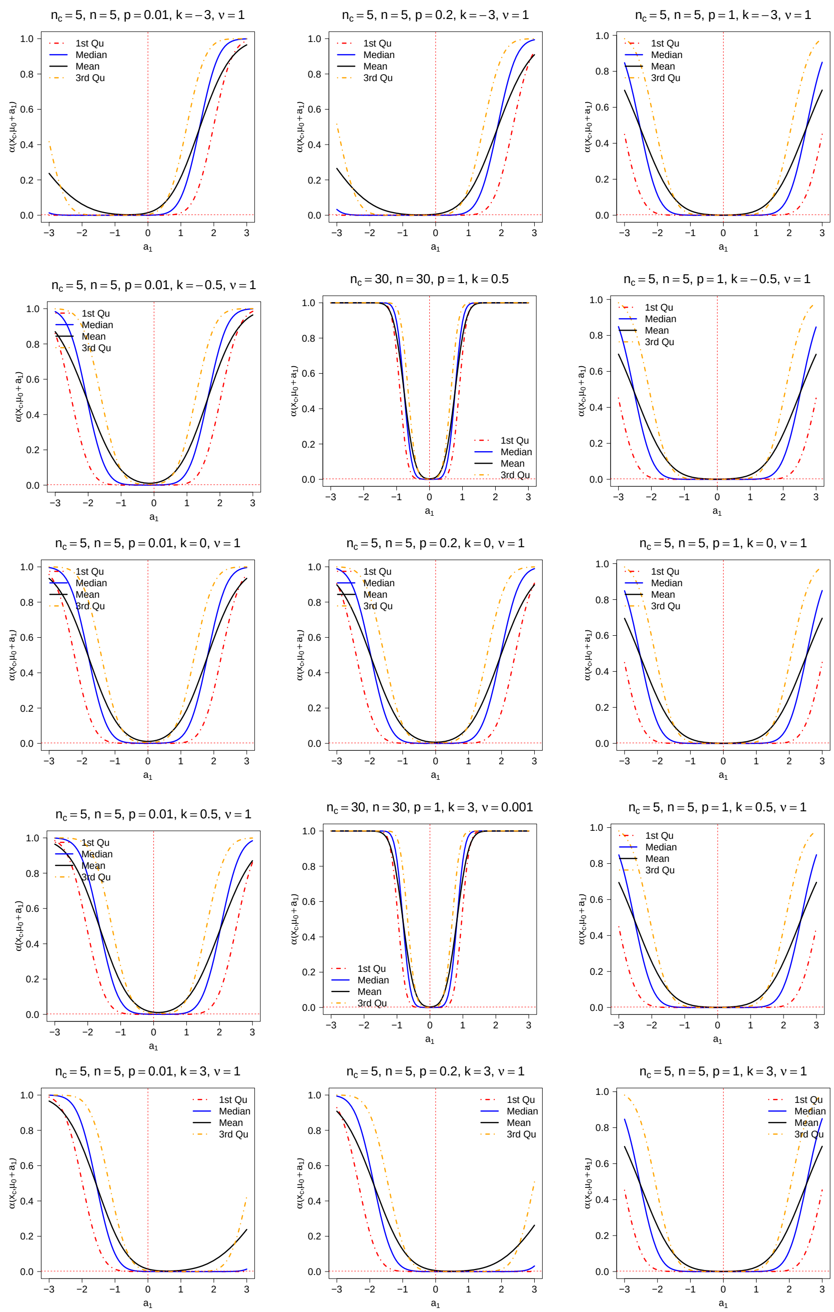

FIGURE 8: Curves of 1st quantile (red dash line), median (blue solid line), mean (black solid line) and 3rd quantile (orange dot line), of $\alpha\left(\bar{X}_c, \mu_0 + a_1, a_2\sigma_0^2\right)$, σ^2 unknown, with $n = n_c = 5$, $\nu = 1$. From top to bottom varying $k = -3, -0.5, 0, 0.5, 3$ and from left to right varying $p = 0.01, 0.2, 1$.

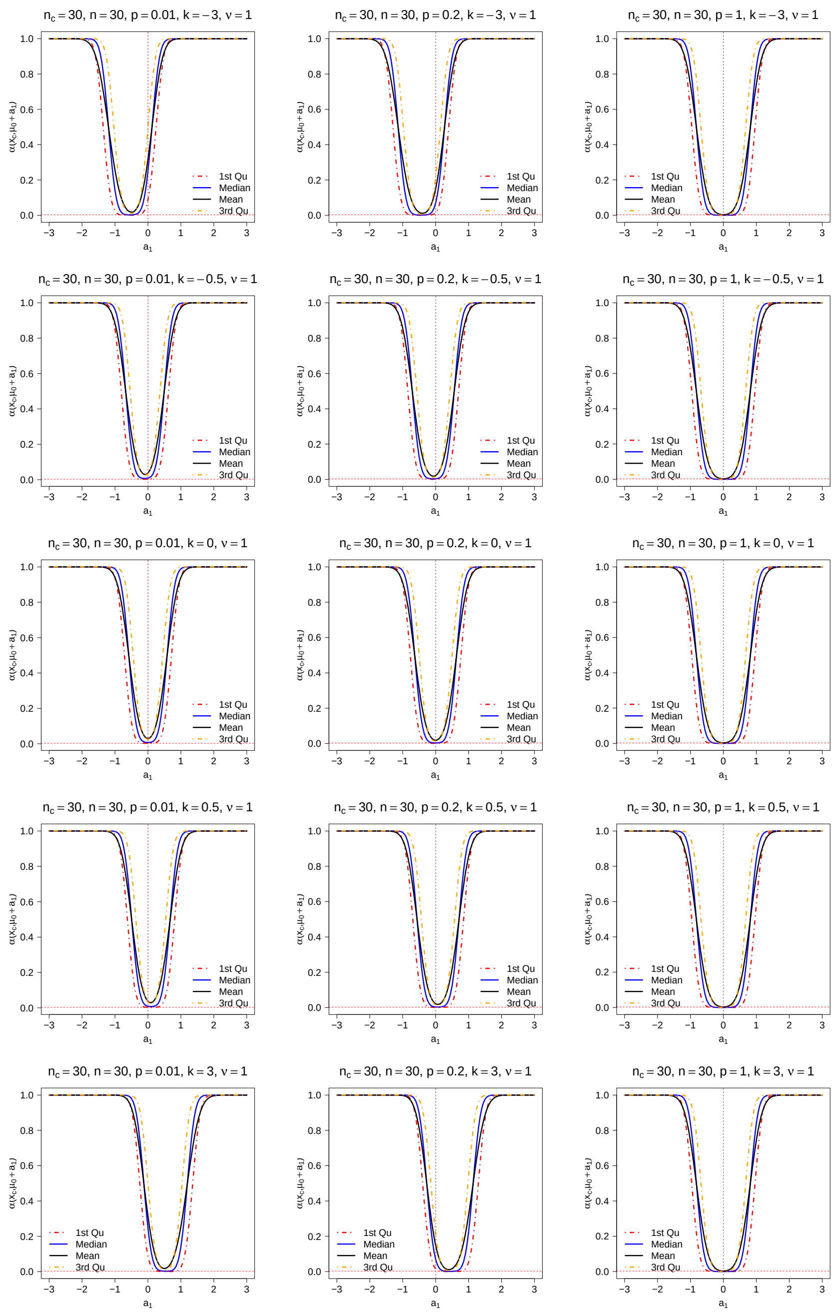

FIGURE 9: Curves of 1st quantile (red dash line), median (blue solid line), mean (black solid line) and 3rd quantile (orange dot line), of $\alpha\left(\bar{X}_c, \mu_0 + a_1, a_2\sigma_0^2\right)$, σ^2 unknown, with $n = n_c = 30$, $\nu = 1$. From top to bottom varying $k = -3, -0.5, 0, 0.5, 3$ and from left to right varying $p = 0.01, 0.2, 1$.

In addition, as expected, the larger the sample size, the greater the power detecting deviations in the process mean. On the other hand, according to the prior distribution for μ, the effect of the prior distribution for σ^2 is notable with small samples.

In general, when $p \to 0$, the control chart may be more sensitive to positive or negative deviations a_1, depending on the magnitude and direction of the deviation of m_0 with respect to $\bar{x}_c$ and the sample sizes as follows:

- If $k < 0$, there is higher power (therefore lower ARL) when deviations are positive ($a_1 > 0$), on the other hand, if the deviations are negative, the power can be less than 0.2 with small samples, even almost null for deviations less than a one standard deviation. We can also see that if $\nu_1 \to 1$, the probability of signal for positive deviations gets bigger than when $\nu \to 0$.

- On the contrary, if $k > 0$, the control chart shows bigger power when deviations are negative ($a_1 < 0$), while in the detection of positive deviations, the power can be less than 0.2 with small samples, even almost null. Further, if $\nu_1 \to 1$, the probability of signal for negative deviations gets bigger than when $\nu \to 0$.

For this case, unknown σ^2, the false alarm rate, $\alpha\left(\bar{x}_c, \mu_0 + a_1, \sigma_0^2\right)$, may be overly large if the sample sizes are large, the prior distribution of μ is very informative and the difference between the prior mean (m_0) and the calibration sample mean ($\bar{x}_c$) is too big. Figure 10 exhibits the behaviour of the ARL for $p = 0.01$, $k = -3,\ 3$, $\nu = 0.001$, $\nu = 1$ and sample sizes $n_c = n = 5$, $n_c = n = 30$, these values turn leads to a ARL-biased chart.

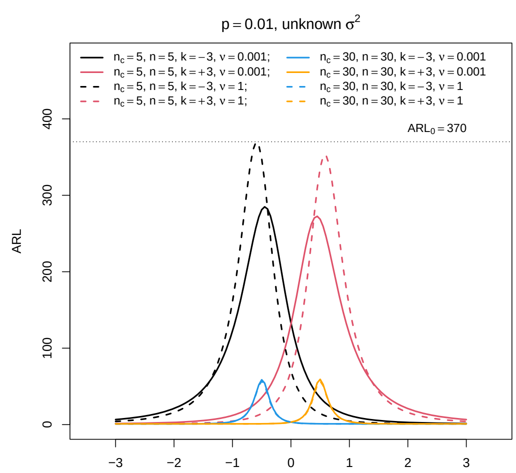

Figure 10: ARL curves with $k = -3,\ 3$, $p = 0.01$, when σ^2 is unknown.

On the other hand, when $p \to 1$, for any k value, the signal probability shows symmetrical behavior around $a_1 = 0$, with a value close to 0.0027. The prior distribution for σ^2 has no visible effects when samples sizes are big. If samples sizes are small we observe some differences between the mean signal probability curves when $\nu = 0.001$ and $\nu = 1$, which leads to differences in the ARL curves, see Figure 11, $p = 1$, $k = -3,\ 3$, $\nu = 0.001$, $\nu = 1$ and sample sizes $n_c = n = 5$, $n_c = n = 30$.

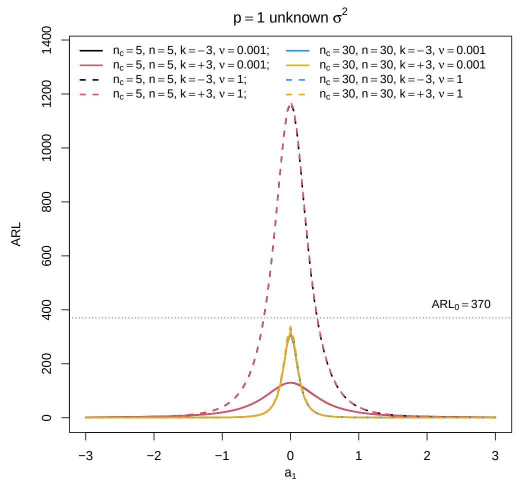

FIGURE 11: ARL curves with $k = -3$, 3, $p = 1$, when σ^2 is unknown.

4. Conclusions

According to the relationship between the probability of detection and ARL, the control charts with predictive limits using very informative prior distributions will no present the maximum of the ARL when the process is in control, which in turn leads to a ARL-biased chart. In addition, when the sample sizes are small the bias is big. An ARL-biased chart is not attractive in practice specially if the bias leads to a high false alarm rate. Therefore, more research is needed using Bayesian approach in the construction of control charts to get a better behavior. Not only is it necessary to determine how to specify the prior distributions, it also seems necessary to evaluate possible corrections to the limits when using sample information for the design of a control chart with an unbiased ARL and satisfying a target value under the process in control.

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