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Additive Outliers in Open-Loop Threshold Autoregressive Models: A Simulation Study

Datos atípicos aditivos en modelos autorregresivos de umbrales: un estudio de simulación

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Abstract

The effect of additive outlier observations is investigated in adapting a non-linearity test and a robust estimation method for the autoregressive coefficients from SETAR(self-exciting threshold autoregressive) models to open-loop models. TAR (threshold autoregressive). Through a Monte Carlo experiment, the power and size of the non-linearity test are studied. Regarding the estimation, the bias and the mean square error ratio between the robust estimator and the least-squares estimator are compared. Additionally, the approximation of the GM estimators' empirical distribution to the univariate normal distribution is evaluated together with the coverage levels of the asymptotic confidence intervals. The results indicate that the adapted non-linearity test has higher power than that based on least squares and does not present distortions in size under the presence of additive outliers. On the other hand, the robust estimation method for autoregressive coefficients exceeds the least-squares one in terms of the mean square error in the presence of this type of observations. These results were analogous to those obtained for SETAR models. Finally, the use of the non-linearity test and the estimation method are illustrated through two real examples.

***Key words*:** additive outliers; open-loop TAR models; generalized method (GM) estimator; nonlinear time series.

Resumen

Se investiga el efecto de observaciones atípicas aditivas en la adaptación de una prueba de no linealidad y un método de estimación robusto para los coeficientes autorregresivos de modelos SETAR(self-exciting threshold

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autoregressive) a modelos open-loop TAR(threshold autoregressive). A través de un experimento Monte Carlo se estudia la potencia y el tamaño de la prueba de no linealidad. Respecto a la estimación, se compara el sesgo y la razón de error cuadrático medio entre el estimador robusto y el de mínimos cuadrados. Adicionalmente, se evalúa la aproximación de la distribución empírica de los estimadores GM de los coeficientes a la distribución normal univariada junto a los niveles de cobertura de los intervalos de confianza asintóticos. Los resultados indican que la prueba de no linealidad adaptada presenta una potencia superior a la basada en mínimos cuadrados y no presenta distorsiones en el tamaño bajo la presencia de datos atípicos aditivos. Por otro lado, el método de estimación robusto para los coeficientes autorregresivos supera al de mínimos cuadrados en términos de error cuadrático medio bajo la presencia de este tipo de observaciones. Estos resultados fueron análogos a los obtenidos para modelos SETAR. Finalmente, se ilustra a través de dos ejemplos reales el uso de la prueba de no linealidad y el método de estimación.

***Palabras clave*:** datos atípicos aditivos; modelos open-loop *TAR*; estimadores *GM*; series de tiempo no lineales.

1. Introduction

Since their appearance in the literature in Tong (1978) and Tong (1990), TAR (threshold autoregressive) models have been widely used to explain phenomena observed in empirical time series and are still very popular. An excellent review was carried out in Hansen (2011), who references more than seventy articles in economics that range from applications of the model to contributions to the theory of estimation and inference. In Chan & Ng (2004), other articles outside this field in which this type of model had been used. The great popularity of TAR models lies in their ease of estimation (which is, in general, the estimation method is conditional least squares) and their ability to capture non-linear behaviors such as asymmetries, limit cycles, and jump phenomena (many of these referenced in the literature, see for instance Franses et al. 2000, Granger & Teräsvirta 1993).

Despite its wide use, one of the topics little studied in the literature is related to outliers data in TAR models; this fact contrasts with the situation in linear models for which the topic has been studied in greater depth (see, for instance, Chen & Liu 1993, Tsay 1988). The few articles on the subject have focused mainly on the case of SETAR(Self Exciting Threshold Autoregressive) models, a particular case of the open-loop TAR model, more specifically the SETAR model is the one obtained by making the time series that determines the change between regimes(the threshold variable) is equal to the same observed time series. This fact can be a limitation in real data analysis because another variable can drive the dynamic of the variable of interest. Then, open-loop TAR models can be used for modeling in that real data applications, see Knotters & De Gooijer (1999), Zhang & Nieto (2015), Gonzalez & Nieto (2020) for examples in univariate open-loop TAR model, and Tsay (1998), Calderón & Nieto (2017), Romero & Calderón (2021) for examples in Multivariate TAR models. The literature of outliers in non-linear time series models has focused mainly on three research lines; the first refers to the detection of non-linearity, the second to robust estimation methods in the presence of outlier data, and the third to methods of detection and modeling of the outlier observations.

In the first place, regarding the detection of non-linearity, Chan & Ng (2004) carried out a simulation study to evaluate some classical tests' properties for the detection of SETAR-type non-linearity existing in the literature; the tests considered are those mentioned in Chan & Tong (1990), Luukkonen et al. (1988), Petruccelli & Davies (1986), Petruccelli (1990), Tsay (1989). In the presence of outlier observations, the results suggest that none of the tests considered is robust since, for these, the empirical sizes generally exceed the nominal size. Based on this, in Hung et al. (2009) is proposed a robust extension to outlier observations of the non-linearity test introduced by Tsay (1989) for the case of SETAR models and show through simulations that the empirical size and power of this new test are adequate. Second, as far as robust estimation methods are concerned, Chan & Cheung (1994) proposed the use of GM (generalized-M) estimators for this type of model; The authors conclude that under the presence of outlier observations, the GM method has a better performance in terms of mean square error than that of conditional least squares. This result was questioned in Giordani (2006), who exemplifies some situations in which the estimators obtained through this method can become inconsistent and inefficient if it is used to estimate the thresholds that determine the model regimes. A few years later, in Zhang et al. (2009) is given conditions under which the estimators obtained through the GM method for the model parameters are consistent. Finally, regarding the third line of research, Battaglia & Orfei (2005) propose a general framework under which outlier observations can be detected and modeled in non-linear time series through the conditional maximum likelihood method; using Monte Carlo exercises, the authors illustrate the performance of this methodology for the case of different non-linear models including the SETAR model.

One of the problems that arise when carrying out traditional estimation procedures for non-linear time series models in the presence of outlier observations is that, as mentioned by Tiao & Tsay (1994), there is always the possibility of confusing effects of this type of observations with non-linear dynamics typical of the time series. Hence, a process that follows a linear model with these types of observations can appear non-linear, and likewise, a linear time series with some outliers could be non-linear. Being able to distinguish these cases is essential since improperly identifying the model can lead to misinference. Therefore, this document seeks to evaluate through simulations the size and power of the non-linearity test in Hung et al. (2009) adapted to the open-loop TAR model. The experimentation is carried out using an autoregressive process as the data-generating process(GDP) for the threshold variable. Additionally, we compare through the mean square error and the bias(that was not studied before) the proposed estimation method of Chan & Cheung (1994) against the classical estimation method (conditional least squares(LS)) for the case of open-loop *TAR* models (adapting the methodology beyond *SETAR* models). We also study for finite samples the empirical normal distribution of the GM estimators for the so-called autoregressive parameters in each regime and how it is the empirical coverage of the confidence intervals for the autoregressive parameters with the adapted estimation methods for open-loop TAR models. We carried out those studies because in Sorour & Tong (1993), it is suggested that the direct extensions from the SETAR model to the open-loop TAR model should be studied. Then the paper proceeds as follows. In the second session, an adaptation of the robust estimation procedure is presented. In the third Section, a non-linearity test of TAR type is implemented to take into account outlier observations. In section 4, a Monte Carlo simulation study is carried out. Finally, a real data application to financial time series showed to illustrate the methodology.

2. Adaptation of the Estimation Procedure and the Non-Linearity Test to the Open-Loop TAR Model

This section presents the adaptation of the estimation procedure of Chan & Cheung (1994) and the non-linearity test of Hung et al. (2009) from the SETAR models to the open-loop TAR models. We first introduce the Open-loop TAR model and how this model can be written as a linear regression model by regimes. Next, it is presented how to take into account the additive outliers in this context.

In Tong (1990, p. 101), we can find the definition of an Open-loop TAR model; however we use a simplification of that model. A time series $\{Y_t\}_{t\in\mathbb{Z}}$ is said to follow a open-loop TAR model, denoted $TAR(Z, k, p_1, \dots, p_k, d)$ if

$$Y_t = \phi_0^{(j)} + \sum_{i=1}^{p_j} \phi_i^{(j)} Y_{t-i} + \sigma_j \epsilon_t \quad si \quad r_{j-1} \leq Z_{t-d} < r_j, \tag{1}$$

where $j = 1, \dots, k$, it is assumed that $\{\epsilon_t\} \overset{iid}{\sim} (0, 1)$, $\{Z_t\}_{t\in\mathbb{Z}}$ is a known stationary time series, we have $r_0 = -\infty < r_1 < \cdots < r_k = \infty$. The parameter p_j is the order of the autoregressive model in the j-th regime and the $\phi_i^{(j)} \quad i = 0, \dots, p_j$ the autoregressive coefficients in this regime($\phi_0^{(j)}$ is the intercept). d is known as the delay parameter, k is the number of regimes and the r_j is known as the thresholds. σ_j represents the scale of ϵ_t at the j-th regime. A particular case of the open-loop *TAR* model, widely used in practice, is the *SETAR* model. In this model, the time series that characterizes the regime change $\{Z_t\}_{t\in\mathbb{Z}}$ is the same time series under analysis $\{Y_t\}_{t\in\mathbb{Z}}$. As it was mentioned in Tsay (1998), the behavior of the threshold variable has a profound impact on the dynamic of the interest variable; therefore, the data-generating process(DGP) for threshold variable can affect the results over the non-linearity test and the estimation procedure in open-loop TAR models in the presence of outliers.

The expressions used in this paper correspond to an open-loop TAR model with two regimes, that is, when $k = 2$ in (1); however, this methodology can be extended to models with more regimes. We first can observe that a 2-regime TAR model

$$Y_t = \begin{cases} \phi_0^{(1)} + \sum_{i=1}^{p_1} \phi_i^{(1)} Y_{t-i} + \sigma_1 \epsilon_t & if \quad Z_t \leq r \\ \phi_0^{(2)} + \sum_{i=1}^{p_2} \phi_i^{(2)} Y_{t-i} + \sigma_2 \epsilon_t & if \quad Z_t > r, \end{cases} \tag{2}$$

can be written as a linear regression model by regimes, that is, for $j = 1, 2$, and assuming that $p_1 = p_2 = p$, then

$$\mathbf{y_j} = \mathbf{X_j} \boldsymbol{\phi_j} + \boldsymbol{\epsilon_j}, \tag{3}$$

$$\begin{aligned} \boldsymbol{\epsilon_j} &= (\epsilon_{ji_1}, \ldots, \epsilon_{ji_{T_j}})', \\ \boldsymbol{\phi_j} &= (\phi_0^{(j)}, \ldots, \phi_p^{(j)})', \\ \mathbf{x_{ji_l-1}} &= (1, Y_{ji_l-1}, \ldots, Y_{ji_l-p})', \\ \mathbf{X_j} &= [\mathbf{x_{ji_1-1}}, \ldots, \mathbf{x_{ji_{T_j}-1}}]', \end{aligned}$$

where observations $\{Y_{p+1}, \ldots, Y_T\}$ can be split in two groups according to the following rule:

$$\begin{cases} Y_t \in 1^{st} \ \ group & if \quad Z_{t-d_0} \leq r_1 \\ Y_t \in 2^{nd} \ \ group & if \quad Z_{t-d_0} > r_1, \end{cases} \tag{4}$$

with $l = 1, \ldots, T_j$, and d_0 is a proposed positive integer value for the delay value. It is necessary to know the threshold value r_1. Therefore $\mathbf{y_1} = (Y_{1i_1}, \ldots, Y_{1i_{T_1}})'$, $\mathbf{y_2} = (Y_{2i_1}, \ldots, Y_{2i_{T_2}})'$ are the observations in the first and second regime respectively ($T_1 + T_2 = T - p$, each one represents the number of observations in each regime). With these, we complete the specification of the linear regression model by regimes (3).

We consider in this paper outlier observations of an additive nature(which are the most studied). Chan & Ng (2004) extended what is suggested by Denby & Martin (1979) and propose to define them as follows: The observed time series is $\{Y_t^*\}_{t\in\mathbb{Z}}$(time series with outlier effect), and the time series of interest is $\{Y_t\}_{t\in\mathbb{Z}}$(time series without outlier effect),

$$Y_t^* = Y_t + \kappa_t, \tag{5}$$

where κ_t is an identically distributed independent random variable that follows a mixed distribution with density $(1-\beta)\delta_0(\cdot) + \beta\phi_\omega(\cdot)$ with $\phi_\omega(\cdot)$ representing the normal distribution of mean 0 and variance ω^2, $\delta_0(\cdot)$ representing a degenerate distribution at 0, β $(0 \leq \beta \leq 1)$ is the percentage of contamination in the time series and ω its magnitude.

2.1. Robust Estimation of the Open-Loop Threshold Autoregressive Models

In this section, we propose an adaptation of the robust methodology proposed in Chan & Cheung (1994) for SETAR to the case of the open-loop TAR model based on a linear regression model by regimes in 3. Note that parameters of the TAR model in (3) can be estimated using least squares estimation in each regime. If the time series is affected by an additive outlier, the estimator is no efficient, and then a robust estimator can be better. The class of estimators GM is an extension of the class of M-robust estimators proposed in classical regression problems (see, for example, Maronna et al. (2019)). It is necessary to take into account two cases when threshold r_1 is known and unknown.

Case threshold r_1 is known.
If r_1 is known, then the GM-estimator $\tilde{\boldsymbol{\phi}}_j$ of $\boldsymbol{\phi}_j$ is defined implicitly for the condition for $j = 1, 2$

$$\sum_{l=1}^{T_j} \eta\left(d^2(\mathbf{x_{ji_l-1}}), \frac{Y_{ji_l} - \mathbf{x_{ji_l-1}}'\tilde{\boldsymbol{\phi}}_j}{\hat{\sigma}_j}\right)\mathbf{x_{ji_l-1}} = \mathbf{0}, \tag{6}$$

where $d^2(\mathbf{x})$ is used to denote the squared Mahalanobis distance, that is, $d^2(\mathbf{x}) = (\mathbf{x}-\mathbf{m})'\mathbf{V}^{-1}(\mathbf{x}-\mathbf{m})$ with $\mathbf{m}$ and $\mathbf{V}$ robust location and dispersion estimators(it is generally suggested to calculate $\mathbf{m}$ through the median and $\mathbf{V}$ using the estimator proposed by Rousseeuw & Van Zomeren (1990)). $\hat{\sigma}_j$ is obtained with a robust scale estimator (for example a M-estimator). The conditions that the function $\eta(\cdot,\cdot)$ must fulfill to have optimal asymptotic properties can be found in (Hampel et al., 1986, p. 315). We propose to use Schweppe-type parametrization of the function $\eta(\cdot,\cdot)$, that is, for $j = 1, 2$

$$\eta\left(d^2(\mathbf{x_{ji_l-1}}), \frac{Y_{ji_l} - \mathbf{x_{ji_l-1}}'\tilde{\boldsymbol{\phi}}_j}{\hat{\sigma}_j}\right) = w(d^2(\mathbf{x_{ji_l-1}}))\psi\left(\frac{Y_{ji_l} - \mathbf{x_{ji_l-1}}'\tilde{\boldsymbol{\phi}}_j}{\hat{\sigma}_j w(d^2(\mathbf{x_{ji_l-1}}))}\right)$$

where the function $\psi(\cdot)$ belongs to the group of functions used in the construction of the class of M-robust estimators (in Table 1 we present some of the functions $\psi(\cdot)$ most used in practice, see Maronna et al. (2019) for plots of that functions). The function $w(\cdot)$ is a function that assigns weights, with rank at $[0, 1]$, and is defined as $w(u) = \frac{\psi(u)}{u}$ if $u \neq 0$ and $w(0) = 1$.

Where $\rho(u)$ is a function such that $\frac{\partial\rho(u)}{\partial u} = \psi(u)$. With Schweppe parametrization and with fixed functions $\rho(\cdot)$ and $\phi(\cdot)$, the equation (6) is non-linear in $\tilde{\boldsymbol{\phi}}_j$, therefore, to find the solution, numerical methods must be used. In general, it is suggested to take a robust initial estimator for $\boldsymbol{\phi_j}$ as the LAD (least absolute deviations) estimator proposed in Rousseeuw (1984). From this estimator $(\tilde{\boldsymbol{\phi}}_j^{(0)})$ we can obtain the residuals for all time ji_l in the regime j as, $\hat{\epsilon}_{ji_l}^{(0)} = Y_{ji_l} - \mathbf{x_{ji_l-1}}'\tilde{\boldsymbol{\phi}}_j^{(0)}$, where the value of $\hat{\sigma}^{(0)}$ can be obtained using the estimator MAD (median absolute deviation) over the residuals. Once you have this information, you can calculate the weights: $w\left(\frac{\hat{\epsilon}^{(0)}_{ji_l}}{\sigma^{(0)}_j w(d^2(\mathbf{x_t}))}\right)$ in the case of the Schweppe. Taking these weights as if they were fixed, the equation (6) can be solved. Weighted least squares are used to obtain the value of $\tilde{\boldsymbol{\phi}}^{(1)}_j$. With this last value, you can find the value of $\hat{\boldsymbol{\sigma}}^{(1)}_{\boldsymbol{j}}$ and calculate the weights again to find the value of $\tilde{\boldsymbol{\phi}}^{(2)}_j$; this procedure is carried out successively until some previously imposed convergence condition is met (for example, that the difference between two successive estimates is small enough). Ultimately, this solution method is known as *IWLS* (iterative weighted least squares). It is important to point out that weighted least squares can be used because $\psi(u) = w(u) \times u$, which is replaced in $\eta(\cdot,\cdot)$ and equation (6).

Table 1: Some functions ψ and their respective functions ρ

Type	Function-ρ $\rho(u)$	Function-ψ $\psi(u)$	Rank of u
Least Squares	$\frac{u^2}{2}$	u	$(-\infty,\infty)$
Huber(k)	$\frac{u^2}{2}$	u	$\lvert u\rvert \le k$
	$k\lvert u\rvert - \frac{k^2}{2}$	$ksgn(u)$	$\lvert u\rvert > k$
Tukey(c)	$\frac{c^2}{6}\left(1-\left(1-\left(\frac{u}{c}\right)^2\right)^3\right)$	$u\left(1-\left(\frac{u}{c}\right)^2\right)^2$	$\lvert u\rvert \le c$
	$\frac{c^2}{6}$	0	$\lvert u\rvert > c$

Following to Maronna et al. (2019, p. 173), we can see that the estimator $\tilde{\boldsymbol{\phi}}_j$ for $j = 1,2$ obtained trough *GM* method is asymptotically normal

$$\sqrt{T_j}(\tilde{\boldsymbol{\phi}}_j - \boldsymbol{\phi}_j) \sim N(0, \Sigma_{\boldsymbol{\phi}}). \tag{7}$$

For this case, the asymptotic covariance matrix of $\tilde{\boldsymbol{\phi}}_{\boldsymbol{j}}$ can be obtained as:

$$\hat{\Sigma}_{\boldsymbol{\phi}_j} = \sigma_j^2 \mathbf{B^{-1}}'\mathbf{C}\mathbf{B^{-1}} \tag{8}$$

$$\mathbf{B} = -E\left[\eta^*\left(d^2(\mathbf{X_j}), \frac{\mathbf{y_j} - \mathbf{X_j}\boldsymbol{\phi_j}}{\sigma_j}\right)\mathbf{X_j}'\mathbf{X_j}\right] \qquad \eta^*(d^2(\mathbf{X_j}), R_t) = \frac{\partial\eta(d^2(\mathbf{X_j}), R_t)}{\partial R_t}$$

$$\mathbf{C} = E\left[\eta\left(d^2(\mathbf{X_j}), \frac{\mathbf{y_j} - \mathbf{X_j}\boldsymbol{\phi_j}}{\sigma_j}\right)^2\mathbf{X_j}'\mathbf{X_j}\right].$$

Case threshold r_1 is unknown.

It can be considered that the estimator *GM* with the Schweppe type parametrization in each of the regimes, it is, in fact, the solution to the optimization problem (9). A global objective function can be defined as:

$$\Gamma(\boldsymbol{\phi_1}, \boldsymbol{\phi_2}, \sigma_1, \sigma_2, r_1) = \sum_{j=1}^{2} \gamma_j(\boldsymbol{\phi_j}, \sigma_j), \tag{9}$$

where $\gamma_j(\boldsymbol{\phi_j}, \sigma_j)$ is the function defined as

$$\gamma_j(\boldsymbol{\phi_j}, \sigma_j) = \sum_{l=1}^{T_j} w(d^2(\mathbf{x_{ji_l-1}}))^2 \rho \left(\frac{Y_{ji_l} - \mathbf{x_{ji_l-1}}' \boldsymbol{\phi}_j}{\sigma_j w(d^2(\mathbf{x_{ji_l-1}}))} \right).$$

A robust estimator of r_1 ($\hat{r}_1$) can be found by searching through the set $\{X_{p_{\alpha\%}}, \dots, X_{p_{(1-\alpha)\%}}\}$ where $X_{p_{\alpha\%}}$ denotes the α-th percentile of the variable Z_t. That is, the value of $\hat{r}_1$ is the solution to:

$$\underset{r_1 \in \{Y_{p_{\alpha\%}}, \dots, Y_{p_{(1-\alpha)\%}}\}}{Argmin} \quad \Gamma(\boldsymbol{\phi_1}, \boldsymbol{\phi_2}, \sigma_1, \sigma_2, r_1). \tag{10}$$

Note that once the value of r_1 is set, the estimators $\hat{\boldsymbol{\phi}}_1, \hat{\boldsymbol{\phi}}_2, \hat{\sigma}_1, \hat{\sigma}_2$ can be easily found according to discussed above. In practice it is suggested to take a value $\alpha = 0.25$ so that there are enough observations in each of the regimes.

Giordani (2006) exemplified some problems where the threshold value estimated using the above procedure may be incorrect. To ensure the consistency of the threshold estimator, the function ψ used must meet certain conditions that can be found in Zhang et al. (2009). In fact, one of the conditions is that the function used is convex, the Tukey function (originally used in the article by Chan & Cheung 1994) of the Table 1 does not fulfill this property while the of Huber does.

3. Non-linearity Test of open-loop TAR Type

The importance of carrying out the non-linearity test allows identifying better a model that best fits the data; this translates into both better forecasts and more accurate inference. Tsay & Chen (2019) mentions several examples of observed time series that seem to be better represented through non-linear models and emphasizes the improvement obtained in forecasting.

The idea behind the non-linearity test for *SETAR* models proposed by Hung et al. (2009) (in fact, it is a generalization of the Tsay (1989) test) is to carry out an autoregression ordered using a *GM* estimator with a function ψ with the Schweppe to obtain predictive residuals that will later be used in the non-linearity test. The adaptation proposed for Open-Loop TAR in this paper consist of arranged autoregression is according to threshold variable Z_{t-d} instead of Y_{t-d}.

The proposed F_{GM} statistic is

$$F_{GM}^{y_{t-1}} = \frac{MSS(T - p - m - (p+1))}{RSS(p+1)}, \tag{11}$$

where

$$MSS = \hat{\boldsymbol{\Psi}}'\mathbf{W}\mathbf{X}(\mathbf{X}'\mathbf{W}\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\hat{\boldsymbol{\Psi}}$$

$$RSS = \hat{\boldsymbol{\Psi}}'\hat{\boldsymbol{\Psi}} - MSS$$

$$\hat{R}_{\pi_i} = \frac{\hat{\epsilon}_{\pi_j}}{\hat{\sigma}_\epsilon} \text{ for } i = m+1, \ldots, T-p$$

$$\hat{\boldsymbol{\Psi}} = (\psi(\hat{R}_{\pi_{m+1}}), \ldots, \psi(\hat{R}_{\pi_{T-p}}))'$$

$$\mathbf{X} = [\mathbf{x}_{\pi_\mathbf{m}}, \ldots, \mathbf{x}_{\pi_{\mathbf{T-p-1}}}]'$$

$$\mathbf{W} = diag(w(d^2(\mathbf{x}_{\pi_\mathbf{m}}))), \ldots, w(d^2(\mathbf{x}_{\pi_{\mathbf{T-p-1}}})),$$

and $\hat{\epsilon}_{\pi_j}$ is the one-step-ahead prediction residual based on the GM estimator, and vectors $\mathbf{x}_{\pi_1}$ for $l = m, \ldots, T-p$, where defined in (3), $\hat{\sigma}_\epsilon$ is the robust variance error estimator, and the matrix $\mathbf{X}$ is obtained from the regressors of the arranged autoregression. Note that p is the order of the AR model proposed for null hypothesis H_0. Under the null hypothesis of an autoregressive linear model, the test statistic in (11) has asymptotically a distribution F (see Hung et al., 2009) with $p+1$ and $(T-p-m-(p+1))$ degrees of freedom. In Tsay (1989) and Hung et al. (2009) proposed to use in practice a value of m equal to $\frac{T}{10}+p$. Note that m is the number of observations that it is used to start the estimation of the parameters in the ordered autoregression. Note that asymptotic convergence of the statistic F_{GM} is not affected by the change of the threshold variable. Finally, in the case where we want to consider another value of delay d on the time series Y_{t-d}, the estimator of this parameter can be selected in such a way that:

$$\underset{v\ inS}{Argmax} \quad F_{GM}^{y_{t-v}}, \tag{12}$$

where S is a previously selected set of values. The value of p used in the construction of the test can be obtained through a preliminary analysis of the *PACF* (partial autocorrelation function) and carrying out the identification of the autoregressive process as if the model out linear.

4. Monte Carlo Results

In order to verify the suitability of the proposed methodology, the results of the simulation exercises related to the adaptation of the non-linearity test of Hung et al. (2009) and the estimation method of Chan & Cheung (1994) are presented applied to the case of TAR models in the presence of additive outlier observations. Also, the effects that additive outlier observations have on this test and the estimation method are evaluated in the following ways: (1) different percentages of contamination of this type of observations, (2) different proportion of outlier observations in each of the model regimes, and (3) coverage levels of the asymptotic confidence intervals constructed under the traditional *GM* methodology.

Following the simulation exercises' design in Chan & Cheung (1994), Hung et al. (2009), a model with two regimes TAR(Z, 2,1,1,0) is considered. This model is also chosen based on the ease with which its estimation process can be carried out and because in practical applications, two or three regimes are generally used (see Hansen, 2011). The transition variable $\{Z_t\}_{t\in\mathbb{Z}}$ follows an AR (1) model. The GDP is then:

$$Y_t = \begin{cases} \phi_0^{(1)} + \phi_1^{(1)} Y_{t-1} + \epsilon_t \quad si \quad Z_t \leq r \\ \phi_0^{(2)} + \phi_1^{(2)} Y_{t-1} + \epsilon_t \quad si \quad Z_t > r \end{cases} \tag{13}$$

$$Z_t = \nu Z_{t-1} + \eta_t \quad |\nu| < 1 \tag{14}$$

$$Y_t^* = Y_t + \kappa_t \tag{15}$$

where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ and $\{\eta_t\}_{t\in\mathbb{Z}}$, are sequences of random variables $\stackrel{iid}{\sim} N(0,1)$. $\{Y_t^*\}_{t\in\mathbb{Z}}$ represents the observed time series (contaminated with additive outliers), κ_t is an independent and identically distributed random variable that follows a mixed distribution with density $(1-\beta)\delta_0(\cdot)+\beta\phi_\omega(\cdot)$, where $\phi_\omega(\cdot)$ representing the normal distribution mean 0 and variance ω^2. $\delta_0(\cdot)$ represents a degenerate distribution at 0, β $(0 \leq \beta \leq 1)$ is the time percentage of contamination in the time series and ω its magnitude. Finally, $\{Y_t\}_{t\in\mathbb{Z}}$ represents the underlying (uncontaminated) time series. For each replication, a sample $\{Y_1, \ldots, Y_T\}$ is generated according to the equation (13). Sample sizes $T = 100$, $T = 200$ are considered. The initial value X_0 is taken equal to zero. The first 1000 observations are discarded in each of the replications,to eliminate possible effects of this initial value on the time series.

Regarding the non-linearity test, the results of the power and size of the test are presented based on the value $F_{GM}^{z_t}$ with different choices for the function $\psi(\cdot)$. The function of *Huber* with values of $k = 1.345$ and $k = 3.291$and that of *Tukey* with values of $c = 4.685$ y $c = 15$. All these choices are based on those selected by Hung et al. (2009)[1]. In both the non-linearity test and the estimation procedure, the *Least Squares* (LS) function is considered to facilitate comparison.

Finally, as regards the values of the model parameters *TAR*, we consider $\nu = \pm 0.5, \pm 0.8$, $\omega = 0, 3, 6, 10$, and $r = 0$. The contamination percentage $\beta = 5\%$ (in exercises presented later we consider $\beta = 10\%$ and $\beta = 1\%$). The initial value for arranged autoregression is $m = \frac{T}{10} + 1$ and for $\phi_0^{(1)}, \phi_1^{(1)}, \phi_0^{(2)}, \phi_1^{(2)}$ following the results of Hung et al. (2009), seven combinations of parameters are considered where $\phi_0^{(1)} = 0.5$, $\phi_0^{(2)} = 0.5$, $\phi_1^{(1)} = 0.5$ and $\phi_1^{(2)}$ varies from -0.8 to 0.8. The coefficient $\phi_1^{(1)}$ remains fixed as in the previously mentioned article[2]. It is important to mention that in the case in which the coefficient $\phi_1^{(2)}$ takes the

[1]In general these values are selected so that estimator obtained with them has a relative asymptotic efficiency in the Gaussian case of 95% y 99.5% respectively.

[2]Hung et al. (2009) suggests that the results of the simulations for the non-linearity test (in the case *SETAR*) are not very sensitive to the choice of the autoregressive coefficient in the first regime.

value of 0.5 the model is reduced to a first-order autoregressive linear model. In Hung et al. (2009) was specified that some of the parameter combinations for the simulation experiment were taken from Tsay (1989). Other combinations were chosen such that there are adequate observations in both regimes for efficient parameter estimation and non-linearity testing.

4.1. Main Results of Non-linearity Test

Table 2 ($T = 100$) and Table 3 ($T = 200$) show the relative empirical frequencies of rejection of the null hypothesis of linearity at significance levels of 10%, 5%, and 1% based on 1000 replications with a contamination percentage of outliers $\beta = 5\%$.

The row in bold shows the empirical size of the test when there are no outliers ($\omega = 0$); this is because when all coefficients are equal, the model is, in fact, linear. We can observe the following: for the function $\psi : LS$ (least squares), the empirical size is 0.011 at a significance level of 1%, 0.054 at a level of 5%, and 0.110 to 10% (as expected). The following three columns show the empirical size for the function $\psi : Huber$ at the same three levels of significance with parameter $k = 1.345$. The following columns show the respective results for different choices of function ψ. In all cases, the empirical sizes coincide with the nominal ones. When all the coefficients are equal, but the parameter ω is different from 0, the results of Table 2 are the empirical size of the test in the presence of outliers. In this case, when the magnitude of the outliers is greater ($\omega = 10$), the empirical size of the test obtained using the least-squares function (the original proposal of Tsay (1989)) is 0.127, 0.236, and 0.298, that is, it becomes higher than the corresponding nominal levels ($10\%, 5\%$, and 1%). On the other hand, when the function $\psi(\cdot)$ used corresponds to the *Huber* (or *Tukey*), the empirical sizes always remain close to the nominal, regardless of the presence of outlier observations ($\omega =$ 3,6,10). As can be seen in Tables 2 and 3, this result is maintained when increasing the sample size from $T = 100$ to $T = 200$.

On the other hand, when the parameter $\phi_1^{(2)}$ takes values other than 0.5, the Table 2 presents the power of the non-linearity test. As this coefficient varies and the value of ω is fixed at 0, the power of the test is obtained when there are no outliers, for example, for the least squares function the empirical power at a significance level of 1% is 0.978, 0.808, 0.450, 0.213 in the parameter values $\phi_1^{(2)} = -0.\ 8$, -0.5, -0.2 and 0 respectively. Now, if ω is set to a value other than 0 and the parameter $\phi_1^{(2)}$ is varied, the power of the non-linearity test is obtained under the presence of outlier observations (remember that ω is related to the magnitude of these, a greater value of ω generates outliers of greater magnitude); by way of illustration, it can be seen in the table that the power of the test with choice of function $\psi : Huber$ and parameter $k = 1,345$ at a significance level of 1% and $\omega = 10$ is $0.840, 0.506$, 0.215, and 0.101 for the values of parameters $\phi_1^{(2)} = -0.8, -0.5, -0.2$, and 0 respectively. The following columns show the power for different choices of function ψ.

Table 2: Empirical relative frequencies of rejection of the linearity hypothesis based on 1000 replications with a model $TAR(Z,2,1,1,0)$, sample size equal to 100.

						LS			Huber $k=1.345$			Huber $k=3.291$			Tukey $c=4.685$			Tukey $c=15$		
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\alpha=$	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
0.5	0.5	0.5	-0.8	0		0.978	0.996	0.999	0.903	0.970	0.988	0.970	0.998	0.999	0.817	0.918	0.955	0.964	0.996	0.999
				3		0.864	0.938	0.965	0.862	0.939	0.973	0.900	0.962	0.982	0.757	0.888	0.930	0.891	0.962	0.982
				6		0.595	0.753	0.811	0.833	0.931	0.970	0.817	0.923	0.958	0.750	0.895	0.943	0.770	0.901	0.942
				10		0.395	0.551	0.644	0.840	0.940	0.965	0.757	0.895	0.938	0.758	0.899	0.937	0.727	0.884	0.926
0.5	0.5	0.5	-0.5	0		0.808	0.932	0.954	0.655	0.837	0.899	0.799	0.911	0.949	0.528	0.743	0.826	0.791	0.912	0.949
				3		0.602	0.768	0.848	0.574	0.766	0.854	0.598	0.794	0.862	0.462	0.678	0.778	0.597	0.786	0.859
				6		0.370	0.574	0.668	0.509	0.756	0.834	0.478	0.710	0.787	0.438	0.674	0.777	0.439	0.689	0.773
				10		0.280	0.447	0.525	0.506	0.750	0.829	0.403	0.675	0.784	0.464	0.675	0.778	0.384	0.642	0.776
0.5	0.5	0.5	-0.2	0		0.450	0.695	0.798	0.325	0.566	0.681	0.416	0.666	0.785	0.262	0.484	0.603	0.418	0.665	0.780
				3		0.296	0.522	0.647	0.256	0.485	0.623	0.255	0.498	0.629	0.218	0.424	0.547	0.264	0.511	0.631
				6		0.246	0.416	0.513	0.241	0.475	0.602	0.193	0.420	0.553	0.225	0.449	0.560	0.167	0.385	0.521
				10		0.208	0.366	0.450	0.215	0.446	0.578	0.163	0.360	0.481	0.214	0.436	0.549	0.152	0.343	0.463
0.5	0.5	0.5	0	0		0.213	0.408	0.544	0.150	0.343	0.492	0.183	0.406	0.533	0.131	0.309	0.431	0.188	0.405	0.529
				3		0.170	0.344	0.467	0.116	0.298	0.451	0.119	0.299	0.429	0.109	0.262	0.387	0.126	0.306	0.432
				6		0.155	0.304	0.419	0.105	0.280	0.400	0.088	0.245	0.345	0.106	0.257	0.382	0.078	0.233	0.334
				10		0.172	0.295	0.385	0.101	0.276	0.406	0.067	0.215	0.307	0.115	0.273	0.375	0.065	0.185	0.297
0.5	0.5	0.5	0.2	0		0.070	0.201	0.313	0.053	0.169	0.251	0.062	0.186	0.292	0.060	0.173	0.255	0.066	0.190	0.295
				3		0.080	0.208	0.301	0.048	0.147	0.232	0.046	0.143	0.225	0.048	0.156	0.240	0.049	0.139	0.246
				6		0.120	0.246	0.341	0.044	0.138	0.218	0.033	0.120	0.201	0.049	0.146	0.211	0.041	0.113	0.189
				10		0.148	0.266	0.357	0.046	0.130	0.218	0.028	0.106	0.176	0.047	0.161	0.224	0.030	0.097	0.165
0.5	0.5	0.5	0.5	0		0.011	0.054	0.110	0.011	0.054	0.100	0.013	0.048	0.098	0.010	0.067	0.125	0.014	0.049	0.103
				3		0.039	0.100	0.158	0.012	0.054	0.100	0.014	0.053	0.098	0.015	0.054	0.119	0.011	0.052	0.097
				6		0.057	0.155	0.231	0.009	0.048	0.089	0.010	0.059	0.108	0.013	0.060	0.111	0.014	0.049	0.088
				10		0.127	0.236	0.298	0.010	0.046	0.083	0.014	0.056	0.091	0.013	0.060	0.100	0.018	0.047	0.074
0.5	0.5	0.5	0.8	0		0.236	0.457	0.591	0.170	0.366	0.501	0.206	0.424	0.563	0.146	0.325	0.436	0.204	0.431	0.560
				3		0.148	0.340	0.465	0.138	0.336	0.461	0.164	0.368	0.480	0.113	0.288	0.411	0.148	0.345	0.467
				6		0.144	0.268	0.372	0.127	0.292	0.403	0.129	0.268	0.379	0.119	0.287	0.389	0.100	0.233	0.337
				10		0.157	0.263	0.346	0.108	0.272	0.389	0.086	0.231	0.336	0.125	0.277	0.388	0.065	0.194	0.297

This table presents the percentages of rejection of the non-linearity hypothesis when using the Hung et al. (2009) test adapted to the case of a TAR model. In the equation (13) the model is presented, the value of ω is related to the magnitude of the outliers, and its definition is presented in the lower part of the equation (5). The value of the ν parameter is equal to 0.5 and the sample size is $T=100$.

TABLE 3: Empirical relative frequencies of rejection of the linearity hypothesis based on 1000 replications with a model $TAR(z, 2, 1, 1, 0)$, sample size equals to 200.

						LS			Huber $k = 1.345$			Huber $k = 3.291$			Tukey $c = 4.685$			Tukey $c = 15$		
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\alpha =$	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
0.5	0.5	0.5	-0.8	0		1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.992	0.998	0.999	1.000	1.000	1.000
				3		0.994	0.998	0.999	0.995	0.999	1.000	0.999	1.000	1.000	0.978	0.995	0.997	0.997	0.999	1.000
				6		0.847	0.924	0.952	0.996	0.998	0.999	0.993	0.999	1.000	0.981	0.995	0.998	0.986	1.000	1.000
				10		0.488	0.660	0.748	0.994	1.000	1.000	0.985	0.997	1.000	0.985	0.996	0.998	0.974	0.996	0.998
0.5	0.5	0.5	-0.5	0		0.994	0.999	1.000	0.965	0.996	0.998	0.995	0.999	0.999	0.896	0.969	0.983	0.995	0.998	0.999
				3		0.906	0.964	0.977	0.928	0.980	0.994	0.943	0.979	0.991	0.844	0.946	0.978	0.932	0.984	0.991
				6		0.593	0.772	0.837	0.919	0.984	0.992	0.878	0.956	0.982	0.849	0.951	0.973	0.862	0.955	0.976
				10		0.334	0.493	0.592	0.906	0.971	0.991	0.828	0.926	0.961	0.864	0.956	0.978	0.805	0.922	0.956
0.5	0.5	0.5	-0.2	0		0.861	0.949	0.971	0.702	0.879	0.939	0.832	0.940	0.973	0.584	0.778	0.857	0.838	0.944	0.975
				3		0.634	0.807	0.867	0.606	0.830	0.906	0.628	0.819	0.895	0.499	0.736	0.837	0.633	0.832	0.896
				6		0.376	0.590	0.691	0.578	0.813	0.881	0.510	0.744	0.831	0.501	0.712	0.825	0.491	0.728	0.832
				10		0.217	0.382	0.463	0.559	0.762	0.854	0.434	0.670	0.767	0.518	0.740	0.825	0.409	0.665	0.778
0.5	0.5	0.5	0	0		0.532	0.754	0.850	0.392	0.620	0.745	0.518	0.736	0.836	0.316	0.532	0.644	0.503	0.733	0.839
				3		0.380	0.599	0.703	0.334	0.556	0.675	0.339	0.551	0.677	0.274	0.474	0.594	0.356	0.577	0.691
				6		0.254	0.421	0.521	0.312	0.519	0.643	0.234	0.445	0.570	0.266	0.471	0.598	0.215	0.430	0.549
				10		0.174	0.311	0.408	0.285	0.522	0.661	0.201	0.403	0.547	0.289	0.482	0.608	0.177	0.380	0.501
0.5	0.5	0.5	0.2	0		0.199	0.378	0.522	0.147	0.319	0.428	0.184	0.370	0.494	0.132	0.294	0.396	0.179	0.374	0.505
				3		0.160	0.328	0.455	0.110	0.273	0.373	0.090	0.251	0.373	0.107	0.250	0.355	0.095	0.259	0.383
				6		0.161	0.298	0.406	0.111	0.256	0.369	0.082	0.200	0.299	0.114	0.254	0.356	0.066	0.211	0.301
				10		0.152	0.267	0.334	0.107	0.269	0.364	0.051	0.184	0.271	0.129	0.269	0.374	0.050	0.167	0.268
0.5	0.5	0.5	0.5	0		0.010	0.059	0.109	0.012	0.061	0.115	0.014	0.051	0.103	0.017	0.078	0.135	0.012	0.056	0.101
				3		0.027	0.096	0.154	0.006	0.051	0.112	0.013	0.058	0.122	0.013	0.064	0.121	0.011	0.059	0.116
				6		0.076	0.187	0.264	0.011	0.053	0.113	0.015	0.052	0.102	0.014	0.079	0.124	0.012	0.041	0.087
				10		0.091	0.179	0.249	0.009	0.053	0.094	0.012	0.050	0.097	0.015	0.067	0.123	0.011	0.033	0.072
0.5	0.5	0.5	0.8	0		0.573	0.774	0.847	0.434	0.656	0.755	0.540	0.752	0.832	0.348	0.557	0.680	0.542	0.749	0.834
				3		0.380	0.595	0.700	0.357	0.606	0.716	0.418	0.666	0.773	0.293	0.516	0.636	0.397	0.629	0.759
				6		0.215	0.382	0.501	0.325	0.553	0.667	0.310	0.548	0.660	0.296	0.507	0.646	0.261	0.475	0.594
				10		0.167	0.305	0.403	0.319	0.559	0.689	0.246	0.478	0.611	0.294	0.542	0.648	0.195	0.421	0.562

This table presents the percentages of rejection of the non-linearity hypothesis when using the Hung et al. (2009) test adapted to the case of a TAR model. In the equation (13) the model is presented, the value of ω is related to the magnitude of the outliers and its definition is presented in the lower part of the equation (5). The value of the ν parameter is equal to 0.5 and the sample size is $T = 200$.

It is worth highlighting some results obtained, for example, in Table 3 for a sample size $T = 200$ and in the case $\omega = 3$. It is observed that the empirical levels for the test (corresponding to the nominals of 1%, 5%, and 10%) using the *LS* function are $(0.076, 0.0187, 0.264)$, while those of the *Huber* function with parameter $k = 1.345$ are $(0.011, 0.053, 0.113)$. This result shows the size problems that additive outlier observations can cause when the least-squares function is used and shows that this problem does not disappear with the increase in the sample size (see Table 2). What we found goes in hand with that pointed out by Hung et al. (2009) for the case of *SETAR* models and verifies the advantages of the test in the presence of outlier observations.

It is observed that the introduction of outlier data ($\omega \neq 0$) slightly decreases the power of the test; however, this effect can be controlled by taking a larger sample size. For example, for the function *Tukey* with parameter $c = 4.685$, $\phi_1^{(2)} = -0.5$, a nominal size of 5% and a sample $T = 100$ the power is (0.743, 0.678, 0.674, 0.675) (see Table 2) for the values of $\omega = 0$, 3,6,10 respectively. On the other hand, when taking a sample of size $T = 200$, the power increases to (0.969, 0.946, 0.951, 0.956).

Regarding the choice of the $\psi(\cdot)$ function and its respective parameters, the results of the simulations in Tables 2 and 3 show that the *Huber* function with parameter $k = 1.345$ generally achieves higher power levels than the *Tukey* function with $c = 4.685$[3]. Similarly, when this function is compared with $k = 3.291$ and that of *Tukey* with $c = 15$ the former seems to give slightly better results. Additionally, unlike what Hung et al. (2009) found, there does not seem to be strong evidence to prefer the use of the *Huber* ($k = 3.291$) and *Tukey* ($c = 15$) to those that use smaller values of to those that k and c parameters.

Additional Monte Carlo Simulation About Non-Linearity Test

In Appendix A, we can find complementary simulation results about the non-linearity test. We consider three scenarios: (1) different values of autoregressive coefficient for the threshold variable $\nu = -0.5$[4]; (2) different percentages of outlier observations $\beta = 10\%$[5], and (3) different percentage of observation in each regime, the threshold value controls this. Each scenario is evaluated for sample size $T = 200$[6]. Scenarios (2) and (3)were not taking into account in Hung et al. (2009). For the scenario (1), we can observe in Table A1 similar results to that obtained when the autoregressive parameter is $\nu = 0.5$ for the non-linearity test, that is, it seems the autoregressive parameter does not affect the results for the

[3]This comparison is valid as long as these parameters guarantee a relative asymptotic efficiency of the estimator in the Gaussian case of 95%.

[4]Results for coefficients $\nu = \pm 0.8$ were not reported in the appendix for the Journal guidelines, however they are available for the authors.

[5]Results for $\beta = 1\%$ were not reported in the appendix for the Journal guidelines, however they are available for the authors.

[6]Results for sample size $T = 100$ were not reported in the appendix for the Journal guidelines, however they are available for the authors.

non-linearity test. For scenario (2), the results of the non-linearity test simulations for different percentages of contamination of outliers based on Table A2 show that if these types of observations are present, the Huber and Tukey functions are better than least-squares since they do not suffer from size problems. If the percentage of contamination is high, then the power is decreased, and it is suggested to use these same functions with low k and c values, while if the percentage is low, the power is kept, and the values of k and c should be high. Huber's function appears to perform better on average than Tukey's. For scenario (3), the main conclusion derived from the results obtained is that the difference in the percentages of observations in each regime generates a significant decrease in the power of the non-linearity test. However, this problem improves with an increase in the sample size see Table A3. In general, we can observe similar results to those obtained in Hung et al. (2009) for SETAR models. However, we can observe a slight increase in the power of robust non-linearity test for the open-loop TAR models compared with the test for SETAR models, when the parameter $\phi_1^{(2)}$ moves from 0.5 to positive or zero values, even when ω is increased.

4.2. Main Results about Estimation of the Autoregressive coefficients

The relative performance of the proposed Robust estimator with respect to the least-squares one is evaluated. First, the mean square error ratio between the estimator GM with function choices ψ : *Huber*, *Tukey* is reported (parameters $k = 1.345$, $c = 4.685$) against the least squares estimator (that is $\psi : LS$)[7]. Second, we obtain the absolute average bias for the estimations is studied. We initially consider a contamination percentage of outlier data of $\beta = 5\%$.

Table 4 presents the mean square error ratios between the GM estimator and the least-squares estimator.On the other hand, Table 5 shows the average biases in absolute value. All results are based on 1000 simulations and for samples of size $T = 100$. Tables 6 and 7 show comparable results for samples of size $T = 200$.

In the case that $\omega = 0$ (that is, when there are no outliers), the estimators obtained through least-squares are better (in terms of mean square error) than those of the GM method for both a sample size $T = 100$ as in the case $T = 200$. For example, for the parameter $\phi_1^{(1)}$, the root of the mean square error ratio almost always exceeds the value of 1.2. On the other hand, under outlier observations, the estimator based on the GM method has a better performance than the least-squares one. In fact, in most cases where outliers exist, the root of the mean square error ratios choosing the function ψ : *Huber* is less than 1. For example, the ratios for the model $\phi_1^{(2)} = -0.8$ with the highest magnitude outliers ($\omega = 10$), and a sample size of $T = 200$ (see Table 6), they become as low as 0.095 and 0.037 for the autoregressive coefficients of each regimen. As the contamination parameter ω increases, the results further favor the estimator based on the GM method.

[7] That is, the root mean square error based on the estimator GM is located in the numerator while the least squares in the denominator.

Table 4: Mean square error ratio between the *GM* and the classical estimator based on 1000 replications with a TAR model $(Z, 2, 1, 1, 0)$, sample size equal to 100, $\nu = 0.5$.

					Huber $k = 1.345$				Tukey $c = 4.685$			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	1.297	1.554	1.176	1.410	1.877	3.071	1.464	2.558
				3	0.917	0.793	0.719	0.485	1.227	1.430	0.870	0.796
				6	0.434	0.283	0.326	0.140	0.563	0.485	0.367	0.207
				10	0.269	0.208	0.195	0.080	0.327	0.311	0.223	0.118
0.5	0.5	0.5	-0.5	0	1.274	1.504	1.179	1.420	1.823	2.782	1.463	2.501
				3	0.877	0.703	0.801	0.772	1.182	1.210	0.963	1.313
				6	0.449	0.335	0.409	0.310	0.608	0.537	0.476	0.479
				10	0.286	0.232	0.222	0.221	0.337	0.351	0.248	0.350
0.5	0.5	0.5	-0.2	0	1.231	1.386	1.154	1.384	1.698	2.580	1.486	2.574
				3	0.806	0.658	0.948	1.119	1.044	1.072	1.236	2.109
				6	0.365	0.298	0.502	0.851	0.463	0.478	0.632	1.523
				10	0.254	0.215	0.309	0.653	0.309	0.340	0.380	1.213
0.5	0.5	0.5	0	0	1.215	1.369	1.245	1.484	1.716	2.546	1.689	2.794
				3	0.799	0.676	1.055	1.377	1.076	1.102	1.469	2.785
				6	0.381	0.313	0.620	1.432	0.462	0.505	0.811	2.761
				10	0.241	0.212	0.340	1.124	0.294	0.327	0.434	2.087
0.5	0.5	0.5	0.2	0	1.231	1.391	1.201	1.416	1.821	2.505	1.747	2.727
				3	0.809	0.723	0.922	1.050	1.090	1.151	1.281	2.011
				6	0.410	0.321	0.540	0.761	0.530	0.500	0.716	1.362
				10	0.255	0.229	0.347	0.684	0.317	0.345	0.468	1.236
0.5	0.5	0.5	0.5	0	1.259	1.427	1.276	1.435	1.886	2.649	2.050	2.830
				3	0.767	0.697	0.804	0.664	1.141	1.179	1.208	1.189
				6	0.423	0.344	0.374	0.342	0.529	0.511	0.508	0.545
				10	0.252	0.241	0.238	0.208	0.328	0.358	0.318	0.327
0.5	0.5	0.5	0.8	0	1.248	1.456	1.288	1.392	1.829	2.633	2.110	2.691
				3	0.745	0.716	0.511	0.433	0.989	1.145	0.701	0.674
				6	0.376	0.311	0.180	0.143	0.493	0.499	0.238	0.217
				10	0.225	0.204	0.104	0.086	0.271	0.299	0.145	0.132

This table presents the mean square error ratio obtained through simulations between the *GM* estimator and the least-squares estimator. In equation (13) the model is presented, the value of ω is related to the outliers' magnitude, and its definition is presented in the lower part of the equation (5). The value of the parameter ν is equal to 0.5, and the sample size is $T = 100$.

Table 5: The *GM* and classical estimators' absolute average bias based on 1000 replications with a TAR model$(Z, 2, 1, 1, 0)$, sample size equal to $100, \nu = 0.5$.

Huber $k = 1.345$					Tukey $c = 4.685$				LS							
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	0.01	0.02	0.00	0.00	0.01	0.03	0.00	0.00	0.00	0.02	0.01	0.01
				3	0.02	0.05	0.01	0.03	0.02	0.04	0.00	0.01	0.05	0.10	0.05	0.13
				6	0.02	0.05	0.02	0.04	0.02	0.04	0.00	0.01	0.12	0.21	0.12	0.29
				10	0.02	0.05	0.02	0.03	0.02	0.04	0.00	0.00	0.14	0.28	0.16	0.41
0.5	0.5	0.5	-0.5	0	0.02	0.03	0.00	0.01	0.03	0.03	0.01	0.01	0.01	0.02	0.00	0.00
				3	0.04	0.05	0.01	0.02	0.03	0.04	0.00	0.01	0.08	0.13	0.05	0.09
				6	0.04	0.05	0.01	0.02	0.03	0.04	0.00	0.00	0.14	0.23	0.10	0.20
				10	0.04	0.05	0.01	0.02	0.03	0.03	0.00	0.00	0.19	0.30	0.13	0.28
0.5	0.5	0.5	-0.2	0	0.01	0.02	0.01	0.01	0.01	0.02	0.01	0.01	0.01	0.02	0.00	0.01
				3	0.02	0.05	0.00	0.01	0.02	0.03	0.00	0.00	0.08	0.13	0.03	0.04
				6	0.03	0.05	0.00	0.00	0.01	0.02	0.01	0.01	0.17	0.24	0.04	0.07
				10	0.03	0.05	0.00	0.00	0.01	0.03	0.01	0.01	0.21	0.31	0.07	0.11
0.5	0.5	0.5	0	0	0.02	0.03	0.01	0.02	0.02	0.03	0.00	0.01	0.02	0.03	0.01	0.02
				3	0.04	0.05	0.00	0.02	0.02	0.03	0.01	0.02	0.10	0.14	0.01	0.01
				6	0.04	0.05	0.00	0.02	0.03	0.03	0.00	0.01	0.18	0.24	0.00	0.02
				10	0.03	0.05	0.01	0.02	0.02	0.03	0.01	0.02	0.23	0.31	0.00	0.02
0.5	0.5	0.5	0.2	0	0.03	0.03	0.02	0.02	0.03	0.03	0.01	0.02	0.03	0.03	0.02	0.02
				3	0.05	0.06	0.03	0.04	0.04	0.05	0.02	0.02	0.11	0.13	0.05	0.07
				6	0.05	0.06	0.03	0.04	0.04	0.04	0.02	0.02	0.21	0.25	0.09	0.11
				10	0.05	0.05	0.03	0.04	0.04	0.04	0.02	0.03	0.26	0.31	0.11	0.14
0.5	0.5	0.5	0.5	0	0.03	0.03	0.02	0.03	0.04	0.04	0.02	0.02	0.03	0.03	0.03	0.03
				3	0.06	0.06	0.05	0.06	0.05	0.05	0.04	0.04	0.13	0.13	0.13	0.13
				6	0.06	0.06	0.06	0.06	0.05	0.04	0.03	0.03	0.23	0.24	0.24	0.23
				10	0.07	0.07	0.05	0.05	0.05	0.05	0.03	0.04	0.31	0.32	0.30	0.31
0.5	0.5	0.5	0.8	0	0.05	0.03	0.06	0.03	0.05	0.03	0.06	0.04	0.05	0.03	0.06	0.03
				3	0.09	0.06	0.10	0.06	0.07	0.05	0.08	0.05	0.17	0.12	0.25	0.16
				6	0.08	0.06	0.10	0.06	0.06	0.04	0.08	0.05	0.29	0.21	0.51	0.32
				10	0.08	0.06	0.10	0.06	0.06	0.04	0.07	0.04	0.41	0.29	0.70	0.43

This table presents the absolute average biases for each of the parameters obtained using the GM estimator and the least-squares estimator. In equation (13), the model is presented, the value of ω is related to the magnitude of the outliers, and its definition is presented in the lower part of the equation (5). The value of the parameter ν is equal to 0.5, and the sample size is $T = 100$.

Table 6: Mean square error ratio between the *GM* and the classical estimator based on 1000 replications with a TAR model $(Z, 2, 1, 1, 0)$, sample size equal to 200, $\nu = 0.5$.

					Huber $k = 1.345$				Tukey $c = 4.685$			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	1.201	1.432	1.142	1.328	1.553	2.562	1.357	2.298
				3	0.795	0.539	0.643	0.285	0.956	0.798	0.735	0.401
				6	0.344	0.157	0.271	0.070	0.383	0.218	0.300	0.089
				10	0.211	0.095	0.141	0.037	0.231	0.137	0.152	0.048
0.5	0.5	0.5	-0.5	0	1.163	1.352	1.187	1.357	1.583	2.393	1.453	2.362
				3	0.695	0.479	0.745	0.449	0.879	0.732	0.884	0.699
				6	0.319	0.171	0.350	0.158	0.371	0.243	0.395	0.234
				10	0.191	0.111	0.216	0.099	0.234	0.157	0.234	0.146
0.5	0.5	0.5	-0.2	0	1.228	1.335	1.202	1.405	1.650	2.372	1.536	2.503
				3	0.721	0.492	0.853	1.023	0.887	0.738	1.068	1.900
				6	0.296	0.170	0.520	0.655	0.350	0.248	0.635	1.127
				10	0.194	0.112	0.300	0.459	0.221	0.154	0.363	0.789
0.5	0.5	0.5	0	0	1.198	1.332	1.154	1.379	1.614	2.257	1.479	2.510
				3	0.658	0.476	0.940	1.284	0.811	0.753	1.210	2.429
				6	0.285	0.178	0.588	1.384	0.333	0.247	0.706	2.450
				10	0.175	0.111	0.304	1.065	0.206	0.160	0.371	1.951
0.5	0.5	0.5	0.2	0	1.268	1.464	1.211	1.403	1.796	2.629	1.654	2.569
				3	0.615	0.484	0.879	0.935	0.780	0.750	1.182	1.682
				6	0.274	0.185	0.516	0.582	0.314	0.260	0.677	1.021
				10	0.170	0.116	0.286	0.413	0.196	0.165	0.356	0.677
0.5	0.5	0.5	0.5	0	1.252	1.405	1.290	1.384	1.782	2.328	1.918	2.430
				3	0.611	0.488	0.600	0.466	0.800	0.713	0.784	0.720
				6	0.245	0.173	0.247	0.163	0.304	0.240	0.324	0.236
				10	0.142	0.115	0.151	0.107	0.169	0.150	0.195	0.157
0.5	0.5	0.5	0.8	0	1.253	1.447	1.302	1.463	1.844	2.660	1.950	2.583
				3	0.615	0.525	0.396	0.318	0.848	0.848	0.495	0.435
				6	0.219	0.173	0.116	0.083	0.280	0.255	0.135	0.109
				10	0.126	0.096	0.064	0.045	0.156	0.143	0.077	0.061

This table presents the mean square error ratio obtained through simulations between the *GM* estimator and the least-squares estimator. In equation (13) the model is presented, the value of ω is related to the outliers' magnitude, and its definition is presented in the lower part of the equation (5). The value of the parameter ν is equal to 0.5, and the sample size is $T = 200$.

TABLE 7: The *GM* and classical estimators' absolute average bias based on 1000 replications with a TAR model$(Z, 2, 1, 1, 0)$, sample size equal to $200, \nu = 0.5$.

					Huber $k = 1.345$				Tukey $c = 4.685$				LS			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	0.01	0.01	0.00	0.01	0.01	0.00	0.00	0.01	0.01	0.01	0.01	0.01
				3	0.03	0.03	0.02	0.03	0.02	0.02	0.01	0.02	0.06	0.10	0.06	0.13
				6	0.03	0.03	0.02	0.04	0.02	0.01	0.01	0.02	0.14	0.22	0.13	0.32
				10	0.03	0.03	0.01	0.03	0.02	0.01	0.01	0.01	0.19	0.30	0.18	0.46
0.5	0.5	0.5	-0.5	0	0.01	0.02	0.00	0.00	0.01	0.02	0.00	0.01	0.01	0.02	0.00	0.00
				3	0.03	0.05	0.01	0.03	0.02	0.03	0.00	0.00	0.08	0.13	0.05	0.11
				6	0.03	0.04	0.01	0.02	0.02	0.02	0.00	0.00	0.16	0.25	0.10	0.24
				10	0.03	0.04	0.02	0.02	0.02	0.02	0.00	0.00	0.21	0.33	0.14	0.32
0.5	0.5	0.5	-0.2	0	0.01	0.02	0.00	0.01	0.02	0.02	0.01	0.01	0.01	0.02	0.00	0.00
				3	0.03	0.05	0.00	0.01	0.02	0.03	0.00	0.00	0.09	0.13	0.02	0.05
				6	0.04	0.05	0.01	0.01	0.02	0.03	0.00	0.01	0.19	0.27	0.06	0.10
				10	0.03	0.05	0.01	0.01	0.02	0.03	0.00	0.01	0.23	0.35	0.06	0.13
0.5	0.5	0.5	0	0	0.01	0.02	0.00	0.00	0.01	0.02	0.00	0.00	0.01	0.02	0.00	0.00
				3	0.03	0.05	0.00	0.00	0.02	0.03	0.00	0.00	0.10	0.14	0.00	0.00
				6	0.04	0.04	0.00	0.00	0.02	0.02	0.00	0.00	0.20	0.27	0.00	0.00
				10	0.03	0.04	0.00	0.01	0.02	0.02	0.00	0.00	0.26	0.34	0.02	0.01
0.5	0.5	0.5	0.2	0	0.02	0.02	0.00	0.01	0.02	0.02	0.00	0.01	0.02	0.02	0.01	0.01
				3	0.04	0.05	0.01	0.02	0.03	0.03	0.01	0.02	0.12	0.14	0.04	0.06
				6	0.04	0.05	0.02	0.03	0.02	0.03	0.01	0.01	0.22	0.26	0.08	0.11
				10	0.04	0.04	0.01	0.03	0.03	0.02	0.00	0.01	0.29	0.35	0.10	0.15
0.5	0.5	0.5	0.5	0	0.01	0.02	0.02	0.02	0.00	0.02	0.02	0.02	0.01	0.02	0.02	0.01
				3	0.04	0.05	0.05	0.05	0.02	0.03	0.04	0.03	0.12	0.13	0.14	0.13
				6	0.03	0.05	0.05	0.05	0.01	0.02	0.03	0.03	0.25	0.26	0.27	0.27
				10	0.03	0.04	0.05	0.04	0.01	0.02	0.03	0.02	0.34	0.34	0.35	0.34
0.5	0.5	0.5	0.8	0	0.00	0.01	0.04	0.02	0.00	0.01	0.05	0.02	0.01	0.01	0.04	0.02
				3	0.04	0.03	0.09	0.05	0.02	0.02	0.07	0.04	0.13	0.10	0.25	0.16
				6	0.04	0.04	0.09	0.05	0.02	0.02	0.06	0.03	0.30	0.23	0.54	0.34
				10	0.04	0.04	0.08	0.05	0.01	0.02	0.05	0.03	0.41	0.31	0.74	0.47

This table presents the absolute average biases for each of the parameters obtained using the GM estimator and the least-squares estimator. In equation (13), the model is presented, the value of ω is related to the magnitude of the outliers, and its definition is presented in the lower part of the equation (5). The value of the parameter ν is equal to 0.5, and the sample size is $T = 200$.

A comparison of the ratios obtained using the function ψ : *Huber* and that of *Tukey* shows that the first choice (*Huber*) seems to perform better for estimating the models' of the autoregressive coefficients. This observation is particularly relevant for the case where there are no outliers. For example, the ratios for a sample size $T = 100$ (see Table 4), the model $\phi_1^{(2)} = -0.5$

are 1.274, $1.504, 1.179, 1.420$ for the *Huber* function while with *Tukey* it is $1.823, 2.782, 1.463, 2.501$.

Regarding the absolute average bias presented in Tables 5 and 7, both estimators (*GM* and *LS*) have a bias very small in the case that there are no outliers. As the magnitude of these observations increases, the bias of the least squares estimator also increases. A case that draws attention is presented in Table 5 for the model $\phi_1^{(2)} = 0.8$ and $\omega = 10$, where absolute average bias of the least-squares estimator reaches 0.7 for the intercept of the second regime and 0.43 for its respective autoregressive coefficient.

The *GM* estimators obtained for the two functions $\psi(\cdot)$ considered have relatively small absolute average bias magnitudes, and in general, these seem to improve with increasing sample size (see Table 7). Unlike the results obtained for the root mean square error ratios, Tables 5 and 7 show that, in general, the magnitude of the bias of the estimator *GM* used by the *Huber* function is greater than the one based on *Tukey*. This suggests that the estimator based on the first has a lower variance in the simulations (so that its mean square error may be smaller as seen in Tables 4 and 6).

Additional Monte Carlo Simulation for the Estimation of the Autoregressive Coefficients

In Appendix B, we can find any other results about parameter estimation. We consider five scenarios: (1) different value of the autoregressive coefficient for the threshold variable ν[8]; (2) different percentage of outlier observations[9], (3) different percentage of observations in each regime, (4) empirical distribution of the GM estimator of the autoregressive coefficients for finite samples, and (5) Percentage of empirical coverage of the finite sample confidence intervals for the autoregressive coefficients. Each scenario is evaluated for sample size $T = 200$[10]. For scenario (1), we can observe in Table B1, similar results to that obtained when the autoregressive parameter is $\nu = -0.5, \pm 0.8$ for the ratio between the mean square error of the *GM* and least-squares estimators, that is, it seems the autoregressive parameter does not affect the results for the estimation performance. For scenario (2), based on Table B2, we can observe similar results to that obtained for the contamination percentage of 5%. In the case of scenario (3), based on Table B3 we can see similar results to those obtained previously when the number of observations in each regime is approximately the same. This corroborates the evidence regarding the improvement obtained (in terms of mean square error) when using the GM estimator instead of the least-squares in the presence of outlier observations, especially when the magnitude of these is large $\omega = 10$. Additionally, the decrease in the number of observations in the second

[8]Results for coefficients $\nu = \pm 0.8$ were not reported in the appendix for Journal guidelines, however they are available for the authors.

[9]Results for $\beta = 1\%$ were not reported in the appendix for the Journal guidelines, however they are available for the authors.

[10]Results for sample size $T = 100$ were not reported in the appendix for the Journal guidelines, however they are available for the authors.

regime worsens the relative performance of the estimator of the coefficients $\phi_0^{(2)}$ and $\phi_1^{(2)}$. In scenario (4), we used Tables B4 and B5 to explore the estimator *GM*'s distribution because the asymptotic normal distribution is guaranteed by Theorem 1 in Hung et al. (2009) for open-loop TAR models. These tables give us the p-value for the univariate Shapiro-Wilk normality test for the distribution of individual *GM* estimator when the sample size is $T = 100$ and $T = 1000$, based on 1000 replications. We can observe that by increasing the sample size, it is possible to approximate the distribution of the *GM* estimator by a normal distribution, in special if the magnitude of the additive outliers is not large. Finally, in scenario (5), we evaluate the coverage percentage of the 95% confidence intervals for the autoregressive parameters of the open-loop TAR model based on the normal distribution, see Table B6. The results show that the empirical coverage percentages are relatively close to the nominal 95%; however, in the vast majority of cases, the empirical levels are lower than the nominal for both functions ψ : Huber and Tukey. This problem seems to improve with increasing sample size. All results about empirical coverage of confidence intervals were possible using the covariance matrix estimator in (8).

Comparisons with the work of Chan & Cheung (1994) are not straightforward; however, the results are analogous because the performance of the GM estimator is better in the presence of outliers. That is, the mean square error ratio between the estimator *GM* and the *LS* tends to decrease in the presence of outliers, and the size of the outlier ω is large, while the ratio tends to decrease when the size of outlier is small or zero. It is important to point out that in Chan & Cheung (1994) is not studied the empirical distributions of the *GM* estimators for the autoregressive parameters in finite samples, neither the empirical coverage of the confidence intervals.

5. Real Data Application

This section presents two real data applications of the non-linearity test and the estimation procedure proposed to study the behavior of the returns of the nominal exchange rate COP/USD. The sample under study is relevant to the extent that the world economic crisis of 2009 generated extreme depreciation of the peso against the dollar Vargas (2011). These movements were widely reflected in the returns; therefore, it could be thought that there are outliers in the observed time series. The weekly returns of the nominal exchange rate are calculated based on daily information taken directly from the Banco de la República website. The sample begins in the first week of 2009 and ends in the last week of 2018; there are 520 observations weekly.

In the first example, we can observe the impact of the outlier in the non-linearity test. The second example is comparable with the real data example BLOWFLY in Hung et al. (2009); the classical non-robust and robust tests gave same results, that is, reject the null hypothesis.

First Example

Analogously to what was proposed in Franses et al. (2000, p. 88) for the use of the *TAR* model, it is considered that the variable determining the regime change is a measure of volatility similar to that proposed by LeBaron (1992) calculated as the median of the absolute value of returns in the last four weeks(previous month)[11]. This measure has been widely used in finance as an approximation to the volatility of exchange rate returns.

Th interest variable Ret_t is:

$$Ret_t = \frac{FX_{7t}}{FX_{1t}} - 1, \tag{16}$$

where FX_{7t} y FX_{1t} represent the nominal exchange rate of the last/first day of the t-th week.

On the other hand, the determining variable of the thresholds is (note that in this case, the value of delay d is equal to 0):

$$Z_t = med(|Ret_{t-1}|, |Ret_{t-2}|, |Ret_{t-3}|, |Ret_{t-4}|). \tag{17}$$

We can observe the plot of two-time series in 1.

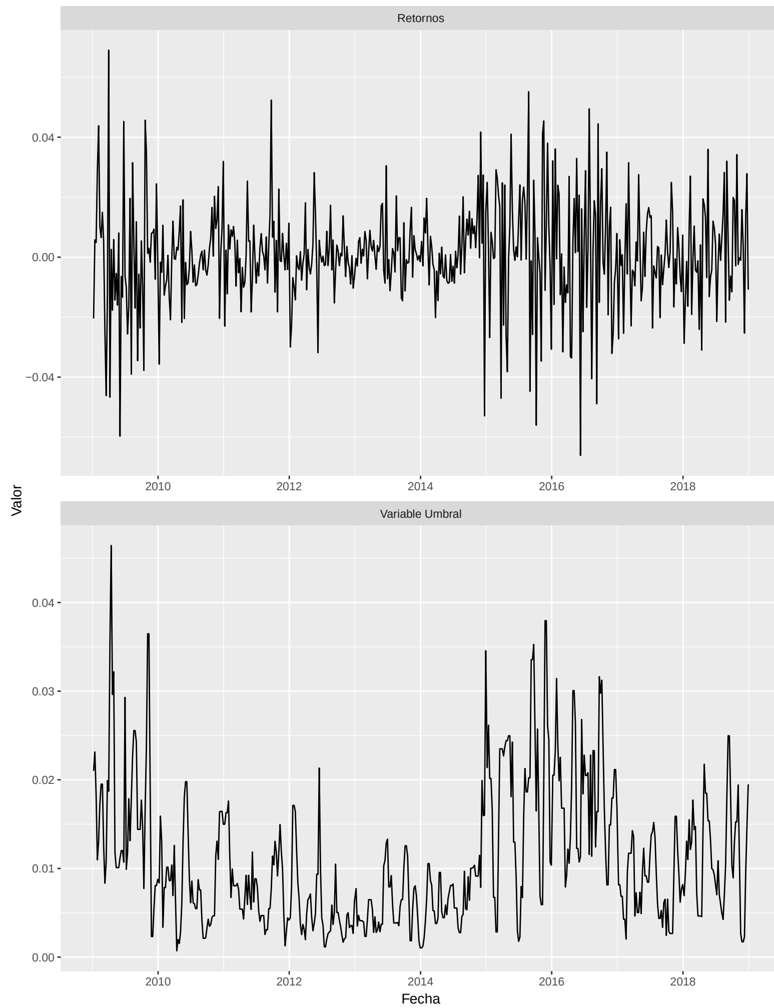
TABLE 8: P-values of the non-linearity test for different orders p of the autoregressive process using volatility measure as a transition variable.

p	LS	Huber $k = 1.345$	Huber $k = 3.291$	Tukey $c = 4.685$	Tukey $c = 15$
1	0.05	0.11	0.03	0.27	0.02
2	0.09	0.11	0.07	0.31	0.08

This table presents the p-values of the non-linearity test proposed by Hung et al. (2009) adapted to the case of open-loop *TAR* models for different values of autoregressive orders. The time series considered is the series of weekly exchange rate returns COP/USD in 2009-2019. The variable used to determine the thresholds is the median of the absolute values of the returns in the last four weeks.

[11] The measure proposed by LeBaron (1992) is the average in the last j weeks, with j a known number.

Figure 1: The behavior of the weekly returns of the COP/USD exchange rate and the volatility measure used as a determinant of the regime change.



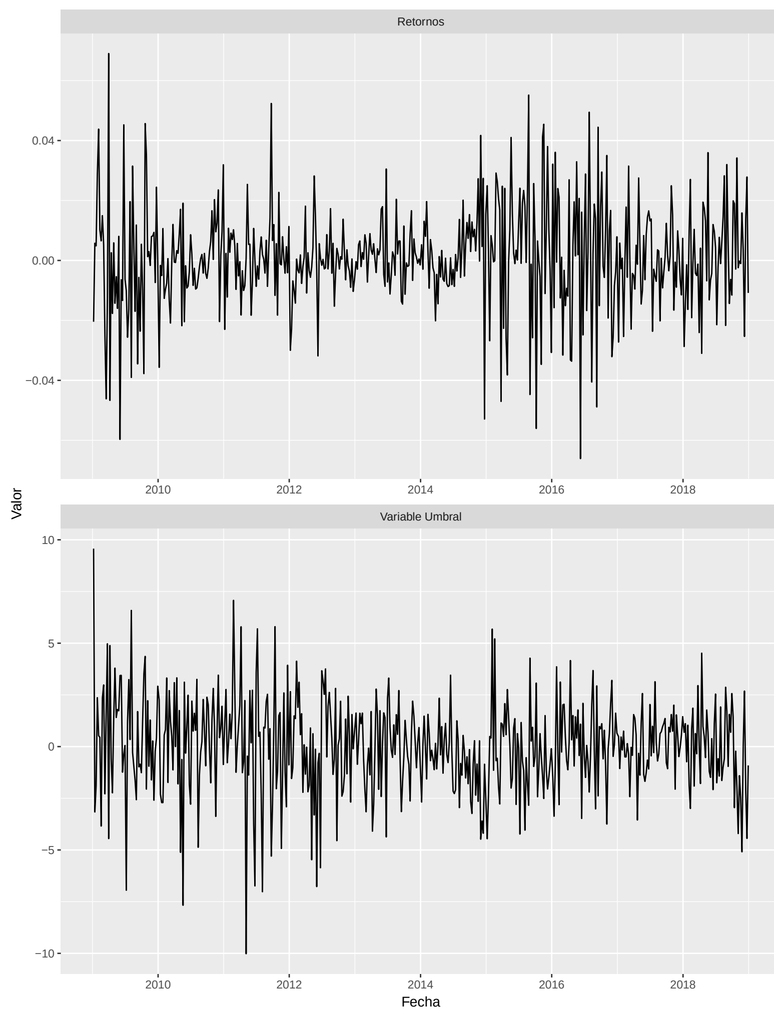
The upper panel presents the weekly returns of the COP/USD exchange rate from the first week of 2009 to the last week of 2018. The lower panel presents the median of the absolute value of the returns in the four weeks before each week.

The results of the non-linearity test are presented in Table 8. As it can be seen for values of the autoregressive order $p = 1, 2$ when using the *LS* function(that is, when the test is not robust to outliers), the null hypothesis of linearity is rejected in favor of the non-linearity type open-loop *TAR* at the level of significance of 10%. However, when testing with functions ψ : *Huber*, *Tukey* with parameters $k = 1.345$, $c = 4.685$, the null hypothesis is not rejected at this level significance. Therefore, we decide not to reject the null hypothesis of linearity.

Second Example

In a relatively recent study, Mohammadi & Jahan-Parvar (2012) find that there may be a relationship between real oil prices and real exchange rates for countries where there is a high production of this product. Since Colombia is a country where this condition is met, the non-linearity test is carried out, but considering that the threshold variable is the change in the actual weekly price of a barrel of Brent oil. The two-time series are shown in Figure 2.

FIGURE 2: Behavior of the weekly returns of the COP/USD exchange rate and the change in the real price of a barrel of Brent oil measured in USD



The upper panel presents the weekly returns of the COP/USD exchange rate from the first week of 2009 to the last week of 2018. The lower panel shows the change in the price of a barrel of Brent oil measured in dollars.

The non-linearity test results for different values of the autoregressive order are presented in Table 9. As it can be seen, it is found that for all lags and using all the $\psi(\cdot)$ functions, there is statistical evidence to reject the null hypothesis of linearity in favor of the TAR-type non-linearity hypothesis. Based on this result, we proceed to estimate the TAR model parameters with two regimes using the function $\psi(\cdot)$: Huber with parameter k equal to 1.3454. In order to identify the autoregressive orders, the estimation procedure is carried out for all the possible combinations $p_1 = 1, \ldots, 5$, and $p_2 = 1, \ldots, 5$. The best model is chosen through the criterion of Akaike information

$$AIC(p_1, p_2) = n_1 \ln(\hat{\sigma}_1^2) + n_2 \ln(\hat{\sigma}_2^2) + 2(p_1 + 1) + 2(p_2 + 1). \tag{18}$$

The optimal value of the threshold r is obtained by carrying out a procedure similar to that specified in (10) once the values of p_1 and p_2 are known. It is assumed that the number of regimens is 2; in empirical applications, it is suggested to carry out this process for a different number of regimens and compare the results obtained.

Table 9: P-values of the non-linearity test for different orders p of the autoregressive process using as threshold variable the changes in the Brent oil price.

p	LS	Huber $k = 1.345$	Huber $k = 3.291$	Tukey $c = 4.685$	Tukey $c = 15$
1	6.7×10^{-16}	8.0×10^{-11}	3.9×10^{-13}	1.2×10^{-7}	5.1×10^{-13}
2	1.1×10^{-15}	1.7×10^{-8}	1.7×10^{-11}	6.3×10^{-5}	4.2×10^{-11}
3	3.8×10^{-15}	3.2×10^{-6}	3.3×10^{-9}	3.8×10^{-5}	8.4×10^{-9}
4	2.8×10^{-14}	3.5×10^{-4}	2.2×10^{-6}	2.2×10^{-2}	1.8×10^{-6}
5	2.2×10^{-14}	7.3×10^{-4}	5.2×10^{-6}	6.5×10^{-3}	4.0×10^{-6}

This table represent the p-values of the non-linearity test proposed by Hung et al. (2009) adapted to the case of the open-loop TAR models for different values of the autoregressive order. The time series considered is the weekly exchange rate return COP/USD in the period 2009-2019. The variable used to determinate the thresholds is the change in the weekly real price of the barrel of Brent oil.

Table 10: Values of the Akaike information criterion for different combination of p_1 y p_2 used by obtaining GM estimator with function ψ : *Huber* $k = 1.345$.

p_1/p_2	1	2	3	4	5
1	-2260.20	-2347.18	-2357.90	-2356.50	-2355.73
2	-2268.12	-2267.31	-2346.77	-2356.56	-2350.59
3	-2276.35	-2262.60	-2253.61	-2274.53	-2347.00
4	-2271.96	-2267.79	-2255.57	-2261.20	-2344.17
5	-2284.75	-2288.07	-2274.72	-2277.78	-2267.48

This table presents the values of the information criterion AIC defined in the equation (18) for the different combinations of p_1 and p_2 selected to carry out the estimation of the open-loop TAR model using the estimator GM with function ψ : *Huber* $k = 1.345$.

The results in Table 10 show that the model selected through Akaike's information criterion is $p_1 = 1$ and $p_2 = 3$. For this model, the estimated threshold value is $\hat{r} = -1.482$. When carrying out the estimation procedure, it is found that the last coefficient in the second regime is not significant; therefore the estimation is carried out for the model $p_1 = 1$ and $p_2 = 2$ and the following results are obtained.

TABLE 11: Estimation results of the *GM* method for the model $TAR(k = 2, p_1 = 1, p_2 = 2, d = 0)$, $r = -1.482$ with function ψ : *Huber* $k = 1.345$.

		ϕ_0	ϕ_1	ϕ_2	σ
Primer régimen	$\hat{\phi}^{(1)}$	0.007***	0.398***	-	0.013
	$\hat{\sigma}_{\phi^{(1)}}$	0.001	0.126	-	-
Segundo régimen	$\hat{\phi}^{(2)}$	-0.0007	0.005	0.135**	0.011
	$\hat{\sigma}_{\phi^{(2)}}$	0.0006	0.061	0.064	-

This table presents the estimation results of the open-loop *TAR* model with two regimes, $p_1 = 1$, $p_2 = 2$, $r = -1.482$ using *GM* method with function ψ : *Huber* $k = 1.345$. The results are presented for each of the regimens. $\hat{\phi}^{(j)}$ y $\hat{\sigma}_{\phi^{(j)}}$ denote the estimates and their respective standard errors for the j-th regime. The latter are calculated using the expression in the equation (8). Significance is evaluated through Z statistics based on the assumption of asymptotic normality.

The results of the proposed model suggest that effectively the behavior of the nominal COP/USD exchange rate changes in response to changes in the real price of Brent oil. When there is a drop during the week above 1,482 dollars (in real terms) in the price of a barrel of Brent oil, the nominal exchange rate COP/USD has a behavior with more remarkable persistence concerning the previous week's value (regime 1). On the other hand, when this price decreases less than this value (or increases), it is observed that this variable has a lower persistence; in fact, its values are determined partly by the value of two weeks ago. The variance in each of these regimes is quite close.

The model can be written as:

$$Ret_t = \begin{cases} 0.007 + 0.398 Ret_{t-1} + 0.013\epsilon_t, & \text{si} \quad Z_t < -1.482 \\ -0.0007 + 0.005 Ret_{t-1} + 0.135 Ret_{t-2} + 0.011\epsilon_t & -1.482 \leq Z_t \end{cases}$$

6. Conclusions

In this investigation, an adaptation of the non-linearity test proposed by Hung et al. (2009) and of the estimation method proposed by Chan & Cheung (1994) for the autoregressive coefficients of the regimes was carried out in the case of open-loop TAR models. Monte Carlo experiments were used to compare the power and size of the non-linearity test with the classical test (which is not robust to outliers) in the presence of additive outliers. The robust estimation method was contrasted with that of least-squares through the mean square error ratios and bias in the presence of these observations. Additionally, different percentages of contamination and the proportion of observations in each regime of the model was studied. The approximation of the empirical distribution of the coefficient estimators was evaluated through the univariate normal distribution and also the empirical coverage levels of the confidence intervals in finite samples for the parameters autoregressive in each regime.

The non-linearity test results show that the classical test (based on least-squares) presents discrepancies in its empirical and nominal size in the presence of additive outlier observations, while the robust test does not. It was observed that the Huber type's robust function has slightly better power than the Tukey type. The results are similar for different percentages of contamination. The non-linearity test's power is susceptible concerning the proportion of observations in each regime; however, this problem is solved with the increase in the sample size.

On the other hand, for the estimation method of the autoregressive coefficients, the simulation exercises show that the robust estimator based on the *GM* methodology has a better performance in terms of mean square error than that of least-squares that when indeed there are additive outliers in the data. These results are maintained by changing the percentages of contamination and the proportion of observations in each regime.

Although the marginal empirical distribution of each of the estimators by the GM method is similar to a univariate normal distribution, the results of the Shapiro-Wilk suggest that this approximation is only correct when the sample size is large enough ($T = 1000$). In any case, the coverage percentages of the asymptotic confidence intervals based on assumption of normality are encouraging even for small sample sizes such as $T = 100$ and $T = 200$, so their use in practical situations seems to be justified.

In the empirical application, the non-linearity test and the adapted estimation procedure were used for the returns of the nominal exchange rate COP/USD time series that can be considered contaminated with outlier observations. The robust non-linearity test results were consistent with different choices of function ψ and the results of the estimation procedure were adequate.

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Appendix A. Additional Monte Carlo Simulation about Non-Linearity Test

Different value of autoregressive coefficient ν for threshold variable. In this section the percentage of contamination is 5% (Table A1). Different contamination percentage β (Table A2). Different Percentage of observations in each regime. In this section the global percentage of contamination is 5% (Table A3).

Appendix B. Additional Monte Carlo Simulation about Estimation of Autoregressive Coefficients

Different value of autoregressive coefficient ν for threshold variable. Different contamination percentage β of outliers (Tables B1, B2). Different percentage of observations in each regime (Table B3). Empirical distribution of the GM estimator of the autoregressive coefficients for finite samples (Tables B4, B5). Percentage of the empirical coverage of the finite sample confidence intervals for the autoregressive coefficients (Table B6).

TABLE A1: Empirical relative frequencies of rejection of the linearity hypothesis based on 1000 replications with a model $TAR(Z, 2, 1, 1, 0)$, sample size equal to 200 and parameter $\nu = -0.5$.

						LS			Huber $k = 1.345$			Huber $k = 3.291$			Tukey $c = 4.685$			Tukey $c = 15$		
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\alpha =$	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
0.5	0.5	0.5	-0.8	0		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.993	0.999	0.999	1.000	1.000	1.000
				3		0.994	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	0.988	0.997	0.999	0.999	1.000	1.000
				6		0.826	0.911	0.936	0.999	1.000	1.000	0.996	0.999	1.000	0.980	0.996	0.998	0.992	0.998	1.000
				10		0.463	0.650	0.730	0.994	0.998	0.999	0.985	0.996	0.997	0.979	0.998	1.000	0.985	0.995	0.997
0.5	0.5	0.5	-0.5	0		0.999	1.000	1.000	0.984	0.998	1.000	0.996	1.000	1.000	0.922	0.976	0.992	0.996	1.000	1.000
				3		0.903	0.972	0.986	0.951	0.986	0.998	0.946	0.990	0.996	0.859	0.953	0.970	0.957	0.993	0.996
				6		0.610	0.770	0.840	0.944	0.986	0.998	0.895	0.969	0.985	0.880	0.956	0.981	0.892	0.973	0.986
				10		0.329	0.500	0.594	0.933	0.981	0.995	0.845	0.950	0.975	0.890	0.973	0.985	0.844	0.949	0.975
0.5	0.5	0.5	-0.2	0		0.902	0.975	0.983	0.743	0.894	0.938	0.874	0.960	0.977	0.617	0.797	0.860	0.875	0.958	0.980
				3		0.628	0.797	0.869	0.643	0.836	0.900	0.655	0.834	0.905	0.535	0.745	0.833	0.660	0.850	0.910
				6		0.355	0.538	0.646	0.616	0.812	0.884	0.518	0.744	0.829	0.525	0.732	0.817	0.529	0.765	0.846
				10		0.223	0.341	0.438	0.577	0.813	0.885	0.427	0.679	0.793	0.534	0.746	0.837	0.437	0.698	0.824
0.5	0.5	0.5	0	0		0.582	0.794	0.873	0.413	0.642	0.750	0.547	0.765	0.849	0.312	0.534	0.658	0.541	0.768	0.848
				3		0.359	0.579	0.678	0.316	0.569	0.691	0.323	0.547	0.672	0.257	0.493	0.605	0.338	0.573	0.700
				6		0.217	0.385	0.472	0.285	0.541	0.655	0.223	0.439	0.569	0.269	0.498	0.602	0.225	0.436	0.581
				10		0.181	0.308	0.396	0.299	0.530	0.672	0.196	0.400	0.530	0.272	0.514	0.628	0.184	0.410	0.557
0.5	0.5	0.5	0.2	0		0.213	0.416	0.549	0.148	0.333	0.441	0.174	0.380	0.519	0.134	0.285	0.408	0.189	0.387	0.519
				3		0.141	0.308	0.413	0.106	0.266	0.386	0.098	0.258	0.359	0.107	0.249	0.369	0.112	0.273	0.377
				6		0.139	0.271	0.362	0.108	0.247	0.379	0.072	0.196	0.309	0.102	0.258	0.366	0.072	0.197	0.322
				10		0.143	0.240	0.315	0.106	0.232	0.336	0.056	0.165	0.247	0.102	0.267	0.364	0.056	0.169	0.254
0.5	0.5	0.5	0.5	0		0.013	0.069	0.121	0.010	0.057	0.108	0.012	0.068	0.117	0.016	0.085	0.135	0.013	0.069	0.114
				3		0.031	0.094	0.166	0.013	0.047	0.092	0.014	0.054	0.099	0.017	0.075	0.123	0.012	0.057	0.090
				6		0.078	0.182	0.253	0.008	0.052	0.103	0.010	0.055	0.102	0.014	0.070	0.131	0.011	0.041	0.088
				10		0.105	0.193	0.260	0.008	0.046	0.099	0.007	0.038	0.080	0.016	0.069	0.124	0.006	0.028	0.067
0.5	0.5	0.5	0.8	0		0.453	0.696	0.797	0.378	0.589	0.719	0.450	0.675	0.783	0.297	0.536	0.644	0.445	0.675	0.785
				3		0.265	0.459	0.555	0.282	0.518	0.630	0.298	0.529	0.657	0.243	0.479	0.594	0.291	0.511	0.649
				6		0.144	0.290	0.376	0.231	0.474	0.609	0.205	0.412	0.550	0.241	0.459	0.585	0.169	0.380	0.509
				10		0.121	0.225	0.294	0.235	0.452	0.591	0.157	0.341	0.463	0.255	0.477	0.597	0.150	0.318	0.441

This table presents the percentages of rejection of the non-linearity hypothesis when using the Hung et al. (2009) test adapted to the case of a TAR model. In the equation (13) the model is presented, the value of ω is related to the magnitude of the outliers, and its definition is presented in the lower part of the equation (5). The value of the ν parameter is equal to -0.5 and the sample size is $T = 200$.

TABLE A2: Empirical relative frequencies of rejection of the linearity hypothesis based on 1000 replications with a model $TAR(Z, 2, 1, 1, 0)$, sample size equal to 200 and contamination percentage 10% $\nu = 0.5$.

						LS			Huber $k = 1.345$			Huber $k = 3.291$			Tukey $c = 4.685$			Tukey $c = 15$		
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\alpha =$	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
0.5	0.5	0.5	-0.8	0		1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.992	0.998	0.999	1.000	1.000	1.000
				3		0.959	0.988	0.992	0.984	0.997	0.998	0.987	0.995	0.996	0.962	0.985	0.995	0.986	0.997	0.997
				6		0.546	0.733	0.819	0.981	0.996	0.998	0.913	0.975	0.989	0.960	0.989	0.997	0.897	0.968	0.989
				10		0.178	0.334	0.444	0.974	0.994	0.997	0.848	0.958	0.975	0.956	0.996	0.997	0.805	0.933	0.963
0.5	0.5	0.5	-0.5	0		0.994	0.999	1.000	0.965	0.996	0.998	0.995	0.999	0.999	0.896	0.969	0.983	0.995	0.998	0.999
				3		0.743	0.893	0.942	0.856	0.959	0.980	0.806	0.936	0.964	0.775	0.917	0.952	0.823	0.936	0.973
				6		0.309	0.495	0.595	0.824	0.939	0.976	0.618	0.798	0.891	0.774	0.908	0.950	0.590	0.800	0.882
				10		0.120	0.242	0.327	0.805	0.922	0.955	0.523	0.737	0.832	0.812	0.927	0.959	0.470	0.711	0.807
0.5	0.5	0.5	-0.2	0		0.861	0.949	0.971	0.702	0.879	0.939	0.832	0.940	0.973	0.584	0.778	0.857	0.838	0.944	0.975
				3		0.427	0.669	0.764	0.499	0.741	0.841	0.418	0.677	0.775	0.421	0.636	0.768	0.433	0.694	0.790
				6		0.158	0.328	0.445	0.464	0.697	0.798	0.271	0.502	0.628	0.420	0.652	0.752	0.236	0.497	0.632
				10		0.080	0.174	0.262	0.428	0.662	0.770	0.208	0.417	0.546	0.435	0.649	0.762	0.169	0.389	0.530
0.5	0.5	0.5	0	0		0.532	0.754	0.850	0.392	0.620	0.745	0.518	0.736	0.836	0.316	0.532	0.644	0.503	0.733	0.839
				3		0.281	0.464	0.591	0.266	0.512	0.634	0.214	0.430	0.540	0.228	0.436	0.565	0.228	0.457	0.570
				6		0.112	0.242	0.339	0.234	0.444	0.552	0.101	0.274	0.399	0.218	0.430	0.542	0.098	0.258	0.382
				10		0.076	0.139	0.200	0.192	0.396	0.528	0.076	0.220	0.311	0.224	0.421	0.536	0.057	0.180	0.288
0.5	0.5	0.5	0.2	0		0.199	0.378	0.522	0.147	0.319	0.428	0.184	0.370	0.494	0.132	0.294	0.396	0.179	0.374	0.505
				3		0.119	0.272	0.371	0.110	0.239	0.341	0.078	0.199	0.287	0.099	0.234	0.328	0.076	0.193	0.309
				6		0.077	0.175	0.270	0.087	0.212	0.308	0.039	0.128	0.207	0.090	0.212	0.308	0.036	0.129	0.208
				10		0.066	0.138	0.201	0.078	0.205	0.298	0.028	0.107	0.166	0.099	0.230	0.331	0.030	0.091	0.155
0.5	0.5	0.5	0.5	0		0.010	0.059	0.109	0.012	0.061	0.115	0.014	0.051	0.103	0.017	0.078	0.135	0.012	0.056	0.101
				3		0.029	0.091	0.155	0.009	0.054	0.109	0.014	0.068	0.128	0.018	0.065	0.109	0.010	0.069	0.128
				6		0.044	0.122	0.199	0.009	0.044	0.083	0.018	0.069	0.134	0.006	0.048	0.094	0.011	0.044	0.098
				10		0.044	0.091	0.153	0.011	0.054	0.090	0.012	0.048	0.097	0.014	0.058	0.120	0.010	0.035	0.068
0.5	0.5	0.5	0.8	0		0.573	0.774	0.847	0.434	0.656	0.755	0.540	0.752	0.832	0.348	0.557	0.680	0.542	0.749	0.834
				3		0.289	0.488	0.614	0.313	0.539	0.667	0.361	0.602	0.714	0.259	0.470	0.591	0.306	0.555	0.655
				6		0.132	0.243	0.359	0.229	0.454	0.573	0.205	0.417	0.547	0.233	0.431	0.557	0.130	0.318	0.441
				10		0.081	0.179	0.268	0.217	0.431	0.553	0.156	0.327	0.446	0.253	0.469	0.579	0.089	0.238	0.346

This table presents the percentages of rejection of the non-linearity hypothesis when using the Hung et al. (2009) test adapted to the case of a *TAR* model with a percentage of contamination of 10%. In the equation (13) the model is presented, the value of ω is related to the magnitude of the outliers, and its definition is presented in the lower part of the equation (5). The value of the ν parameter is equal to 0.5 and the sample size is $T = 200$.

TABLE A3: Empirical relative frequencies of rejection of the linearity hypothesis based on 1000 replications with a model $TAR(Z, 2, 1, 1, 0)$, sample size equal to 200, threshold value $r = 0.6$ and $\nu = 0.5$.

						LS			Huber $k = 1.345$			Huber $k = 3.291$			Tukey $c = 4.685$			Tukey $c = 15$		
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\alpha =$	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
0.5	0.5	0.5	-0.8	0		0.969	0.990	0.998	0.859	0.955	0.976	0.968	0.989	0.997	0.765	0.880	0.935	0.965	0.988	0.995
				3		0.890	0.944	0.968	0.810	0.924	0.951	0.884	0.958	0.976	0.722	0.872	0.922	0.880	0.957	0.975
				6		0.619	0.768	0.831	0.801	0.921	0.952	0.824	0.923	0.961	0.729	0.866	0.922	0.807	0.913	0.952
				10		0.365	0.558	0.643	0.775	0.907	0.943	0.764	0.900	0.941	0.712	0.872	0.923	0.736	0.883	0.939
0.5	0.5	0.5	-0.5	0		0.850	0.942	0.970	0.670	0.842	0.906	0.823	0.936	0.965	0.574	0.776	0.853	0.821	0.928	0.961
				3		0.640	0.807	0.871	0.576	0.781	0.864	0.617	0.803	0.889	0.485	0.722	0.814	0.618	0.820	0.895
				6		0.410	0.587	0.702	0.549	0.771	0.851	0.516	0.733	0.826	0.509	0.714	0.814	0.491	0.703	0.818
				10		0.254	0.413	0.512	0.571	0.760	0.837	0.465	0.698	0.792	0.518	0.744	0.831	0.433	0.682	0.770
0.5	0.5	0.5	-0.2	0		0.494	0.731	0.827	0.333	0.584	0.709	0.464	0.701	0.811	0.283	0.495	0.632	0.471	0.706	0.807
				3		0.327	0.577	0.694	0.265	0.523	0.655	0.286	0.531	0.652	0.228	0.460	0.583	0.276	0.536	0.665
				6		0.243	0.444	0.560	0.247	0.487	0.638	0.200	0.441	0.562	0.236	0.448	0.576	0.187	0.415	0.544
				10		0.185	0.326	0.406	0.255	0.484	0.607	0.161	0.378	0.497	0.263	0.472	0.575	0.150	0.348	0.475
0.5	0.5	0.5	0	0		0.252	0.466	0.596	0.188	0.377	0.493	0.243	0.445	0.588	0.170	0.333	0.455	0.240	0.456	0.581
				3		0.196	0.380	0.506	0.156	0.340	0.445	0.154	0.324	0.435	0.154	0.302	0.422	0.157	0.335	0.450
				6		0.182	0.331	0.438	0.148	0.313	0.412	0.095	0.240	0.359	0.137	0.301	0.406	0.078	0.219	0.350
				10		0.146	0.269	0.364	0.131	0.291	0.403	0.072	0.213	0.322	0.151	0.291	0.408	0.057	0.190	0.288
0.5	0.5	0.5	0.2	0		0.084	0.224	0.325	0.073	0.208	0.294	0.074	0.220	0.323	0.085	0.207	0.293	0.075	0.221	0.324
				3		0.089	0.221	0.329	0.065	0.177	0.255	0.045	0.147	0.239	0.067	0.178	0.259	0.047	0.147	0.252
				6		0.125	0.253	0.352	0.057	0.165	0.264	0.035	0.128	0.205	0.075	0.183	0.260	0.034	0.123	0.206
				10		0.139	0.248	0.322	0.057	0.164	0.259	0.029	0.107	0.190	0.074	0.189	0.286	0.019	0.097	0.169
0.5	0.5	0.5	0.5	0		0.010	0.059	0.109	0.012	0.061	0.115	0.014	0.051	0.103	0.017	0.078	0.135	0.012	0.056	0.101
				3		0.027	0.096	0.154	0.006	0.051	0.112	0.013	0.058	0.122	0.013	0.064	0.121	0.011	0.059	0.116
				6		0.076	0.187	0.264	0.011	0.053	0.113	0.015	0.052	0.102	0.014	0.079	0.124	0.012	0.041	0.087
				10		0.091	0.179	0.249	0.009	0.053	0.094	0.012	0.050	0.097	0.015	0.067	0.123	0.011	0.033	0.072
0.5	0.5	0.5	0.8	0		0.262	0.452	0.585	0.160	0.335	0.461	0.223	0.421	0.555	0.139	0.294	0.403	0.213	0.419	0.550
				3		0.180	0.365	0.465	0.142	0.306	0.427	0.183	0.394	0.512	0.113	0.258	0.379	0.162	0.352	0.481
				6		0.133	0.239	0.349	0.117	0.282	0.403	0.127	0.317	0.423	0.112	0.280	0.377	0.095	0.237	0.377
				10		0.136	0.242	0.324	0.109	0.262	0.388	0.099	0.254	0.358	0.115	0.263	0.379	0.077	0.205	0.314

This table presents the percentages of rejection of the non-linearity hypothesis when using the Hung et al. (2009) test adapted to the case of a TAR model. In the equation (13) the model is presented, the value of ω is related to the magnitude of the outliers, and its definition is presented in the lower part of the equation (5). The value of the ν parameter is equal to 0.5 and the sample size is $T = 200$ and the threshold value r is 0.6 so that approximately 30% of the observations remain in the second regime.

Table B1: Mean square error ratio between the GM and the classical estimator based on 1000 replications with a TAR model $(Z, 2, 1, 1, 0)$, sample size equal to 200, $\nu = -0.5$.

					Huber $k = 1.345$				Tukey $c = 4.685$			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	1.165	1.354	1.185	1.375	1.400	2.419	1.465	2.378
				3	0.870	0.499	0.638	0.275	1.011	0.812	0.749	0.384
				6	0.438	0.141	0.237	0.070	0.474	0.214	0.264	0.096
				10	0.251	0.079	0.145	0.043	0.271	0.118	0.162	0.059
0.5	0.5	0.5	-0.5	0	1.176	1.323	1.188	1.380	1.438	2.383	1.522	2.496
				3	0.831	0.461	0.721	0.488	0.995	0.717	0.866	0.779
				6	0.408	0.154	0.346	0.165	0.447	0.230	0.392	0.259
				10	0.209	0.096	0.200	0.102	0.213	0.134	0.220	0.151
0.5	0.5	0.5	-0.2	0	1.226	1.395	1.191	1.307	1.575	2.450	1.546	2.204
				3	0.763	0.486	0.908	0.948	0.933	0.772	1.153	1.637
				6	0.337	0.175	0.484	0.555	0.389	0.250	0.602	0.933
				10	0.199	0.108	0.284	0.410	0.223	0.158	0.338	0.654
0.5	0.5	0.5	0	0	1.180	1.346	1.218	1.419	1.540	2.407	1.656	2.560
				3	0.652	0.464	0.983	1.348	0.792	0.724	1.331	2.469
				6	0.295	0.159	0.627	1.173	0.342	0.245	0.825	2.195
				10	0.179	0.117	0.314	1.019	0.196	0.180	0.406	1.990
0.5	0.5	0.5	0.2	0	1.288	1.537	1.206	1.427	1.793	2.810	1.656	2.471
				3	0.644	0.489	0.880	0.980	0.818	0.769	1.174	1.709
				6	0.274	0.174	0.466	0.565	0.340	0.264	0.588	0.922
				10	0.174	0.118	0.342	0.429	0.209	0.173	0.416	0.686
0.5	0.5	0.5	0.5	0	1.274	1.390	1.261	1.416	1.835	2.432	1.815	2.579
				3	0.587	0.454	0.637	0.474	0.768	0.691	0.848	0.753
				6	0.228	0.153	0.246	0.161	0.298	0.231	0.303	0.235
				10	0.145	0.098	0.160	0.103	0.182	0.145	0.194	0.152
0.5	0.5	0.5	0.8	0	1.274	1.415	1.333	1.454	1.969	2.635	2.003	2.531
				3	0.591	0.530	0.386	0.294	0.821	0.861	0.503	0.414
				6	0.212	0.157	0.111	0.077	0.270	0.223	0.128	0.099
				10	0.120	0.091	0.069	0.046	0.153	0.131	0.084	0.061

This table presents the mean square error ratio obtained through simulations between the GM estimator and the least-squares estimator. In equation (13) the model is presented, the value of ω is related to the outliers' magnitude, and its definition is presented in the lower part of the equation (5). The value of the parameter ν is equal to -0.5, and the sample size is $T = 200$.

Table B2: Mean square error ratio between the *GM* and the classical estimator based on 1000 replications with a TAR model $(Z, 2, 1, 1, 0)$, sample size equal to 200, $\nu = 0.5$ and percentage of contamination 10%.

					Huber $k = 1.345$				Tukey $c = 4.685$			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	1.201	1.432	1.142	1.328	1.553	2.562	1.357	2.298
				3	0.633	0.373	0.504	0.202	0.708	0.448	0.532	0.214
				6	0.271	0.133	0.210	0.065	0.269	0.139	0.212	0.056
				10	0.165	0.089	0.117	0.037	0.161	0.096	0.108	0.033
0.5	0.5	0.5	-0.5	0	1.163	1.352	1.187	1.357	1.583	2.393	1.453	2.362
				3	0.528	0.358	0.586	0.321	0.622	0.452	0.675	0.410
				6	0.250	0.149	0.274	0.137	0.262	0.160	0.297	0.165
				10	0.161	0.109	0.161	0.095	0.165	0.120	0.159	0.107
0.5	0.5	0.5	-0.2	0	1.228	1.335	1.202	1.405	1.650	2.372	1.536	2.503
				3	0.548	0.355	0.714	0.785	0.617	0.454	0.886	1.425
				6	0.239	0.152	0.374	0.518	0.233	0.166	0.424	0.858
				10	0.163	0.110	0.211	0.384	0.163	0.118	0.239	0.642
0.5	0.5	0.5	0	0	1.198	1.332	1.154	1.379	1.614	2.257	1.479	2.510
				3	0.509	0.344	0.825	1.246	0.564	0.428	1.077	2.585
				6	0.225	0.156	0.425	1.297	0.220	0.175	0.500	2.339
				10	0.147	0.110	0.224	1.069	0.143	0.121	0.258	2.012
0.5	0.5	0.5	0.2	0	1.268	1.464	1.211	1.403	1.796	2.629	1.654	2.569
				3	0.484	0.359	0.744	0.740	0.545	0.455	0.984	1.291
				6	0.222	0.161	0.378	0.466	0.225	0.188	0.450	0.764
				10	0.139	0.108	0.204	0.387	0.134	0.117	0.233	0.576
0.5	0.5	0.5	0.5	0	1.252	1.405	1.290	1.384	1.782	2.328	1.918	2.430
				3	0.460	0.351	0.451	0.342	0.522	0.417	0.520	0.423
				6	0.199	0.155	0.203	0.146	0.211	0.170	0.219	0.165
				10	0.129	0.115	0.135	0.109	0.129	0.119	0.143	0.121
0.5	0.5	0.5	0.8	0	1.253	1.447	1.302	1.463	1.844	2.660	1.950	2.583
				3	0.446	0.363	0.287	0.234	0.545	0.485	0.296	0.252
				6	0.164	0.139	0.097	0.073	0.176	0.154	0.090	0.070
				10	0.112	0.093	0.059	0.044	0.107	0.102	0.052	0.041

This table presents the mean square error ratio obtained through simulations between the *GM* estimator and the least-squares estimator. In equation (13) the model is presented, the value of ω is related to the outliers' magnitude, and its definition is presented in the lower part of the equation (5). The value of the parameter ν is equal to 0.5, and the sample size is $T = 200$ with a percentage of contamination of 10%.

Table B3: Mean square error ratio between the *GM* and the classical estimator based on 1000 replications with a TAR model $(Z, 2, 1, 1, 0)$, sample size equal to 200, $\nu = 0.5$ and threshold value $r = 0.6$.

					Huber $k = 1.345$				Tukey $c = 4.685$			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	1.186	1.397	1.150	1.380	1.615	2.574	1.461	2.534
				3	0.683	0.418	0.650	0.387	0.827	0.625	0.765	0.620
				6	0.244	0.126	0.330	0.119	0.294	0.181	0.374	0.175
				10	0.148	0.073	0.183	0.070	0.171	0.101	0.207	0.104
0.5	0.5	0.5	-0.5	0	1.266	1.375	1.172	1.398	1.781	2.402	1.493	2.422
				3	0.622	0.402	0.779	0.651	0.786	0.570	0.949	1.060
				6	0.237	0.125	0.387	0.234	0.279	0.173	0.456	0.363
				10	0.140	0.076	0.233	0.162	0.159	0.102	0.270	0.243
0.5	0.5	0.5	-0.2	0	1.170	1.324	1.205	1.305	1.529	2.305	1.559	2.292
				3	0.579	0.381	0.922	1.109	0.687	0.545	1.201	2.058
				6	0.216	0.120	0.499	0.737	0.242	0.165	0.621	1.309
				10	0.133	0.074	0.283	0.586	0.148	0.100	0.322	0.957
0.5	0.5	0.5	0	0	1.231	1.421	1.204	1.452	1.676	2.512	1.681	2.776
				3	0.561	0.375	1.003	1.434	0.692	0.553	1.432	2.863
				6	0.209	0.121	0.580	1.064	0.238	0.168	0.802	2.096
				10	0.128	0.075	0.356	1.104	0.146	0.105	0.472	2.091
0.5	0.5	0.5	0.2	0	1.202	1.318	1.274	1.437	1.659	2.181	1.883	2.706
				3	0.511	0.382	0.973	1.161	0.627	0.540	1.363	2.150
				6	0.212	0.130	0.572	0.680	0.247	0.169	0.761	1.193
				10	0.139	0.080	0.328	0.547	0.165	0.110	0.444	0.974
0.5	0.5	0.5	0.5	0	1.219	1.425	1.286	1.405	1.787	2.497	1.966	2.478
				3	0.505	0.385	0.745	0.648	0.638	0.556	1.043	1.049
				6	0.180	0.126	0.337	0.279	0.212	0.164	0.431	0.417
				10	0.104	0.076	0.235	0.197	0.122	0.103	0.310	0.277
0.5	0.5	0.5	0.8	0	1.211	1.426	1.234	1.352	1.806	2.577	1.800	2.547
				3	0.499	0.403	0.495	0.398	0.638	0.595	0.664	0.637
				6	0.174	0.124	0.166	0.120	0.200	0.160	0.206	0.174
				10	0.095	0.072	0.106	0.077	0.113	0.096	0.120	0.111

This table presents the mean square error ratio obtained through simulations between the *GM* estimator and the least-squares estimator. In equation (13) the model is presented, the value of ω is related to the outliers' magnitude, and its definition is presented in the lower part of the equation (5). The value of the parameter ν is equal to 0.5, and the sample size is $T = 200$ and threshold value $r = 0.6$ so that approximately 30% of the observations remain in the second regime.

Table B4: *P*-values of the Shapiro-Wilk test of normality for the empirical distributions of the *GM* estimators based on 1000 replications of the estimation process with a TAR model$(Z, 2, 1, 1, 0)$, sample size equal to 100.

					Huber $k = 1.345$				Tukey $c = 4.685$			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	0.00	0.00	0.95	0.00	0.00	0.00	0.19	0.00
				3	0.00	0.00	0.58	0.00	0.00	0.00	0.01	0.00
				6	0.00	0.00	0.88	0.00	0.00	0.00	0.18	0.00
				10	0.00	0.00	0.32	0.00	0.00	0.00	0.02	0.00
0.5	0.5	0.5	-0.5	0	0.00	0.25	0.04	0.01	0.00	0.02	0.00	0.00
				3	0.00	0.02	0.37	0.03	0.00	0.00	0.01	0.00
				6	0.00	0.01	0.01	0.00	0.00	0.02	0.00	0.00
				10	0.00	0.12	0.08	0.01	0.00	0.00	0.04	0.00
0.5	0.5	0.5	-0.2	0	0.00	0.02	0.13	0.04	0.00	0.24	0.00	0.01
				3	0.00	0.00	0.31	0.40	0.00	0.02	0.00	0.00
				6	0.01	0.55	0.09	0.33	0.00	0.03	0.02	0.04
				10	0.02	0.03	0.12	0.54	0.00	0.03	0.00	0.16
0.5	0.5	0.5	0	0	0.00	0.14	0.04	0.03	0.00	0.00	0.00	0.00
				3	0.00	0.01	0.00	0.03	0.00	0.00	0.00	0.00
				6	0.00	0.00	0.03	0.07	0.00	0.00	0.00	0.27
				10	0.00	0.01	0.09	0.00	0.00	0.00	0.00	0.00
0.5	0.5	0.5	0.2	0	0.00	0.00	0.02	0.22	0.00	0.00	0.04	0.03
				3	0.00	0.00	0.20	0.25	0.00	0.00	0.09	0.20
				6	0.00	0.01	0.02	0.24	0.00	0.00	0.00	0.01
				10	0.00	0.00	0.07	0.37	0.00	0.00	0.00	0.01
0.5	0.5	0.5	0.5	0	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00
				3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
				6	0.00	0.00	0.00	0.06	0.00	0.00	0.00	0.00
				10	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00
0.5	0.5	0.5	0.8	0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
				3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
				6	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
				10	0.00	0.14	0.00	0.00	0.00	0.08	0.00	0.00

This table present the p-values obtained when carrying out the Shapiro-Wilk normality test on the empirical distribution of the coefficients estimated through the *GM* method obtained through simulations. In equation (13) the model is presented, the value of ω is related to the magnitude of the outliers and its definition is presented in the lower part of equation (5). The value of the parameter ν is equal to 0.5, the sample size is $T = 100$, and the threshold value r is 0.

TABLE B5: P-values of the Shapiro-Wilk test of normality for the empirical distributions of the GM estimators based on 1000 replications of the estimation process with a TAR model$(Z, 2, 1, 1, 0)$, sample size equal to 1000.

					Huber $k = 1.345$				Tukey $c = 4.685$			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	0.74	0.08	0.87	0.05	0.10	0.28	0.72	0.04
				3	0.03	0.29	0.83	0.46	0.01	0.21	0.36	0.04
				6	0.84	0.14	0.36	0.02	0.28	0.08	0.36	0.03
				10	0.52	0.00	0.48	0.03	0.36	0.05	0.13	0.12
0.5	0.5	0.5	-0.5	0	0.99	0.50	0.37	0.00	0.66	0.73	0.04	0.00
				3	0.76	0.08	0.59	0.08	0.69	0.12	0.11	0.04
				6	0.96	0.43	0.58	0.01	0.82	0.59	0.56	0.04
				10	0.86	0.48	0.80	0.05	0.73	0.04	0.12	0.01
0.5	0.5	0.5	-0.2	0	0.23	0.83	0.28	0.01	0.17	0.92	0.78	0.06
				3	0.11	0.95	0.27	0.04	0.40	0.82	0.42	0.35
				6	0.41	0.73	0.33	0.30	0.11	0.65	0.99	0.36
				10	0.21	0.85	0.36	0.61	0.09	0.37	0.70	0.11
0.5	0.5	0.5	0	0	0.95	0.70	0.14	0.48	0.73	0.57	0.08	0.37
				3	0.20	0.86	0.57	0.66	0.58	0.94	0.07	0.20
				6	0.28	0.17	0.14	0.02	0.12	0.46	0.29	0.04
				10	0.52	0.06	0.55	0.51	0.29	0.02	0.18	0.83
0.5	0.5	0.5	0.2	0	0.56	0.53	0.00	0.75	0.41	0.74	0.00	0.20
				3	0.67	0.64	0.01	0.51	0.92	0.55	0.00	0.02
				6	0.44	0.29	0.04	0.79	0.19	0.54	0.02	0.08
				10	0.02	0.52	0.08	0.57	0.05	0.85	0.00	0.46
0.5	0.5	0.5	0.5	0	0.04	0.03	0.14	0.40	0.01	0.00	0.56	0.24
				3	0.15	0.08	0.28	0.38	0.01	0.00	0.24	0.10
				6	0.12	0.01	0.14	0.22	0.01	0.01	0.04	0.52
				10	0.54	0.18	0.64	0.07	0.23	0.05	0.80	0.17
0.5	0.5	0.5	0.8	0	0.85	0.35	0.07	0.74	0.24	0.86	0.01	0.70
				3	0.75	0.78	0.02	0.19	0.06	0.67	0.00	0.63
				6	0.53	0.22	0.06	0.58	0.36	0.54	0.05	0.76
				10	0.04	0.04	0.11	0.26	0.16	0.05	0.02	0.51

This table present the p-values obtained when carrying out the Shapiro-Wilk normality test on the empirical distribution of the coefficients estimated through the GM method obtained through simulations. In equation (13) the model is presented, the value of *omega* is related to the magnitude of the outliers and its definition is presented in the lower part of equation (5). The value of the parameter ν is equal to 0.5, the sample size is $T = 1000$, and the threshold value r is 0.

TABLE B6: Empirical coverage of the 95% confidence intervals constructed from the *GM* estimator based on 1000 replications with a TAR model$(Z, 2, 1, 1, 0)$, sample size equal to 200.

					Huber $k = 1.345$				Tukey $c = 4.685$			
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$	$\hat{\phi}_0^{(1)}$	$\hat{\phi}_1^{(1)}$	$\hat{\phi}_0^{(2)}$	$\hat{\phi}_1^{(2)}$
0.5	0.5	0.5	-0.8	0	0.94	0.95	0.95	0.94	0.94	0.93	0.95	0.94
				3	0.92	0.91	0.94	0.92	0.93	0.93	0.94	0.93
				6	0.93	0.91	0.95	0.92	0.93	0.92	0.95	0.94
				10	0.93	0.91	0.94	0.93	0.93	0.94	0.94	0.94
0.5	0.5	0.5	-0.5	0	0.95	0.94	0.95	0.93	0.94	0.93	0.96	0.93
				3	0.93	0.91	0.95	0.94	0.94	0.93	0.95	0.94
				6	0.93	0.92	0.95	0.94	0.94	0.92	0.97	0.94
				10	0.93	0.92	0.95	0.95	0.94	0.93	0.95	0.94
0.5	0.5	0.5	-0.2	0	0.94	0.93	0.94	0.93	0.95	0.93	0.94	0.93
				3	0.95	0.91	0.94	0.94	0.95	0.93	0.94	0.95
				6	0.93	0.92	0.95	0.93	0.94	0.93	0.94	0.93
				10	0.95	0.92	0.94	0.93	0.95	0.93	0.94	0.95
0.5	0.5	0.5	0	0	0.95	0.96	0.94	0.94	0.95	0.95	0.93	0.95
				3	0.95	0.93	0.95	0.94	0.94	0.94	0.93	0.94
				6	0.95	0.93	0.94	0.94	0.95	0.94	0.94	0.94
				10	0.94	0.93	0.94	0.95	0.94	0.93	0.93	0.94
0.5	0.5	0.5	0.2	0	0.95	0.94	0.93	0.94	0.95	0.94	0.94	0.94
				3	0.94	0.93	0.93	0.94	0.95	0.94	0.93	0.95
				6	0.94	0.93	0.95	0.93	0.94	0.94	0.94	0.93
				10	0.94	0.93	0.93	0.92	0.95	0.95	0.93	0.93
0.5	0.5	0.5	0.5	0	0.94	0.93	0.94	0.93	0.95	0.94	0.93	0.93
				3	0.92	0.91	0.93	0.92	0.94	0.94	0.93	0.93
				6	0.92	0.92	0.93	0.91	0.93	0.93	0.94	0.93
				10	0.94	0.93	0.94	0.91	0.94	0.94	0.94	0.93
0.5	0.5	0.5	0.8	0	0.95	0.93	0.93	0.93	0.95	0.93	0.93	0.94
				3	0.94	0.92	0.92	0.92	0.94	0.93	0.92	0.93
				6	0.93	0.91	0.92	0.91	0.95	0.93	0.92	0.92
				10	0.94	0.91	0.93	0.92	0.95	0.93	0.94	0.93

This table presents the empirical coverage of the confidence intervals obtained through simulations based on the *GM* estimator and following those proposed in subsection 2.1. In equation (13) the model is presented, the value of ω is related to the magnitude of the outliers and its definition is presented in the lower part of equation (5). The value of the parameter ν is equal to 0.5, the sample size is $T = 200$, and the threshold value r is 0.

ccc