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Bayesian Estimation of Morgenstern Type Bivariate Rayleigh Distribution using Some Types of Ranked Set Sampling

Estimación bayesiana de la distribución de Rayleigh bivariada de tipo Morgenstern utilizando algunos tipos de muestreo por conjuntos clasificados

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Abstract

In this paper we consider Bayesian estimation based on bivariate ranked set sample, in which units are ranked based on measurements made on an easily and exactly measurable auxiliary variable X which is correlated with the study variable Y. We obtain Bayes estimator for the scale parameter of the study variate Y, when (X, Y) follows a Morgenstern type bivariate Rayleigh distribution. The Bayes estimators are considered based on bivariate ranked set sampling, extreme ranked set sampling and maximum ranked set sampling with unequal sample. The accuracy of estimation methods in this paper is illustrated using simulation study. Finally, a real data set is analyzed.

***Key words*:** Bayesian estimation; Concomitant ranked set samplingremark Extreme ranked set samplingremark Maximum ranked set sampling and Rayleigh distribution.

Resumen

En este artículo consideramos la estimación bayesiana basada en una muestra de conjuntos clasificados bivariados, en la que las unidades se clasifican según las mediciones realizadas en una variable auxiliar X fácil y exactamente medible que se correlaciona con la variable de estudio Y. Obtuvimos el estimador de Bayes para el parámetro de escala de la variante de estudio Y, cuando (X, Y) sigue una distribución de Rayleigh bivariada de tipo Morgenstern. Los estimadores de Bayes se consideran basados en un

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muestreo conjunto bivariado, un muestreo conjunto extremo clasificado y un muestreo conjunto máximo clasificado con muestra desigual. La precisión de los métodos de estimación en este documento se ilustra mediante el estudio de simulación. Finalmente, se analiza un conjunto de datos real.

***Palabras clave*:** Estimación bayesianaremark muestreo conjunto concomitante clasificadoremark muestreo conjunto extremo clasificadoremark muestreo conjunto máximo clasificado y distribución de Rayleigh.

1. Introduction

Farlie-Gumbel-Morgenstern (FGM) distribution is a class of bivariate distributions that first introduced by Morgenstern (1956) and extended it to the multivariate case by Farlie (1960). Some authors studied on FGM class of distribution in literature: For example Bivariate logistic distributions by Gumbel (1960*b*), Morgenstern type bivariate gamma distribution by D'Este (1981) and Tahmasebi & Jafari (2015*a*), uniform distribution by Bairamov & Bekci (1999), Tahmasebi & Jafari (2017) and Singh & Mehta.V (2015), exponential distribution by Gumbel (1960*a*), Balasubramanian & Beg (1997) and Chacko & Thomas (2011), generalized exponential distribution by Tahmasebi & Jafari (2014) and Tahmasebi & Jafari (2015*b*).

Let the n pairs (X_i, Y_i) be a random sample from a bivariate distribution with cdf $F(x, y)$. If the (X_i, Y_i) are ordered by X-variates (in increasing order of magnitude), then the Y values associated with the $X_{r:n}$, $1 \leq r \leq n$ of X (denoted as $Y_{[r:n]}$) is called the concomitant of the rth order statistics.

The paired (X, Y) is said to be a bivariate FGM distributed if the bivariate cumulative distribution function (cdf) $F(x, y)$ is an absolutely continuous such that:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))], \quad -1 \leq \alpha \leq 1.$$

The corresponding pdf is given by:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1+\alpha(1-2F_X(x))(1-2F_Y(y))], \quad -1 \leq \alpha \leq 1.$$

Therefore, the conditional cumulative distribution function (cdf) and pdf of Y given $X = x$ are obtained by:

$$F_{Y|X}(y|x) = F_Y(y)[1+\alpha(1-2F_X(x))(1-F_Y(y))],$$

and

$$f_{Y|X}(y|x) = f_Y(y)[1+\alpha(1-2F_X(x))(1-2F_Y(y))],$$

respectively.

Morgenstern type bivariate Rayleigh distribution (MTBRD) with the cdf as

$$F_{X,Y}(x,y) = (1-e^{-\frac{x^2}{2\sigma_1^2}})(1-e^{-\frac{y^2}{2\sigma_2^2}})(1+\alpha e^{-\frac{x^2}{2\sigma_1^2}-\frac{y^2}{2\sigma_2^2}}), \quad x, y > 0,$$

and the pdf as

$$f_{X,Y}(x,y) = \frac{xy}{\sigma_1^2\sigma_2^2} e^{-\frac{x^2}{2\sigma_1^2}-\frac{y^2}{2\sigma_2^2}}[1+\alpha(2e^{-\frac{x^2}{2\sigma_1^2}}-1)(2e^{-\frac{y^2}{2\sigma_2^2}}-1)],$$

are considered by Tahmasebi, Eskandarzadeh & Almaspoor (2017) to obtain unbiased estimator and BLUE of parameter σ_2 using ranked set sampling (RSS). This sampling technique proposed by McIntyre (1952) and has studied in different areas that actual measurements of the variable of study are time consuming or expensive. The RSS is an alternative to simple random sampling (SRS) that can be used to improve the cost efficiency and precision.

In order to obtain a ranked set sample of size n from the population, we draw n independent sets, each is based on a simple random sample of size n from the population and order each set by visual judgment (without actual measurement). Then the rth observation of rth set is measured.

Estimation of some distribution parameters based on RSS are considered in literature for example: location-scale family of distribution by Stokes (1995), two-parameter exponential distribution by Lam et al. (1994), bivariate normal distribution by Al-Saleh & Al-Ananbeh (2005) and Al-Saleh & Al-Ananbeh (2007), Morgenstern type bivariate exponential distribution by Chacko & Thomas (2008), Downton's bivariate exponential distribution by Al-Saleh & Diab (2009), and Morgenstern type bivariate gamma distribution by Tahmasebi & Jafari (2015*b*).

A modified of ranked set sampling procedure called extreme ranked set sampling (ERSS) in which only the largest or the smallest judgment ranked unit of each set is chosen for quantification introduced by Stokes (1980) and applied by Samawi et al. (1996). Comparison of Bayesian estimation based on RSS and ERSS from Morgenstern type bivariate exponential distribution considered by Chacko (2017). Eskandarzadeh et al. (2018), considered measures of information for an another modification of RSS called maximum ranked set sampling with unequal samples (MRSSU). In this method n sets via SRS are taken from the population such that the size of ith set is i, and the maximum of each set is measured.

The organization of this article is as follows. In Section 2, different Bayes estimators for σ_2 when the association parameter α is known in MTBRD are obtained by using bivariate RSS, ERSS and MRSSU methods, respectively. We consider the Bayesian estimation of σ_2 based on the above methods when α is unknown in Section 3. Also, the efficiency of all estimators are evaluated by simulation study in Section 4. In addition, a data set have been analyzed for illustrative purposes in Section 5. Finally, the paper is concluded in 6.

We also consider the square-root inverted-gamma (S.R.IGamma) prior for σ_2 which has the form

$$\pi(\sigma_2|a,b) = \frac{a^b}{\Gamma(b)2^{b-1}}(\sigma_2)^{-2b-1}e^{-\frac{a}{2\sigma_2^2}}, \quad \sigma_2 > 0, \tag{1}$$

where $a > 0$ and $b > 0$. Also, $E(\sigma_2) = (a/2)^{1/2}\frac{\Gamma(b-1/2)}{\Gamma(b)}$. When $a = b = 0$, (1) be the non-informative prior of σ_2. This prior pdf was first proposed by Bernardo (1994).

2. Bayesian estimation of σ_2 when α is known

In this Section, we obtain different Bayes estimators of σ_2 when the association parameter α is known in MTBRD by using bivariate RSS and some its modifications.

2.1. Bayesian estimation based on bivariate RSS

RSS is useful in cases where measuring the main variable Y is much more difficult than to ranking. However, even if ranking of the interest variable Y is difficult, but there is an easily ranked axillary variable X as the ranking criterion available, then it may be used to judgment ranking the study variable Y. This sampling method is called bivariate RSS and introduced by Stokes (1977) as follows:

Step 1. Draw n independent sets, each is based on a simple random bivariate sample of size n from the population (X, Y).

Step 2. Rank the units within each sample with respect to an axillary variable X together with the associated variable Y.

Step 3. Measure the rth observation of the rth set $(X_{(r)r}, Y_{[r]r})$, $r = 1, 2, \ldots, n$, where $X_{(r)r}$ is the rth order statistics of the rth sample and $Y_{[r]r}$ is the corresponding measurement made on the study variable Y of the same unit.

The pdf and cdf of $X_{(r)r}$, denoted by $f_{r:n}(x)$ and $F_{r:n}(x)$ and are given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \quad -\infty < x < \infty,$$

and

$$F_{r:n}(x) = \sum_{i=r}^{n}\binom{n}{i}[F(x)]^{i}[1-F(x)]^{n-i}, \quad -\infty < x < \infty,$$

respectively.

Let (X, Y) has a MTBRD as defined in (1) and $Y_{[r]r}$, $r = 1, 2, 3, \ldots, n$, be the observations of the concomitant variable Y based on bivariate RSS with an axillary variable X. From Scaria & Nair (1999), the cdf and pdf of $Y_{[r]r}$ are given by

$$\begin{aligned} F_{[r]r}(y) &= \int_{-\infty}^{+\infty} F_{Y|X}(y|x) f_{r:n}(x)dx \\ &= (1-e^{-\frac{y^2}{2\sigma_2^2}})\left[1+\delta_r(e^{-\frac{y^2}{2\sigma_2^2}}-1)\right], \quad y>0, \end{aligned} \tag{2}$$

$$\begin{aligned} f_{[r]r}(y) &= \int_{-\infty}^{+\infty} f_{Y|X}(y|x) f_{r:n}(x)dx \\ &= \frac{y}{\sigma_2^2}e^{-\frac{y^2}{2\sigma_2^2}}[1+\delta_r(2e^{-\frac{y^2}{2\sigma_2^2}}-1)], \quad y>0, \end{aligned} \tag{3}$$

respectively, where $\delta_r = \frac{\alpha(n-2r+1)}{n+1}$. Therefore, the mean and variance of $Y_{[r]r}$ is given by

$$E[Y_{[r]r}] = \sigma_2 \beta_r, \quad Var[Y_{[r]r}] = \sigma_2^2 \lambda_r, \tag{4}$$

where $\beta_r = \sqrt{\frac{\pi}{2}} + \delta_r \frac{\sqrt{\pi}}{2}(1-\sqrt{2})$ and $\lambda_r = \frac{4-\pi}{2} - \frac{\pi\delta_r^2(1-\sqrt{2})^2}{4} - \delta_r[1 + \frac{\pi(\sqrt{2}-2)}{2}]$. If we denote

$$t_k(r) = (-1)^{k-1}\left[(2-k) - k\alpha\left(\frac{n-2r+1}{n+1}\right)\right],$$

then, the cdf and pdf defined in (2) and (3) can be written as follows

$$F_{[r]r}(y|\sigma_2) = \sum_{k=1}^{2} t_k(r)\frac{1}{k}\left(1 - e^{-\frac{ky^2}{2\sigma_2^2}}\right),$$

$$f_{[r]r}(y|\sigma_2) = \sum_{k=1}^{2} t_k(r)\frac{y}{\sigma_2^2}e^{-\frac{ky^2}{2\sigma_2^2}}.$$

Since $Y_{[r]r}, r = 1, 2, \dots, n$ are independent we have the joint pdf function of the ranked set sample $\mathbf{Y}^{RSS} = (Y_{[1]1}, Y_{[2]2}, \dots, Y_{[n]n})$ as follows

$$\begin{aligned} f^{RSS}(\mathbf{y}|\sigma_2) &= \prod_{r=1}^{n} f_{[r]r}(y_r|\sigma_2) = \prod_{r=1}^{n}\sum_{k=1}^{2} t_k(r)\frac{y_r}{\sigma_2^2}e^{-\frac{ky_r^2}{2\sigma_2^2}} \\ &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left[\prod_{r=1}^{n} t_{i_r}(r)\frac{y_r}{\sigma_2^2}e^{-\frac{i_r y_r^2}{2\sigma_2^2}}\right] \\ &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} t_{i_r}(r)y_r\right]\frac{1}{\sigma_2^{2n}}e^{-\frac{\sum_{r=1}^{n} i_r y_r^2}{2\sigma_2^2}}\right\}. \end{aligned}$$

Let us assume that the pdf of σ_2 follows the S.R.IGamma prior. Hence, the posterior pdf can be obtained as follows

$$\begin{aligned} \pi^{RSS}(\sigma_2|\mathbf{y}) &= \frac{f^{RSS}(\mathbf{y}|\sigma_2)\pi(\sigma_2)}{\int_0^\infty f^{RSS}(\mathbf{y}|\sigma_2)\pi(\sigma_2)d\sigma_2} \\ &= \frac{\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} t_{i_r}(r)y_r\right]\frac{1}{\sigma_2^{2n+2b+1}}\exp\left\{\frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma_2^2}\right\}\right\}}{\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} t_{i_r}(r)y_r\right]\left[\sum_{r=1}^{n} i_r y_r^2 + a\right]^{-n-b}\Gamma(n+b)2^{n+b-1}\right\}}, \end{aligned} \tag{5}$$

and the Bayesian estimate of σ_2 based on the square error loss (SEL) function is obtained from (5) as

$$\hat{\sigma_2}^{RRS} = \int_0^\infty \sigma_2 \pi^{RSS}(\sigma_2|\mathbf{y})d\sigma_2$$
$$= \frac{\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} t_{i_r}(r)y_r\right]\left[\sum_{r=1}^{n} i_r y_r^2 + a\right]^{-n-b+\frac{1}{2}} \Gamma(n+b-\frac{1}{2})\right\}}{\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} t_{i_r}(r)y_r\right]\left[\sum_{j=1}^{n} i_r y_r^2 + a\right]^{-n-b} 2^{(\frac{1}{2})}\Gamma(n+b)\right\}}. \tag{6}$$

The Bayesian estimate of σ_2 based on the linear-exponential (LINEX) loss function is given by

$$\tilde{\sigma_2}^{RRS} = -\frac{1}{C}\ln(E[e^{-C\sigma_2}|\mathbf{Y}^{RRS}]), \tag{7}$$

where

$$E[e^{-C\sigma_2}|\mathbf{Y}^{RRS}] = \int_{\sigma_2} e^{-C\sigma_2}\pi^{RSS}(\sigma_2|\mathbf{y})d\sigma_2. \tag{8}$$

Note that the integral (8) cannot be found analytically.

The Bayes estimators of σ_2 based on non-informative prior is obtained by putting $a = b = 0$ in (6).

2.2. Bayesian Estimation Based on Bivariate Upper Extreme RSS

We can provided a ranked set sample of size n by each sample measurement of Y which is taken on the unit that has the maximum value for the X variable. Let $Y_{[n]r}$ be concomitant of largest order statistics $X_{(n)r}$ of the rth sample for $r = 1, 2, \ldots, n$. Then, $Y_{[n]1}, Y_{[n]2}, \ldots, Y_{[n]n}$ are the observations of the interest variable Y based on bivariate upper extreme ranked set sample (ERSS). In other words, the procedure of bivariate ERSS is as follows:

Step 1. Draw n independent sets, each is based on a simple random bivariate sample of size n from the population (X, Y).

Step 2. Rank the units within each sample with respect to an auxiliary variable X together with the associated variable Y.

Step 3. Measure the nth observation of the rth set $(X_{(n)r}, Y_{[n]r})$, $r = 1, 2, \ldots, n$, where $X_{(n)r}$ is the largest order statistics of the rth sample and $Y_{[n]r}$ is the concomitant of $X_{(n)r}$.

From (4), the mean and variance of $Y_{[n]r}$ are given as $E(Y_{[n]r}) = \sigma_2\beta_n$ and $Var[Y_{[n]r}] = \sigma_2^2\lambda_n$, respectively. Also, for $1 \leq r < s \leq n$, $Cov[Y_{[n]r}, Y_{[n]s}] = 0$.

From (2) and (3) the cdf and pdf of $Y_{[n]r}, r = 1, 2, \ldots, n$ are given by

$$F_{[n]r}(y|\sigma_2) = (1 - e^{-\frac{y^2}{2\sigma_2^2}}) \left[1 - \alpha \left(\frac{n-1}{n+1}\right)(e^{-\frac{y^2}{2\sigma_2^2}} - 1)\right],$$

$$f_{[n]r}(y|\sigma_2) = \frac{y}{\sigma_2^2} e^{-\frac{y^2}{2\sigma_2^2}} \left[1 - \alpha \left(\frac{n-1}{n+1}\right)(2e^{-\frac{y^2}{2\sigma_2^2}} - 1)\right], \quad y > 0. \tag{9}$$

If we denote

$$w_k = (-1)^{k-1} \left[(2-k) + k\alpha \left(\frac{n-1}{n+1}\right)\right],$$

then the pdf defined in (9) can be written as follows

$$f_{[n]r}(y|\sigma_2) = \sum_{k=1}^{2} w_k \frac{y}{\sigma_2^2} e^{-\frac{ky^2}{2\sigma_2^2}}.$$

Now, the joint pdf function of the ERSS, $\mathbf{Y}^{ERSS} = (Y_{[n]1}, Y_{[n]2}, \ldots, Y_{[n]n})$ is given by

$$\begin{aligned} f^{ERSS}(\mathbf{y}|\sigma_2) &= \prod_{r=1}^{n} f_{[n]r}(y_r|\sigma_2) = \prod_{r=1}^{n} \sum_{k=1}^{2} w_k \frac{y_r}{\sigma_2^2} e^{-\frac{ky_r^2}{2\sigma_2^2}} \\ &= \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \left[\prod_{r=1}^{n} w_{i_r} \frac{y_r}{\sigma_2^2} e^{-\frac{i_r y_r^2}{2\sigma_2^2}}\right] \\ &= \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \left\{ \left[\prod_{r=1}^{n} w_{i_r} y_r\right] \frac{1}{\sigma_2^{2n}} e^{-\frac{\sum_{r=1}^{n} i_r y_r^2}{2\sigma_2^2}} \right\}. \end{aligned} \tag{10}$$

Let us assume that the pdf prior of σ_2 is given by (1). Hence, the posterior pdf can be obtained as follows

$$\pi^{ERRS}(\sigma_2|\mathbf{y}) = \frac{\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \left\{ \left[\prod_{r=1}^{n} w_{i_r} y_r\right] \frac{1}{\sigma_2^{2n+2b+1}} \exp\left\{ \frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma_2^2} \right\} \right\}}{\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \left\{ \left[\prod_{r=1}^{n} w_{i_r} y_r\right] \left[\sum_{r=1}^{n} i_r y_r^2 + a\right]^{-n-b} \Gamma(n+b) 2^{n+b-1} \right\}}, \tag{11}$$

and the Bayesian estimate of σ_2 is obtained from (11) as follows

$$\hat{\sigma_2}^{ERRS} = \frac{\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \left[\prod_{r=1}^{n} w_{i_r} y_r\right] \left[\sum_{r=1}^{n} i_r y_r^2 + a\right]^{-n-b+\frac{1}{2}} \Gamma(n+b-\frac{1}{2})}{\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \left[\prod_{r=1}^{n} w_{i_r} y_r\right] \left[\sum_{j=1}^{n} i_r y_r^2 + a\right]^{-n-b} 2^{(\frac{1}{2})} \Gamma(n+b)}.$$

2.3. Bayesian Estimation Based on Bivariate MRSSU

In the previous methods, one chooses n^2 independent bivariate units, arranges them randomly into n sets with equal size, but in this method selecting $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, independent bivariate units, arrange them randomly into n sets such that the size of rth set is r.

The procedure of MRSSU for bivariate (X, Y), where X is an auxiliary variable and Y the study variable is as follows:

Step 1. Draw n independent sets, each is based on a simple random bivariate sample, such that the size of rth set is r.

Step 2. Rank the units within each sample with respect to an auxiliary variable X together with the associated variable Y.

Step 3. In the rth sample of size r, select the unit $(X_{r:r}, Y_{[r]:r})$, $r = 1, 2, \ldots, n$, where $X_{r:r}$ is the maximum of the rth sample and $Y_{[r]:r}$ be concomitant of the largest order statistics $X_{r:r}$.

Also, the following procedure can represent bivariate MRSSU:

$$
\begin{array}{cccccc}
\underline{(X_{1:1}, Y_{[1]:1})} & & & & \rightarrow & (X_{1:1}, Y_{[1]:1}) \\
(X_{1:2}, Y_{[1]:2}) & \underline{(X_{2:2}, Y_{[2]:2})} & & & \rightarrow & (X_{2:2}, Y_{[2]:2}) \\
\vdots & \vdots & \vdots & \vdots & \rightarrow & \vdots \\
(X_{1:n}, Y_{[1]:n}) & (X_{2:n}, Y_{[2]:n}) & \cdots & \underline{(X_{n:n}, Y_{[n]:n})} & \rightarrow & (X_{n:n}, Y_{[n]:n})
\end{array}
$$

As mentioned earlier, $X_{r:r}$'s are independent and non-identically distributed (INID). The pdf and cdf of $X_{r:r}$'s are derived as follows:

$$f_{r:r}(x) = rf(x)[F(x)]^{r-1}, \quad -\infty < x < \infty,$$

and

$$F_{r:r}(x) = [F(x)]^{r}, \quad -\infty < x < \infty,$$

respectively. $Y_{[1]:1}, Y_{[2]:2}, \ldots, Y_{[n]:n}$ be the observations of the concomitant variable Y based on bivariate MRSSU. From (2) and (3) the cdf and pdf of $Y_{[r]:r}, r = 1, 2, \ldots, n$ are given by

$$F_{[r]:r}(y|\sigma_2) = (1 - e^{-\frac{y^2}{2\sigma_2^2}}) \left[1 - \alpha\left(\frac{r-1}{r+1}\right)(e^{-\frac{y^2}{2\sigma_2^2}} - 1)\right],$$

$$f_{[r]:r}(y|\sigma_2) = \frac{y}{\sigma_2^2} e^{-\frac{y^2}{2\sigma_2^2}} \left[1 - \alpha\left(\frac{r-1}{r+1}\right)(2e^{-\frac{y^2}{2\sigma_2^2}} - 1)\right], \quad y > 0, \tag{12}$$

respectively. If we denote

$$v_k(r) = (-1)^{k-1}\left[(2-k) + k\alpha\left(\frac{r-1}{r+1}\right)\right],$$

then the pdf defined in (12) can be written as follows

$$f_{[r]:r}(y|\sigma_2) = \sum_{k=1}^{2} v_k(r)\frac{y}{\sigma_2^2}e^{-\frac{ky^2}{2\sigma_2^2}}.$$

Now, the joint pdf function of $\mathbf{Y}^{MRSSU} = (Y_{[1]:1}, Y_{[2]:2}, \dots, Y_{[n]:n})$ is given by

$$\begin{aligned} f^{MRSSU}(\mathbf{y}|\sigma_2) &= \prod_{r=1}^{n} f_{[r]:r}(y_r|\sigma_2) = \prod_{r=1}^{n}\sum_{k=1}^{2} v_k(r)\frac{y_r}{\sigma_2^2}e^{-\frac{ky_r^2}{2\sigma_2^2}} \\ &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left[\prod_{r=1}^{n} v_{i_r}(r)\frac{y_r}{\sigma_2^2}e^{-\frac{i_r y_r^2}{2\sigma_2^2}}\right] \\ &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left[\prod_{r=1}^{n} v_{i_r}(r)y_r\right]\frac{1}{\sigma_2^{2n}}e^{-\frac{\sum_{r=1}^{n} i_r y_r^2}{2\sigma_2^2}}. \end{aligned} \quad (13)$$

By using the prior pdf of σ_2 in (1), the posterior pdf can be obtained as follows

$$\pi^{MRSSU}(\sigma_2|\mathbf{y}) = \frac{\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} v_{i_r}(r)y_r\right]\frac{1}{\sigma_2^{2n+2b+1}}\exp\left\{\frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma_2^2}\right\}\right\}}{\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} v_{i_r}(r)y_r\right]\left[\sum_{r=1}^{n} i_r y_r^2 + a\right]^{-n-b}\Gamma(n+b)2^{n+b-1}\right\}},$$

and the Bayesian estimate of σ_2 is given by

$$\hat{\sigma_2}^{MRSSU} = \frac{\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\dots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} v_{i_r}(r)y_r\right]\left[\sum_{r=1}^{n} i_r y_r^2 + a\right]^{-n-b+\frac{1}{2}}\Gamma(n+b-\frac{1}{2})\right\}}{\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\dots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} v_{i_r}(r)y_r\right]\left[\sum_{j=1}^{n} i_r y_r^2 + a\right]^{-n-b} 2^{(\frac{1}{2})}\Gamma(n+b)\right\}}.$$

3. Bayesian Estimation of σ_2 when α is Unknown

In this Section, we consider Bayesian estimation of σ_2 based on bivariate RSS, ERSS and MRSSU when the association parameter α is unknown. Since α lies between -1 and 1 we assume a $Uniform(-1,1)$ prior for α and the pdf prior of σ_2 is given by (1), independent of α. Using the joint pdf function given in (2.1), (10), (13) and (based on RSS, ERSS and MRSSU respectively) and the joint prior densities of σ_2 and α, we obtain the joint pdf of $(data, \sigma_2, \alpha)$ as follows

$$f(data|\sigma_2, \alpha) \times \pi_1(\sigma_2) \times \pi_2(\alpha).$$

Thus the joint posterior pdf of σ_2 and α is given by

$$\pi(\sigma_2, \alpha|\mathbf{y}) = \frac{f(\mathbf{y}|\sigma_2, \alpha)\pi_1(\sigma_2)\pi_2(\alpha)}{\iint f(\mathbf{y}|\sigma_2, \alpha)\pi_1(\sigma_2)\pi_2(\alpha)d\alpha d\sigma_2}. \quad (14)$$

Therefore, the Bayes estimator of any function of α and σ_2 say $g(\alpha, \sigma_2)$, under squared error loss function is given by

$$E_{\sigma_2,\alpha}[g(\sigma_2,\alpha)|\mathbf{Y}] = \frac{\iint g(\sigma_2,\alpha) f(\mathbf{y}|\sigma_2,\alpha)\pi_1(\sigma_2)\pi_2(\alpha)d\alpha d\sigma_2}{\iint f(\mathbf{y}|\sigma_2,\alpha)\pi_1(\sigma_2)\pi_2(\alpha)d\alpha d\sigma_2}. \tag{15}$$

It is not possible to compute the integral in (15). Thus we propose the MCMC technique to generate sample from the posterior distribution and then compute the Bayes estimator of σ_2 and α. We use Gibbs sampling procedure to generate sample from the posterior pdf.

The joint posterior pdf of σ_2 and α given in (14) based on RSS can be written as

$$\begin{aligned} \pi^{RSS}(\sigma_2,\alpha|\mathbf{y}) \quad &\propto \quad f^{RSS}(\mathbf{y}|\sigma_2,\alpha)\pi_1(\sigma_2)\pi_2(\alpha) \\ \propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2} &\left\{ \left[\prod_{r=1}^{n} t_{i_r}(r) y_r\right] \frac{1}{\sigma_2^{2n+2b+1}} \exp\left\{ \frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma^2} \right\} \right\}. \end{aligned} \tag{16}$$

From (16) the conditional posterior pdf of σ_2 is given by

$$\pi^{RSS}(\sigma_2|\mathbf{y},\alpha) \propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2} \left\{ \left[\prod_{r=1}^{n} t_{i_r}(r) y_r\right] \frac{1}{\sigma_2^{2n+2b+1}} \exp\left\{ \frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma^2} \right\} \right\},$$

and the conditional posterior pdf of α is given by

$$\pi^{RSS}(\alpha|\mathbf{y},\sigma_2) \propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2} \left\{ \left[\prod_{r=1}^{n} t_{i_r}(r) y_r\right] \exp\left\{ \frac{-\sum_{r=1}^{n} i_r y_r^2}{2\sigma^2} \right\} \right\}.$$

Since upper ERSS and MRSSU are considered when $\alpha > 0$, we assume a $Uniform(0,1)$ prior for α and the pdf (1) for σ_2. Similarly, the joint posterior densities of σ_2 and α based on ERSS and MRSSU can be written as follows $\pi^{ERSS}(\sigma_2,\alpha|\mathbf{y}) \propto f^{ERSS}(\mathbf{y}|\sigma_2,\alpha)\pi_1(\sigma_2)\pi_2(\alpha)$

$$\propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2} \left\{ \left[\prod_{r=1}^{n} w_{i_r}(r) y_r\right] \frac{1}{\sigma_2^{2n+2b+1}} \exp\left\{ \frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma^2} \right\} \right\}, \tag{17}$$

$\pi^{MRSSU}(\sigma_2,\alpha|\mathbf{y}) \propto f^{MRSSU}(\mathbf{y}|\sigma_2,\alpha)\pi_1(\sigma_2)\pi_2(\alpha)$

$$\propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2} \left\{ \left[\prod_{r=1}^{n} v_{i_r}(r) y_r\right] \frac{1}{\sigma_2^{2n+2b+1}} \exp\left\{ \frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma^2} \right\} \right\}, \tag{18}$$

respectively.

From (17) and (18) the conditional posterior densities of σ_2 based on ERSS and MRSSU are given by

$$\pi^{ERSS}(\sigma_2|\mathbf{y},\alpha) \propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} w_{i_r}(r)y_r\right]\frac{1}{\sigma_2^{2n+2b+1}}\exp\left\{\frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma^2}\right\}\right\}, \quad (19)$$

$$\pi^{MRSSU}(\sigma_2|\mathbf{y},\alpha) \propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} v_{i_r}(r)y_r\right]\frac{1}{\sigma_2^{2n+2b+1}}\exp\left\{\frac{-\sum_{r=1}^{n} i_r y_r^2 - a}{2\sigma^2}\right\}\right\}.$$

Also, the conditional posterior densities of α are given by

$$\pi^{ERSS}(\alpha|\mathbf{y},\sigma_2) \propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} w_{i_r}(r)y_r\right]\exp\left\{\frac{-\sum_{r=1}^{n} i_r y_r^2}{2\sigma^2}\right\}\right\},$$

$$\pi^{MRSSU}(\alpha|\mathbf{y},\sigma_2) \propto \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\cdots\sum_{i_n=1}^{2}\left\{\left[\prod_{r=1}^{n} v_{i_r}(r)y_r\right]\exp\left\{\frac{-\sum_{r=1}^{n} i_r y_r^2}{2\sigma^2}\right\}\right\}.$$

Since no full conditional densities are known analytically based on concomitant RSS, ERSS and obviously based on bivariate MRSSU, we use the Metropolis-Hastings (M-H) algorithm to draw samples from the joint posterior distributions. By assuming truncated normal proposal distribution $q_\alpha \propto N(\alpha^{(i-1)}, \tau^2)I(0 < \alpha < 1)$ for α and $q_{\sigma_2} \propto N({\sigma_2}^{(i-1)}, \tau^2)I(\sigma_2 > 0)$ for σ_2, the sampler works as follows:

1. We begin with an initial guess value $\theta^{(0)} = (\sigma_2^{(0)}, \alpha^{(0)})$ and set $i = 1$.
2. Accepted the generated $\alpha^{(i)}$ from its proposal pdf with probability:

$$\rho_1 = \min\left\{\frac{\pi(\alpha^{(i)}|\sigma_2^{(i-1)},\mathbf{y})}{\pi(\alpha^{(i-1)}|\sigma_2^{(i-1)},\mathbf{y})}\frac{q_\alpha(\alpha^{(i-1)}|\sigma_2^{(i-1)},\alpha^{(i)},\mathbf{y})}{q_\alpha(\alpha^{(i)}|\sigma_2^{(i-1)},\alpha^{(i-1)},\mathbf{y})}, 1\right\}.$$

3. Accepted the generated $\sigma_2^{(i)}$ from its proposal pdf with probability:

$$\rho_2 = \min\left\{\frac{\pi(\sigma_2^{(i)}|\alpha^{(i)},\mathbf{y})}{\pi(\sigma_2^{(i-1)}|\alpha^{(i)},\mathbf{y})}\frac{q_{\sigma_2}({\sigma_2}^{(i-1)}|\sigma_2^{(i)},\mathbf{y})}{q_{\sigma_2}({\sigma_2}^{(i)}|\sigma_2^{(i-1)},\mathbf{y})}, 1\right\}.$$

4. Set $i = i + 1$.
5. Repeat steps 2-4, N times.

The value of τ should be fixed so that the optimized value of ρ is between $(0.3, 0.7)$. Now the approximation of Bayesian estimation of σ_2 can be obtained using the following equations

$$\hat{\sigma_2} = \hat{E}(\sigma_2|\mathbf{y}) = \frac{1}{N-M}\sum_{i=M+1}^{N}{\sigma_2}^{(i)},$$

where, M is the burn-in period. Also, for α, we have

$$\hat{\alpha} = \hat{E}(\alpha|\mathbf{y}) = \frac{1}{N-M}\sum_{i=M+1}^{N}\alpha^{(i)}.$$

***Note* 1.** When the posteriors are not known analytically, then one can use the method of Lindley (1980) as another technique for providing the Bayes estimator for the parameter.

4. Simulation Study

The simulation study are based on 1000 Monte Carlo runs, and 5000 samples generated by using the M-H algorithm and discard the first $M = 1000$ values as burn-in period. The mean square error and bias of Bayesian estimation of σ_2 are obtained based on concomitant RSS, ERSS and MRSSU for values $\sigma_2 = .5$, $\alpha \in 0.25(0.25)0.75$. We assume that the prior distribution of σ_2 follows S.R.IGamma(1,1). Also, the non informative prior is obtained by putting $a = 0$ and $b = 0$ in (1).

The simulation study is repeated for $n = 5(1)8$ when α is known and then for $n = 2(1)5$ when α is unknown and the results are presented in Table 1 and Table 2, respectively.

From Table 1 it can be observed that in most cases the estimates of σ_2 based on concomitant ERSS and then MRSSU have the least MSE and bias. It is also observed that the bias and MSE of all estimators based on RSS, ERSS and MRSSU decrease with increasing n. From this Table a much variation between estimators based on S.R.IGamma(1,1) prior and non-informative prior cannot be seen. Also the estimators based on both priors follow the same efficiency pattern but estimators based on S.R.IGamma(1,1) give a slight improvement over non-informative prior.

Note that, when α is unknown, from Table 2 it is observed that the Bayes estimators for σ_2 based on concomitant MRSSU have MSE and bias smaller than based on the other scheme sampling. Thus one can conclude that the estimators of σ_2 based on MRSSU perform better than estimators based on RSS and ERSS when α is positive. But the bias and MSE of almost all of estimators of α based on concomitant RSS are smaller than that of estimators based on ERSS and MRSSU. In addition, as n increases, the MSE and bias values for both parameters σ_2 and α decrease.

Table 1: MSE and biased of Bayes estimators of σ_2 when α is known.

n	α	Sampling	S.R.IGamma		non-informative	
			MSE	biased	MSE	biased
5	0.25	RSS	0.0137	0.0728	0.0157	0.0325
		ERSS	0.0133	0.0689	0.0156	0.0308
		MRSSU	0.0126	0.0689	0.0147	0.0295
	0.5	RSS	0.0132	0.0707	0.0154	0.030
		ERSS	0.0107	0.0555	0.0130	0.0279
		MRSSU	0.0117	0.0655	0.0132	0.0283
	0.75	RSS	0.0160	0.0817	0.0141	0.0270
		ERSS	0.0102	0.0511	0.0114	0.0230
		MRSSU	0.0106	0.0603	0.0130	0.0261
6	0.25	RSS	0.0115	0.0671	0.0122	0.0323
		ERSS	0.0079	0.0503	0.0112	0.0291
		MRSSU	0.0103	0.0506	0.0116	0.0287
	0.5	RSS	0.0111	0.0602	0.0141	0.0206
		ERSS	0.0093	0.0496	0.0127	0.0198
		MRSSU	0.0099	0.0564	0.0092	0.0195
	0.75	RSS	0.0124	0.0694	0.0134	0.0225
		ERSS	0.0087	0.0519	0.0092	0.0212
		MRSSU	0.0100	0.0432	0.0103	0.0207
7	0.25	RSS	0.0098	0.0574	0.0121	0.0189
		ERSS	0.0080	0.0412	0.0098	0.0161
		MRSSU	0.0080	0.0482	0.0083	0.0185
	0.5	RSS	0.0084	0.0491	0.0096	0.0171
		ERSS	0.0059	0.0351	0.0104	0.0169
		MRSSU	0.0087	0.0470	0.0064	0.0160
	0.75	RSS	0.0081	0.0477	0.0115	0.0161
		ERSS	0.0052	0.0309	0.0072	0.0118
		MRSSU	0.0081	0.0438	0.0081	0.0125
8	0.25	RSS	0.0084	0.0485	0.0092	0.0182
		ERSS	0.0074	0.0300	0.0071	0.0122
		MRSSU	0.0081	0.0486	0.0080	0.0142
	0.5	RSS	0.0083	0.0424	0.0082	0.0184
		ERSS	0.0059	0.0319	0.0073	0.0147
		MRSSU	0.0057	0.0357	0.0079	0.0173
	0.75	RSS	0.0073	0.0406	0.0073	0.0156
		ERSS	0.0047	0.0244	0.0065	0.0125
		MRSSU	0.0069	0.0403	0.0066	0.0127

Table 2: MSE and biased of Bayes estimators when α is unknown.

n	α	Sampling	σ_2		α	
			MSE	biased	MSE	biased
2	0.25	RSS	0.0304	0.1422	0.0612	0.2460
		ERSS	0.0279	0.1343	0.0595	0.2431
		MRSSU	0.0187	0.1110	0.0611	0.2460
	0.5	RSS	0.0416	0.1751	0.0632	-0.2500
		ERSS	0.0358	0.1494	0.0663	-0.2570
		MRSSU	0.0251	0.1286	0.0637	-0.2508
	0.75	RSS	0.0366	0.1565	0.0640	-0.2516
		ERSS	0.0363	0.1629	0.0695	-0.2625
		MRSSU	0.0187	0.1132	0.0688	-0.2579
3	0.25	RSS	0.0248	0.1132	0.0638	0.2513
		ERSS	0.0157	0.0768	0.0536	0.2298
		MRSSU	0.0122	0.0766	0.0598	0.2427
	0.5	RSS	0.0274	0.1185	0.0007	-0.0027
		ERSS	0.0190	0.0953	0.0013	-0.0177
		MRSSU	0.0122	0.0798	0.0013	-0.0112
	0.75	RSS	0.0229	0.1131	0.0640	-0.2516
		ERSS	0.0256	0.1221	0.0695	-0.2625
		MRSSU	0.0163	0.0936	0.0688	-0.2579
4	0.25	RSS	0.0196	0.0982	0.0652	0.2540
		ERSS	0.0100	0.0565	0.0503	0.2212
		MRSSU	0.0093	0.0515	0.0592	0.2414
	0.5	RSS	0.0131	0.0642	0.0008	0.0018
		ERSS	0.0182	0.0861	0.0020	-0.0242
		MRSSU	0.0096	0.0544	0.0012	-0.0068
	0.75	RSS	0.0149	0.0815	0.0600	-0.243
		ERSS	0.0202	0.1014	0.0728	-0.2678
		MRSSU	0.0102	0.0696	0.0686	-0.2600
5	0.25	RSS	0.01017	0.0616	0.0647	0.2532
		ERSS	0.0088	0.0464	0.0502	0.2186
		MRSSU	0.0079	0.0377	0.0562	0.2307
	0.5	RSS	0.0114	0.0652	0.0007	0.0073
		ERSS	0.0109	0.0578	0.0037	-0.0296
		MRSSU	0.0082	0.0473	0.0022	-0.0120
	0.75	RSS	0.0146	0.0799	0.0621	-0.2475
		ERSS	0.0142	0.0773	0.0709	-0.2627
		MRSSU	0.0095	0.0566	0.0722	-0.2646

5. Real Data Analysis

In order to compare the performance of Bayesian estimates of the parameter σ_2 we consider a bivariate data set from the 44 women data included the body weight(kg) (X) and the resting metabolic rate (RMR) (kcal/24 hr) (Y) reported by Owen & et al. (1986). Since correlation between X and Y of the available observations is 0.74, in addition to RSS we derived the Bayes estimator based on ERSS and MRSSU. By taking samples of size 4 from two data sets based on concomitant RSS, ERSS and MRSSU, Bayesian estimates of σ_2 are obtained. The obtained RSS, ERSS and MRSSU observations are given in Table 3.

TABLE 3: The data based on RSS, ERSS and MRSSU from X and Y; $n = 4$.

RSS		ERSS		MRSSU	
$X_{(r)r}$	$Y_{[r]r}$	$X_{(n)r}$	$Y_{[n]r}$	$X_{r:r}$	$Y_{[r]:r}$
55.0	1034	83.4	1248	104.5	1414
57.8	1090	143.3	1708	143.3	1708
86.2	1466	125.2	1630	123.1	1640
62.3	1402	62.3	1402	74.8	1273

For the Bayesian estimations we used the prior pdf in (1) with $a = b = 0.0001$ to σ_2. The results are reported in Table 4. We can find that the estimated values for σ_2 based on different samplings are close together.

To obtain an estimator for the mean of study variable Y, we have obtained the sample mean $\bar{Y}$ of the 44 available Y observations and is given by $\bar{Y} = 1339.84$. From Table 4, we can see that Bayes estimators based on MRSSU is closer to the $\bar{Y}$ than the other estimators. Note that when α is unknown while an estimator of σ_2 using MCMC method is obtained one can also get an estimator for α. This obtained estimator for α is included in Table 4. Since the sample correlation between all the available X and Y observations is 0.74, as given in Tahmasebi, Jafari & Ahsanullah (2017) a rough estimate of α is obtained as 1. From Table 4 it can be seen that the estimator of α based on RSS is more close to estimator based on sample correlation.

TABLE 4: The Bayesian estimations of σ_2 and α.

Sampling scheme-loss function	σ_2^*	Estimator for mean $\sigma_2^*\sqrt{\frac{\pi}{2}}$	α^*
RSS	982.90	1231.88	0.8338
ERSS	1063.49	1332.89	0.5475
MRSSU	1069.68	1340.64	0.5179

6. Conclusions

In this paper, Bayesian estimation of the scale parameter σ_2 for Morgenstern type bivariate Rayleigh distribution based on RSS, ERSS and MRSSU are considered. When the association parameter α is unknown Bayes estimators are derived using MCMC method. By a simulation study difference between the estimation procedures are compared. Simulation results demonstrate that when α is known, the Bayes estimates of σ_2 based on ERSS are performed better than their RSS and MRSSU counterparts. It is also observed that the bias and MSE of all estimators decrease with increasing n. Also, when α is unknown the performance of the Bayes estimators for σ_2 based on MRSSU is better than based on the other this study scheme sampling. But the bias and MSE of almost all of estimators of α based on RSS are smaller than that of estimators based on ERSS and MRSSU. In addition, as n increases, the MSE and bias values of all estimators for both parameters σ_2 and α decrease.

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