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A Reparameterized Weighted Lindley Distribution: Properties, Estimation and Applications

Una distribución de Lindley ponderada reparametrizada: propiedades, estimación y aplicaciones

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Abstract

In this paper, we discuss several mathematical properties and estimation methods for a reparameterized version of the weighted Lindley (RWL) distribution. The RWL distribution can be particularly useful for modeling reliability (survival) data with bathtub-shaped or increasing hazard rate function. The inferential procedure to obtain the parameter estimates is conducted via the maximum likelihood approach considering random right-censoring. Extensive numerical simulations are carried out to investigate and evaluate the performance of the proposed estimation method. Finally, the potentiality of the RWL model is analyzed by employing two real data sets.

***Key words*:** Lindley distribution; Monte Carlo simulation; Random right-censoring data; Weighted Lindley distribution.

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Resumen

En este artículo, discutimos varias propiedades matemáticas y métodos de estimación para una versión reparametrizada de la distribución ponderada de Lindley (RWL). La distribución RWL puede ser particularmente útil para modelar datos de confiabilidad (supervivencia) con función de tasa de riesgo en forma de bañera o creciente. El procedimiento inferencial para obtener las estimaciones de los parámetros se realiza mediante el enfoque de máxima verosimilitud considerando la censura aleatoria a la derecha. Se realizan extensas simulaciones numéricas para investigar y evaluar el rendimiento del método de estimación propuesto. Finalmente, la utilidad del modelo RWL se analiza mediante el uso de dos conjuntos de datos reales.

***Palabras clave*:** Datos censurados aleatorios a la derecha; Distribución de Lindley; Distribución ponderada de Lindley; Simulación Monte Carlo.

1. Introduction

The Lindley distribution is a lifetime distribution that was introduced in the context of fiducial distributions and Bayes theorem (Lindley 1958). Ghitany et al. (2008) studied its mathematical properties, such as moments, failure rate, mean residual life, entropy function, and asymptotic distribution of the extreme order statistics and inferential procedures. Moreover, the authors showed that such distribution outperforms the exponential model in many situations, which allowed its application in diverse areas, such as biology, engineering, and medicine.

Since the Lindley distribution has only one parameter and accommodates solely increasing hazard function, it does not provide enough flexibility for analyzing different types of lifetime data. To increase the flexibility for modeling purposes, many generalizations based on this distribution have been proposed in the recent literature. For example, the generalized Lindley (Zakerzadeh & Dolati 2009), two-parameter weighted Lindley (Ghitany et al. 2011), extended Lindley (Bakouch et al. 2012), transmuted two-parameter Lindley (Kemaloglu & Yilmaz 2017), Weibull Lindley distribution (Asgharzadeh et al. 2018), Weibull Marshall-Olkin Lindley (Afify et al. 2020), among other distributions. The two-parameter weighted Lindley (WL) distribution has become increasingly popular for modeling survival or reliability data with bathtub-shaped and increasing hazard rate functions (Ali 2015, Ramos et al. 2017, Louzada & Ramos 2017). Some generalizations of this distribution can be found, e.g., in Asgharzadeh et al. (2016), Ramos & Louzada (2016), Shanker et al. (2019), and references cited therein.

In recent years, some traditional distributions have been reparameterized in terms of their mean to model real problems; see, e.g., Cepeda & Gamerman (2005), Santos-Neto et al. (2016), Rigby et al. (2019), Bourguignon & Gallardo (2020). In our bibliographical review, we noted that not much attention has been paid to parameterizations of the Lindley distribution, as well as its generalizations, except the work proposed by Mazucheli et al. (2016). The authors introduced an alternative parameterization for the WL distribution in the context of orthogonal parameters (Cox & Reid 1987). Such reparameterizated WL (RWL) distribution is very useful because one of its parameters is the mean, which is interpretable in many problems. For instance, in medical studies, it gives the mean survival time of patients, which can be related to a set of covariates. In additon, the other parameter can be interpreted as a precison parameter. Thus, if covariates are present in the data set, we can model practical situations where the precision is not constant (Cepeda & Gamerman 2005, Santos-Neto et al. 2016, Bourguignon & Gallardo 2020). Another appealing advantage of using the RWL distribution is due to computational stability. On the other hand, by using such distribution, we have that the sample mean is an unbiased estimator for the population mean, and hence, the precision parameter can be readily estimated by using a one-dimensional numerical method, avoiding numerical problems.

Although Mazucheli et al. (2016) have proposed the RWL distribution, the authors did not study its properties and, in addition, they also did not consider the maximum likelihood estimation for the parameters under censored data. Our objective in this paper is to derive and discuss many mathematical properties of this distribution, including its moments, quantile function, characteristic function, mean and median deviations, hazard rate function, mean residual life function, and Laplace transform function. Also, we show that the second parameter of this distribution can be interpreted as a precision parameter, which can be useful in further studies. The inference for the model parameters is conducted under the classical (or frequentist) framework via the maximum likelihood method assuming the presence of uncensored and random censored data. Numerical simulations are carried out in order to investigate the performance of the maximum likelihood estimators (MLEs) under different sample sizes and proportions of censored data. Finally, the applicability of the RWL distribution is illustrated in two real data sets.

The remainder of this paper is organized as follows. Section 2 reviews the RWL distribution. Section 3 present some properties of the RWL distribution, such as moments, quantile function, characteristic function, mean and median deviations, mean residual life function, and Laplace transform. Section 4 discusses the inferential procedure based on MLEs for complete and censored data. Section 5 presents a simulation study to evaluate the performance of the proposed estimators. Section 6 illustrates the relevance of the RWL distribution on two real data sets. Section 7 summarizes the present study.

2. The RWL Distribution

The original parameterization of the probability density function (PDF) of the WL distribution introduced by Ghitany et al. (2011) is given by

$$f(y;\lambda,\phi)=\frac{\lambda^{\phi+1}}{(\lambda+\phi)\Gamma(\phi)}y^{\phi-1}(1+y)\exp\{-\lambda y\},\quad y>0, \tag{1}$$

where $\lambda>0$ and $\phi>0$ are shape parameters and $\Gamma(\phi)=\int_0^\infty y^{\phi-1}\exp\{-y\}dy$ is the gamma function.

Recently, Mazucheli et al. (2016) proposed a new parameterization of the WL distribution, which allows diverse features of data modeling to be considered. The RWL distribution is obtained by transforming (λ, ϕ) into (μ, ϕ), where

$$\mu = \frac{\phi(\lambda + \phi + 1)}{\lambda(\lambda + \phi)}$$

is the mean of the original parameterization (1).

Therefore, the WL distribution reparameterized by its mean has PDF expressed as

$$f(y; \mu, \phi) = \frac{[a(\mu, \phi)]^{\phi+1}}{(2\mu)^{\phi}\Gamma(\phi)\,[a(\mu, \phi) + 2\mu\phi]} y^{\phi-1}(1 + y) \exp\left\{-\frac{a(\mu, \phi)}{2\mu} y\right\}, \quad y > 0, \tag{2}$$

where $a(\mu, \phi) = \phi(1-\mu) + \sqrt{\phi^2(\mu - 1)^2 + 4\mu\phi(\phi + 1)}$, $\mu > 0$ is the mean and $\phi > 0$ is the shape parameter. From now on, we will use the notation $Y \sim \text{RWL}(\mu, \phi)$ to indicate that the random variable Y has the RWL distribution.

Figure 1 presents examples of the PDF (2) considering different values of ϕ when μ is fixed, and different values of μ when ϕ is fixed. Note that decreasing and unimodal behavior can be seen for the PDF. Moreover, ϕ controls the shape of the PDF, as well as the different degrees of asymmetry and kurtosis, whereas μ is the mean and also a scale parameter of the distribution.

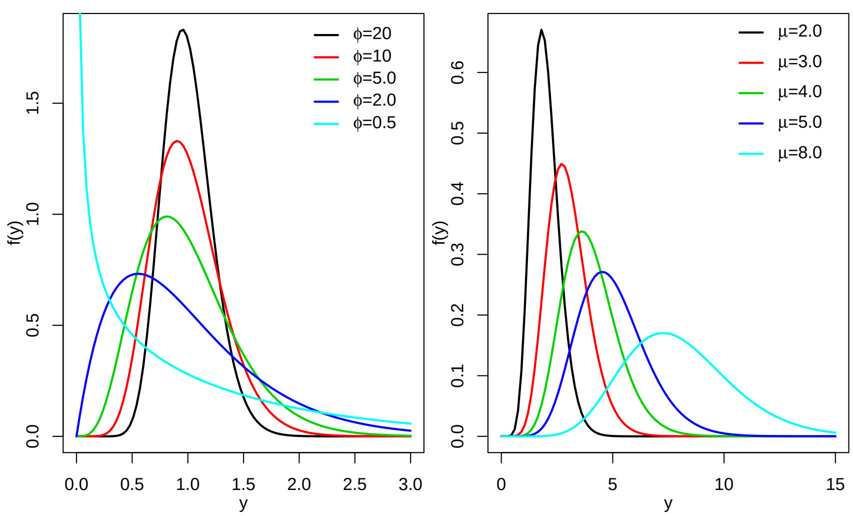

FIGURE 1: Plots of PDF of the RWL distribution. Left panel: $\mu = 1$ fixed and different values of ϕ. Right panel: $\phi = 10$ fixed and different values of μ.

As in original parameterization (1), we also can write the PDF (2) as a two-component mixture:

$$f(y; \mu, \phi) = \left(\frac{a(\mu, \phi)}{a(\mu, \phi) + 2\mu\phi}\right) f_1(y; \mu, \phi) + \left(\frac{2\mu\phi}{a(\mu, \phi) + 2\mu\phi}\right) f_2(y; \mu, \phi), \tag{3}$$

where $f_j(y;\mu,\phi) = \left(\frac{a(\mu,\phi)}{2\mu}\right)^{\phi+j-1} \frac{y^{\phi+j-2}}{\Gamma(\phi+j-1)} \exp\left\{-\frac{a(\mu,\phi)}{2\mu}y\right\}$, $y > 0$, is the PDF of the gamma distribution with shape parameter $\phi + j - 1$ and scale parameter $a(\mu,\phi)/2\mu$, for $j = 1, 2$. This mixture representation is useful in order to get the properties of the RWL distribution, since the properties of the gamma distribution are well known in the statistical literature (Johnson et al. 1994). For instance, the corresponding survival and hazard rate functions of the RWL distribution are easily found and given, respectively, by

$$S(y;\mu,\phi) = \frac{1}{\Gamma(\phi)}\left[\Gamma\left(\phi, \frac{a(\mu,\phi)}{2\mu}y\right) + \frac{[a(\mu,\phi)y]^{\phi}\exp\left\{-\frac{a(\mu,\phi)}{2\mu}y\right\}}{(2\mu)^{\phi-1}\left[a(\mu,\phi)+2\mu\phi\right]}\right] \tag{4}$$

and

$$h(y;\mu,\phi) = \frac{[a(\mu,\phi)]^{\phi+1}\, y^{\phi-1}(1+y)\exp\left\{-\frac{a(\mu,\phi)}{2\mu}y\right\}}{2\mu\left[(2\mu)^{\phi-1}\left[a(\mu,\phi)+2\mu\phi\right]\Gamma\left(\phi, \frac{a(\mu,\phi)}{2\mu}y\right) + [a(\mu,\phi)y]^{\phi}\exp\left\{-\frac{a(\mu,\phi)}{2\mu}y\right\}\right]} \tag{5}$$

where, for all $c > 0$ and $d \geq 0$,

$$\Gamma(c,d) = \int_d^{\infty} y^{c-1}e^{-y}dy$$

is the upper incomplete gamma function.

Figure 2 shows different shapes for the hazard rate function of the RWL distribution, considering distinct values of μ and ϕ. It can be noted that the hazard rate function has monotonically increased ($\phi \geq 1$) and bathtub ($\phi < 1$) shapes for all $\mu > 0$ (as the original WL distribution; see Ghitany et al. (2011)).

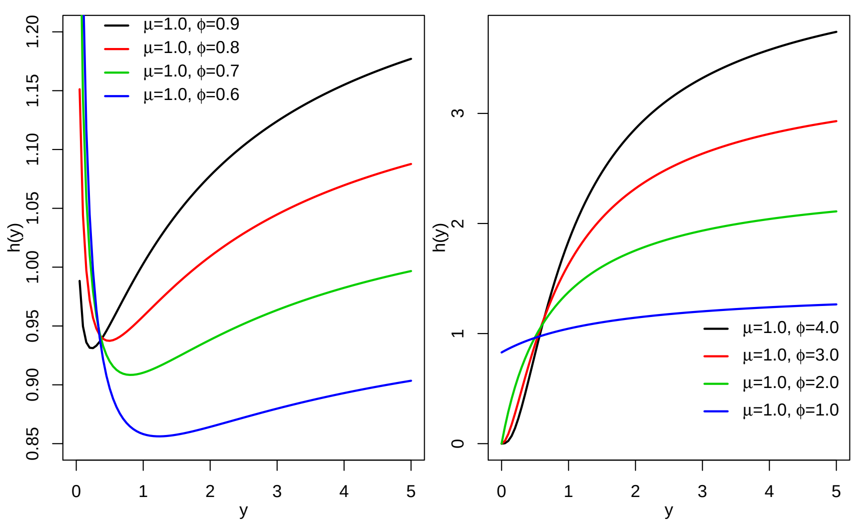

FIGURE 2: Plots of the hazard rate function of the RWL distribution.

3. Some Properties of the RWL Distribution

In this section, we present some mathematical properties of the RWL distribution, such as r-th moments, characteristic function, and Laplace transform, among others.

3.1. Quantile Function

The quantile function of a probability distribution is useful in statistical applications and Monte Carlo simulation. From (4), we have that the cumulative distribution function (CDF) of the RWL distribution is given by

$$F(y;\mu,\phi) = 1 - \frac{1}{\Gamma(\phi)}\left[\Gamma\left(\phi, \frac{a(\mu,\phi)}{2\mu}y\right) + \frac{[a(\mu,\phi)y]^{\phi}\exp\left\{-\frac{a(\mu,\phi)}{2\mu}y\right\}}{(2\mu)^{\phi-1}\left[a(\mu,\phi)+2\mu\phi\right]}\right]$$

Hence, the p-quantile, y_p, is obtained by solving the following equation:

$$\frac{[a(\mu,\phi)y_p]^{\phi}\exp\left\{-\frac{a(\mu,\phi)}{2\mu}y_p\right\}}{(2\mu)^{\phi-1}\left[a(\mu,\phi)+2\mu\phi\right]} = \Gamma(\phi)(1-p) - \Gamma\left(\phi, \frac{a(\mu,\phi)}{2\mu}y_p\right) \tag{6}$$

for $0 < p < 1$. Observe that if $p = 0.5$ we get the median of the RWL distribution.

Note that the quantile function does not have a closed mathematical expression. In this case, the `uniroot` function of the `R` software can be used to find out the desired quantiles of the data; see R Core Team (2020) and Brent (1973).

3.2. Moments

Many important characteristics and properties of a probability distribution can be obtained through its moments, such as mean, variance, skewness, and kurtosis.

Theorem 1. *If $Y \sim RWL(\mu,\phi)$, then the r-th power, logarithmic and negative moments are given, respectively, by*

(i) $$E\left[Y^r\right] = \left[\frac{2\mu}{a(\mu,\phi)}\right]^r \frac{\left[a(\mu,\phi)+2\mu\phi+2\mu r\right]\Gamma(\phi+r)}{\left[a(\mu,\phi)+2\mu\phi\right]\Gamma(\phi)}$$

(ii) $$E\left[\log\left(Y^r\right)\right] = r\left[\psi(\phi) + \frac{2\mu}{a(\mu,\phi)+2\mu\phi} - \log\left(\frac{a(\mu,\phi)}{2\mu}\right)\right];$$

(iii) $$E\left[Y^{-r}\right] = \left[\frac{a(\mu,\phi)}{2\mu}\right]^r \frac{\Gamma(\phi-r)\left[a(\mu,\phi)+2\mu(\phi-r)\right]}{\left[a(\mu,\phi)+2\mu\phi\right]\Gamma(\phi)}$$

where $\psi(k) = \dfrac{d}{dk}\log\left(\Gamma(k)\right)$ *is the digamma function.*

Proof. We will only prove the item (i) of Theorem 1 because the proof for the other remaining items follows similarly. In fact, let us use the mixture representation given in Equation (3). We then have

$$E[Y^r] = \left(\frac{a(\mu,\phi)}{a(\mu,\phi)+2\mu\phi}\right)\int_0^\infty y^r f_1(y;\mu,\phi)dy + \left(\frac{2\mu\phi}{a(\mu,\phi)+2\mu\phi}\right)\int_0^\infty y^r f_2(y;\mu,\phi)dy$$

where $f_j(y;\mu,\phi)$ is the PDF of the gamma distribution with shape parameter $\phi+j-1$ and scale parameter $\frac{a(\mu,\phi)}{2\mu}$, for $j=1,2$. Note that

$$\int_0^\infty y^r f_j(y;\mu,\phi)dy = \left(\frac{2\mu}{a(\mu,\phi)}\right)^r \frac{\Gamma(\phi+j+r-1)}{\Gamma(\phi+j-1)}, \quad j=1,2.$$

Thus, after some algebraic manipulations, we finish the proof of this theorem. □

Corollary 1. *The mean and variance of the random variable $Y \sim RWL(\mu,\phi)$ are given, respectively, by*

$$E[Y] = \mu \tag{7}$$

and

$$Var[Y] = \left[\frac{2\mu}{a(\mu,\phi)}\right]^2 \frac{[a(\mu,\phi)+2\mu\phi+4\mu]\,\phi(\phi+1)}{a(\mu,\phi)+2\mu\phi} - \mu^2. \tag{8}$$

Proof. These results can be obtained easily from item (i) of Theorem 1, considering $r=1$ and $r=2$ with some algebraic manipulations. □

Figure 3 displays a plot of ϕ against $Var[Y]$ for $\mu=1$ fixed. According to this figure, we can see that the ϕ parameter can be interpreted as a precision parameter. The variance increases as ϕ tend to zero, and it decreases when ϕ goes to infinity.

The coefficient of variation (CV) is used to analyze the dispersion in terms of their average value when two or more data sets have different units of measure. Thus, we can say that the CV is a way of expressing the variability of the data, excluding the influence of the variable's order of magnitude. Often, the CV is given in percentage.

It follows from Corollary 1 that the CV of $Y \sim \text{RWL}(\mu,\phi)$ is given by

$$CV[Y] = \frac{\sqrt{Var[Y]}}{E[Y]} = \sqrt{\frac{4\,[a(\mu,\phi)+2\mu\phi+4\mu]\,\phi(\phi+1)}{[a(\mu,\phi)]^2[a(\mu,\phi)+2\mu\phi]} - 1}.$$

The next corollary gives us the harmonic mean of the RWL distribution. This measure of central tendency can be useful in many real problems; see, e.g., Hasna & Alouini (2004), Limbrunner et al. (2000) and Raftery et al. (2006).

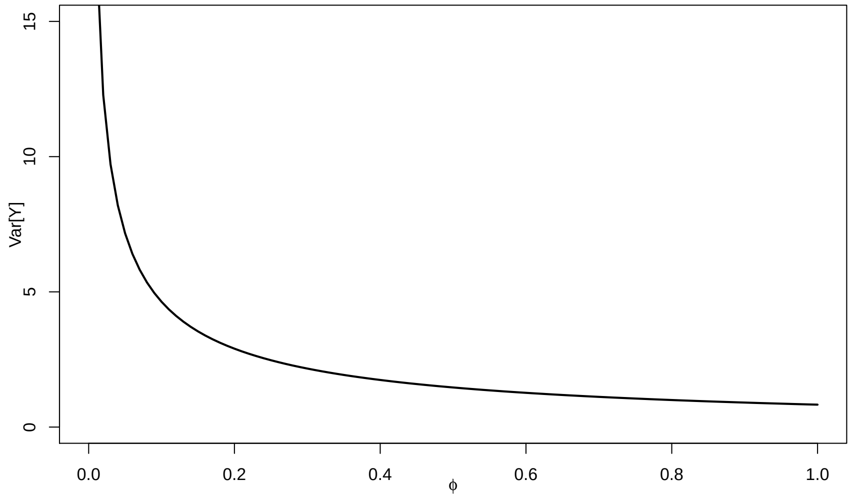

FIGURE 3: Plot of ϕ against $Var[Y]$ for $\mu = 1$ fixed.

Corollary 2. *The harmonic mean of the random variable* $Y \sim RWL(\mu, \phi)$ *is given by*

$$H_m = \left(E\left[\frac{1}{Y}\right]\right)^{-1} = \frac{2\mu\Gamma(\phi)[a(\mu,\phi) + 2\mu\phi]}{a(\mu,\phi)\Gamma(\phi-1)[a(\mu,\phi) + 2\mu(\phi-1)]}.$$

Proof. This result can be established by using the item (iii) of Theorem 1 with $r = 1$ and then taking the reciprocal of the resulting expression. □

Another way to characterize a distribution is by using its characteristic function (CF). The CF of a random variable is also known as Fourier transform of its PDF and has applications in the most diverse areas of scientific knowledge; see, e.g., Manolakis et al. (2005), Yu (2004) and Lukacs (1972).

Theorem 2. *If* $Y \sim RWL(\mu, \phi)$*, then its CF is given by*

$$\Psi_Y(s) = \left(\frac{1}{a(\mu,\phi) + 2\mu\phi}\right)\left(1 - \frac{2\mu is}{a(\mu,\phi)}\right)^{-\phi}\left[a(\mu,\phi) + 2\mu\phi\left(1 - \frac{2\mu is}{a(\mu,\phi)}\right)^{-1}\right],$$

for all $s \in \mathbb{R}$*, where* $i = \sqrt{-1}$ *is the imaginary unit.*

Proof. In fact, by the representation of mixture given in Equation (3), we have

$$\begin{aligned}\Psi_Y(s) = E[e^{isY}] &= \int_0^\infty e^{isy} f(y;\mu,\phi)dy \\ &= \left(\frac{a(\mu,\phi)}{a(\mu,\phi)+\phi}\right)\int_0^\infty e^{isy} f_1(y;\mu,\phi)dy \\ &\quad + \left(\frac{2\mu\phi}{a(\mu,\phi)+\phi}\right)\int_0^\infty e^{isy} f_2(y;\mu,\phi)dy.\end{aligned}$$

Now, as $f_j(y)$ is the PDF of gamma distribution with parameters $\phi + j - 1$ and $a(\mu,\phi)/2\mu$, $j = 1, 2$, we then obtain

$$\Psi_Y(s) = \left(\frac{a(\mu,\phi)}{a(\mu,\phi)+2\mu\phi}\right)\left(1-\frac{2\mu is}{a(\mu,\phi)}\right)^{-\phi} + \left(\frac{2\mu\phi}{a(\mu,\phi)+2\mu\phi}\right)\left(1-\frac{2\mu is}{a(\mu,\phi)}\right)^{-\phi-1}.$$

Now, after some algebraic manipulations, we get the desired result. □

The r-th power moments could also be obtained by using the CF. In fact, let $\Psi_Y^{(r)}(0)$ be the r-th derivative of $\Psi_Y(s)$ with respect to s, evaluated at the point $s = 0$. Then,

$$\Psi_Y^{(r)}(0) = \left.\frac{d\Psi_Y^{(r)}(s)}{ds}\right|_{s=0} = i^r E\left[Y^r\right].$$

3.3. Mean Residual Life Function

The mean residual life (MRL) function represents the expected additional lifetime given that a component has survived or not failed until time y. The MRL function is defined by

$$r(y;\boldsymbol{\theta}) = E\left[Y - y | Y > y\right] = \frac{1}{S(y;\boldsymbol{\theta})}\int_y^\infty t f(t;\boldsymbol{\theta})dt - y,$$

where $f(y;\boldsymbol{\theta})$ and $S(y;\boldsymbol{\theta})$ are, respectively, the PDF and survival function of the random variable Y, and $\boldsymbol{\theta}$ is the parameter vector.

Proposition 1. *The MRL function of the random variable $Y \sim RWL(\mu,\phi)$ is given by*

$$r(y;\mu,\phi) = \frac{2\mu}{[a(\mu,\phi)+2\mu\phi]\Gamma(\phi)S(y;\mu,\phi)}\left[\Gamma\left(\phi+1,\frac{a(\mu,\phi)y}{2\mu}\right) + \frac{2\mu\Gamma\left(\phi+2,\frac{a(\mu,\phi)y}{2\mu}\right)}{a(\mu,\phi)}\right] - y,$$

where $S(y;\mu,\phi)$ is the survival function defined in Equation (4).

Proof. By using the mixture representation given in Equation (3), we have

$$\int_y^\infty t f(t;\boldsymbol{\theta})dt = \left(\frac{a(\mu,\phi)}{a(\mu,\phi)+2\mu\phi}\right)\int_y^\infty t f_1(t;\mu,\phi)dt + \left(\frac{2\mu\phi}{a(\mu,\phi)+2\mu\phi}\right)\int_y^\infty t f_2(t;\mu,\phi)dt \tag{9}$$

where $f_j(t;\mu,\phi) = \left(\frac{a(\mu,\phi)}{2\mu}\right)^{\phi+j-1}\frac{t^{\phi+j-2}}{\Gamma(\phi+j-1)}\exp\left\{-\frac{a(\mu,\phi)}{2\mu}t\right\}$, for $j = 1, 2$.

Now, for $j = 1, 2$,

$$\begin{aligned}\int_y^\infty t f_j(t;\mu,\phi)dt &= \frac{[a(\mu,\phi)]^{\phi+j-1}}{(2\mu)^{\phi+j-1}\Gamma(\phi+j-1)}\int_y^\infty t^{\phi+j-1}\exp\left\{-\frac{a(\mu,\phi)}{2\mu}t\right\}dt\\ &= \frac{2\mu}{a(\mu,\phi)\Gamma(\phi+j-1)}\int_{\frac{a(\mu,\phi)y}{2\mu}}^\infty z^{\phi+j-1}\exp\{-z\}dz,\\ &= \frac{2\mu\Gamma\left(\phi+j,\frac{a(\mu,\phi)y}{2\mu}\right)}{a(\mu,\phi)\Gamma(\phi+j-1)}.\end{aligned} \tag{10}$$

where $z = \frac{a(\mu,\phi)t}{2\mu}$. Substituting Equation (10) into Equation (9), we can get the result after some algebraic manipulations. □

Figure 4 shows the possible shapes for the MRL function of the RWL distribution. Note that as the hazard rate function is bathtub-shaped (increasing), the MRL function has upside-down bathtub (decreasing) shape according to Bryson & Siddiqui (1969) and Olcay (1995).

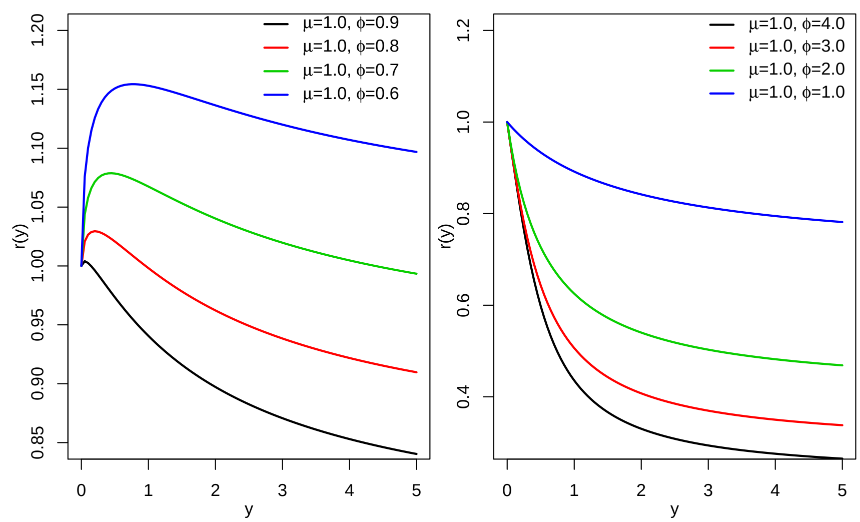

Figure 4: Plots of the MRL function of the RWL distribution.

3.4. Mean and Median Deviations

Mean and median deviations are useful for measuring the amount of scattering in a population. They are defined as follows.

Consider a random variable Y with PDF $f(y)$ and let μ and m denote, respectively, the mean and median of Y, that is, $\mu = E[Y]$ and $m = Median[Y]$. Then, the mean and median deviations are defined, respectively, by

$$\delta_1 = \int_0^\infty |y-\mu| f(y)dy \quad \text{and} \quad \delta_2 = \int_0^\infty |y-m| f(y)dy.$$

After some algebraic manipulations, we find the following simplified expressions for δ_1 and δ_2:

$$\delta_1 = 2\left[\mu F(\mu) - \zeta(\mu)\right] \quad \text{and} \quad \delta_2 = m - 2\zeta(m), \tag{11}$$

where $F(.)$ is the CDF of Y and $\zeta(.)$ is defined as

$$\zeta(s) = \int_0^s y f(y) dy, \quad s > 0.$$

Proposition 2. *The mean and median deviations for a random variable $Y \sim RWL(\mu, \phi)$ are given, respectively, by*

$$\delta_1 = 2\left[\mu F(\mu) - \frac{2\mu}{\left[a(\mu,\phi) + 2\mu\phi\right]\Gamma(\phi)}\left(\gamma\left(\phi+1, \frac{a(\mu,\phi)}{2}\right) + \frac{2\mu\gamma\left(\phi+2, \frac{a(\mu,\phi)}{2}\right)}{a(\mu,\phi)}\right)\right] \tag{12}$$

and

$$\delta_2 = m - \frac{2\mu}{\left[a(\mu,\phi) + 2\mu\phi\right]\Gamma(\phi)}\left[\gamma\left(\phi+1, \frac{a(\mu,\phi)}{2\mu}m\right) + \frac{2\mu\gamma\left(\phi+2, \frac{a(\mu,\phi)}{2\mu}m\right)}{a(\mu,\phi)}\right], \tag{13}$$

where $F(.)$ is the CDF of the RWL distribution given in Equation (6) and

$$\gamma(b, c) = \int_0^c y^{b-1} e^{-y} dy$$

is the lower incomplete gamma function.

***Proof*.** It is enough to solve the integral

$$\zeta(s) = \int_0^s y f(y; \mu, \phi) dy, \quad s > 0,$$

where $f(.)$ is the PDF given in Equation (2). In fact, using the two-component mixture given in Equation (3), we have

$$\zeta(s) = \left(\frac{a(\mu,\phi)}{a(\mu,\phi) + 2\mu\phi}\right)\int_0^s y f_1(y;\mu,\phi) dy + \left(\frac{2\mu\phi}{a(\mu,\phi) + 2\mu\phi}\right)\int_0^s y f_2(y;\mu,\phi) dy.$$

where $f_j(y;\mu,\phi) = \left(\frac{a(\mu,\phi)}{2\mu}\right)^{\phi+j-1} \frac{y^{\phi+j-2}}{\Gamma(\phi+j-1)} \exp\left\{-\frac{a(\mu,\phi)}{2\mu} y\right\}$, for $j = 1, 2$.

Let $z = \frac{a(\mu,\phi)}{2\mu} y$, so $dz = \frac{a(\mu,\phi)}{2\mu} dy$. Thus, after some algebraic manipulations, we get

$$\zeta(s) = \frac{2\mu}{\left[a(\mu,\phi) + 2\mu\phi\right]\Gamma(\phi)}\left(\gamma\left(\phi+1, \frac{a(\mu,\phi)}{2\mu}s\right) + \frac{2\mu\gamma\left(\phi+2, \frac{a(\mu,\phi)}{2\mu}s\right)}{a(\mu,\phi)}\right). \tag{14}$$

Now, the results given in Equations (12) and (13) follow easily by using Equations (11) and (14). □

3.5. Laplace Transform

The Laplace transform of a PDF is useful in several applications of mathematics, engineering and statistics, such as frailty models, machine learning, complex differential equations, signal processing, control systems, among others. The Laplace transform of a nonnegative random variable Y, at $s \in \mathbb{C}$, is defined by

$$Q(s) = \int_0^\infty e^{-sy} f(y;\boldsymbol{\theta}) dy,$$

where $f(y;\boldsymbol{\theta})$ is the PDF of Y and $\boldsymbol{\theta}$ is the associated parameter vector.

Proposition 3. *The Laplace transform of the RWL distribution at a complex argument s is given by*

$$Q(s) = \left(\frac{1}{a(\mu,\phi) + 2\mu\phi}\right)\left(\frac{a(\mu,\phi)}{2\mu s + a(\mu,\phi)}\right)^{\phi+1} \left[a(\mu,\phi) + 2\mu(s+\phi)\right].$$

Proof. Let $\boldsymbol{\theta} = (\mu, \phi)$. Then,

$$\begin{aligned} Q(s) &= \int_0^\infty e^{-sy} f(y;\boldsymbol{\theta}) dy \\ &= \left(\frac{a(\mu,\phi)}{a(\mu,\phi) + 2\mu\phi}\right) \int_0^\infty e^{-sy} f_1(y;\boldsymbol{\theta}) dy \\ &\quad + \left(\frac{2\mu\phi}{a(\mu,\phi) + 2\mu\phi}\right) \int_0^\infty e^{-sy} f_2(y;\boldsymbol{\theta}) dy, \end{aligned} \tag{15}$$

where $f_j(y;\mu,\phi) = \left(\frac{a(\mu,\phi)}{2\mu}\right)^{\phi+j-1} \frac{y^{\phi+j-2}}{\Gamma(\phi+j-1)} \exp\left\{-\frac{a(\mu,\phi)}{2\mu} y\right\}$, for $j = 1, 2$.

Now, note that

$$\int_0^\infty e^{-sy} f_j(y;\mu,\phi) dy = \left(\frac{a(\mu,\phi)}{a(\mu,\phi) + 2\mu s}\right)^{\phi+j-1}. \tag{16}$$

Thus, by substituting Equation (16) into Equation (15) and making some algebraic manipulations, we obtain

$$Q(s) = \left(\frac{1}{a(\mu,\phi) + 2\mu\phi}\right)\left(\frac{a(\mu,\phi)}{2\mu s + a(\mu,\phi)}\right)^{\phi+1} \left[a(\mu,\phi) + 2\mu(s+\phi)\right].$$

□

4. Estimation

We consider the situation where the lifetime is not completely observed and is subject to random right-censoring. The mechanism of random right-censoring is what most occurs in practical problems and it generalizes the Type I and Type II right-censoring mechanisms.

Let C_i denote the censoring time, and Y_i be the lifetime of interest for the i-th sampling unit. Suppose that the random variables C_i and Y_i are independent. We then observe $y_i = \min(Y_i, C_i)$ and $\nu_i = I(Y_i \leq C_i)$, where $\nu_i = 1$ if Y_i is the observed lifetime and $\nu_i = 0$ if it is the censoring time. From n pairs of times and censoring indicators $(y_1, \nu_2), (y_2, \nu_2), \ldots, (y_n, \nu_n)$, the observed likelihood function for $\boldsymbol{\theta} = (\mu, \phi)^\top$ under non-informative censoring is given by

$$L(\boldsymbol{\theta}; \boldsymbol{y}) = \prod_{i=1}^{n} \left[f(y_i; \boldsymbol{\theta})\right]^{\nu_i} \left[S(y_i; \boldsymbol{\theta})\right]^{1-\nu_i}, \tag{17}$$

where $f(y_i; \boldsymbol{\theta})$ and $S(y_i; \boldsymbol{\theta})$ are the PDF and survival function of the RWL distribution, defined in Equations (2) and (4), respectively.

Since $h(y_i; \boldsymbol{\theta}) = \dfrac{f(y_i; \boldsymbol{\theta})}{S(y_i; \boldsymbol{\theta})}$, we then have that the likelihood function (17) reduces to

$$L(\boldsymbol{\theta}; \boldsymbol{y}) = \prod_{i=1}^{n} \left[h(y_i; \boldsymbol{\theta})\right]^{\nu_i} S(y_i; \boldsymbol{\theta}),$$

where $h(y_i; \boldsymbol{\theta})$ is the hazard rate function of the RWL distribution, given in Equation (5). Therefore, the log-likelihood function for $\boldsymbol{\theta}$ can be expressed as

$$\begin{aligned}
\ell(\boldsymbol{\theta}; \boldsymbol{y}) = {} & d(\phi+1)\log[a(\mu,\phi)] + (\phi-1)\sum_{i=1}^{n} \nu_i \log(y_i) \\
& + \sum_{i=1}^{n} \nu_i \log(1+y_i) - \frac{a(\mu,\phi)}{2\mu}\sum_{i=1}^{n} \nu_i y_i \\
& - \sum_{i=1}^{n} \nu_i \log\Bigg[(2\mu)^{\phi-1}\left[a(\mu,\phi)+2\mu\phi\right]\Gamma\left(\phi, \frac{a(\mu,\phi)}{2\mu} y_i\right) \\
& \qquad + \left[a(\mu,\phi) y_i\right]^{\phi} \exp\left\{-\frac{a(\mu,\phi)}{2\mu} y_i\right\}\Bigg] \\
& - d\log(2\mu) - n\log(\Gamma(\phi)) + \sum_{i=1}^{n} \log\Bigg[\Gamma\left(\phi, \frac{a(\mu,\phi)}{2\mu} y_i\right) \\
& \qquad + \frac{\left[a(\mu,\phi) y_i\right]^{\phi} \exp\left\{-\frac{a(\mu,\phi)}{2\mu} y_i\right\}}{(2\mu)^{\phi-1}\left[a(\mu,\phi)+2\mu\phi\right]}\Bigg]
\end{aligned} \tag{18}$$

where $d < n$ is the observed number of failures and $a(\mu, \phi)$ is defined as previously.

The MLE of parameter vector $\boldsymbol{\theta}$ can be found by maximizing the log-likelihood function given in Equation (18). In this work, we used the R function `maxLik`, which is available in the package of the same name; see Henningsen & Toomet (2011), to carry out such optimization procedure.

When we have uncensored data, $\nu_i = 1, \forall i$. In this case, the MLE of μ is the sample mean, $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} y_i$, whereas the MLE of ϕ is found by maximizing the following log-likelihood function:

$$\ell(\phi; \boldsymbol{y}) \propto n(\phi+1)\log[a(\hat{\mu},\phi)] + (\phi-1)\sum_{i=1}^{n}\log(y_i) + \sum_{i=1}^{n}\log(1+y_i)$$
$$- \frac{a(\hat{\mu},\phi)}{2\hat{\mu}}\sum_{i=1}^{n} y_i - n\phi\log(2\hat{\mu}) - n\log(\Gamma(\phi)) - n\log(a(\hat{\mu},\phi) + 2\hat{\mu}\phi),$$

which can be made by using, for example, the `maxLik` function.

Under mild conditions, it can be shown that the MLE $\hat{\boldsymbol{\theta}}$ is consistent and follows an asymptotic bivariate normal distribution with mean vector $\boldsymbol{\theta}$ and covariance matrix equal to the inverse of the expected Fisher information matrix $\mathcal{I}(\boldsymbol{\theta})$, that is,

$$(\hat{\mu},\hat{\phi}) \xrightarrow{D} \mathrm{N}_2\left((\mu,\phi), \mathcal{I}^{-1}(\mu,\phi)\right) \quad \text{as} \quad n \to \infty,$$

where $\xrightarrow{D}$ denotes convergence in distribution. Unfortunately, the exact expected Fisher information matrix is difficult to be obtained for the RWL distribution. In this case, we can approximate it by its observed version obtained from the `maxLik` package results. Hence, we can construct approximate $100(1-\alpha)\%$ confidence intervals for the individual parameters, as well as hypothesis tests, through the estimated marginal distributions (both normal).

5. Results Based on Computation

In this section, we perform a Monte Carlo simulation study to verify the asymptotic behavior of MLEs of the RWL distribution parameters under different sample sizes and percentages of censoring. All the analyses were carried out using the `R` software, and the seed used in the pseudo-random number generators was 2020. Specifically, the random samples of size n from the RWL distribution with parameters μ and ϕ were generated using the following steps:

1. Generate $U_i \sim \text{Uniform}(0,1)$, for $i = 1, 2, \ldots, n$;
2. Generate $X_i \sim \text{Gamma}\left(\phi, \frac{a(\mu,\phi)}{2\mu}\right)$, for $i = 1, 2, \ldots, n$;
3. Generate $W_i \sim \text{Gamma}\left(\phi+1, \frac{a(\mu,\phi)}{2\mu}\right)$, for $i = 1, 2, \ldots, n$;
4. If $U_i \leq \dfrac{a(\mu,\phi)}{a(\mu,\phi)+2\mu\phi}$, then set $Y_i = X_i$, otherwise set $Y_i = W_i$, for $i = 1, 2, \ldots, n$.

The following performance criteria were considered: mean relative estimate (MRE) and mean squared error (MSE), which are given, respectively, by

$$\text{MRE}_i = \frac{1}{N}\sum_{j=1}^{N}\frac{\hat{\theta}_{i,j}}{\theta_i} \quad \text{and} \quad \text{MSE}_i = \frac{1}{N}\sum_{j=1}^{N}\left(\hat{\theta}_{i,j}-\theta_i\right)^2, \quad i=1,2,$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top = (\mu, \phi)^\top$ is the parameter vector and $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2)^\top = (\hat{\mu}, \hat{\phi})^\top$ is its MLE, while $N = 10,000$ is the number of estimates obtained through the proposed approach.

According to these criteria, it is expected that the MRE and MSE return values closer to one and zero, respectively. We also compute the coverage probabilities (CPs) of the 95% confidence intervals. For a large number of experiments using 95% confidence intervals, the relative frequencies of these intervals that covered the true values of $\boldsymbol{\theta}$ should be closer to 0.95. The CPs were calculated using the numeric observed information matrix obtained from the `maxLik` package results.

We considered a sample size $n \in \{20, 50, 100, 200, 400\}$ and $\boldsymbol{\theta} \in \{(0.5, 0.7), (2, 5)\}$, with censoring percentages of 0%, 25% and 50%. We selected these values for $\boldsymbol{\theta}$ in order to get, respectively, bathtub-shaped and increasing hazard rate functions. The censored data were generated using the same procedure as in Bayoud (2012). Let p_j, $j = 1, 2, \ldots, N$, denote the proportion of censored data in the j-th sample, then according to this procedure it is expected that the mean for the proportions of censored data ($\hat{E}[p]$) will be approximately 0.000, 0.250 and 0.500.

Under these scenarios, we report the values of the empirical MREs, MSEs, and CPs in Tables 1 and 2. According to these tables, we can see that the MSEs of all estimators tend to zero as the sample size increases, suggesting that all estimators are consistent with the parameters. In contrast, the MRE values tend to one, meaning that the estimators are asymptotically unbiased for the parameters, as expected. We can also see that, as the censoring percentage increases, the MREs and MSEs of the MLEs also increase, as expected. Furthermore, we observe that, as n increases, the CPs tend to the nominal level (0.95). Therefore, in general, all of these results show the excellent performance of the MLEs of the corresponding parameters.

6. Real Data Examples

In this section, we illustrate the proposed methodology on electrical appliances data (Section 6.1), as well as on lifetimes of an agricultural machine (Section 6.2).

We compared the results obtained by the RWL distribution with the corresponding ones achieved with the use of other two-parameter lifetime distributions reparameterized by their mean. Namely, the reparameterized gamma (Louzada & Ramos 2018), reparameterized inverse gamma (Bourguignon & Gallardo 2020), and reparameterized Birnbaum-Saunders (Santos-Neto et al. 2012) distributions. We present the PDFs of these distributions as follows:

Table 1: MRE, MSE, CP and expected censoring proportion estimates for $N = 10,000$ samples of sizes $n \in \{20, 50, 100, 200, 400\}$, with 0%, 25% and 50% of random censored data, for $\mu = 0.5$ and $\phi = 0.7$.

		$\mu = 0.5$			$\phi = 0.7$			
	n	MRE	MSE	CP	MRE	MSE	CP	$\hat{E}[p]$
0%	20	1.001	0.015	0.910	1.147	0.080	0.964	-
	50	1.000	0.006	0.932	1.052	0.021	0.950	-
	100	0.999	0.003	0.942	1.025	0.009	0.955	–
	200	1.001	0.002	0.949	1.013	0.004	0.948	-
	400	1.001	0.001	0.944	1.006	0.002	0.950	-
25%	20	1.024	0.027	0.904	1.153	0.098	0.956	0.250
	50	1.011	0.010	0.931	1.052	0.025	0.954	0.250
	100	1.007	0.004	0.946	1.025	0.011	0.952	0.250
	200	1.005	0.002	0.948	1.012	0.005	0.950	0.250
	400	1.002	0.001	0.945	1.007	0.002	0.955	0.249
50%	20	1.093	0.088	0.888	1.192	0.168	0.959	0.500
	50	1.031	0.020	0.919	1.063	0.034	0.957	0.500
	100	1.017	0.009	0.938	1.030	0.014	0.953	0.500
	200	1.010	0.004	0.945	1.015	0.007	0.952	0.500
	400	1.004	0.002	0.946	1.008	0.003	0.951	0.499

Table 2: MRE, MSE, CP and expected censoring proportion estimates for $N = 10,000$ samples of sizes $n \in \{20, 50, 100, 200, 400\}$, with 0%, 25% and 50% of random censored data, for $\mu = 2$ and $\phi = 5$.

		$\mu = 2$			$\phi = 5$			
	n	MRE	MSE	CP	MRE	MSE	CP	$\hat{E}[p]$
0%	20	1.002	0.038	0.929	1.150	4.548	0.942	-
	50	1.002	0.015	0.936	1.066	1.457	0.949	-
	100	1.001	0.008	0.943	1.032	0.622	0.952	-
	200	1.001	0.004	0.950	1.017	0.292	0.952	-
	400	1.001	0.002	0.945	1.009	0.141	0.951	-
25%	20	1.004	0.049	0.927	1.180	6.610	0.925	0.249
	50	1.002	0.019	0.943	1.074	1.806	0.953	0.250
	100	1.001	0.009	0.948	1.035	0.791	0.952	0.250
	200	1.001	0.005	0.948	1.019	0.369	0.951	0.250
	400	1.001	0.002	0.946	1.011	0.178	0.947	0.250
50%	20	1.010	0.080	0.927	1.232	11.910	0.902	0.498
	50	1.003	0.029	0.942	1.092	2.621	0.943	0.500
	100	1.002	0.014	0.947	1.045	1.155	0.951	0.501
	200	1.001	0.007	0.950	1.024	0.537	0.949	0.500
	400	1.001	0.003	0.948	1.013	0.252	0.951	0.501

- **Reparameterized gamma (RG) distribution:**
  According to Louzada & Ramos (2018), the PDF of the RG distribution is given by

$$f(y; \mu, \phi) = \frac{1}{\Gamma(\phi)} \left(\frac{\phi}{\mu}\right)^{\phi} y^{\phi-1} \exp\left\{-\frac{\phi}{\mu} y\right\}, \quad y > 0,$$

where $\mu > 0$ is the mean parameter and $\phi > 0$ is the precision parameter.

- **Reparameterized inverse gamma (RIG) distribution:**
According to Bourguignon & Gallardo (2020), the PDF of the RIG distribution is given by

$$f(y;\mu,\phi)=\frac{[\mu(1+\phi)]^{\phi+2}}{\Gamma(\phi+2)}y^{-\phi-3}\exp\left\{-\frac{\mu(1+\phi)}{y}\right\},\quad y>0,$$

where $\mu>0$ is the mean parameter and $\phi>0$ is the dispersion parameter.

- **Reparameterized Birnbaum-Saunders (RBS) distribution:**
Presented by Santos-Neto et al. (2012), it has PDF given by

$$f(y;\mu,\phi)=\frac{\exp\{\phi/2\}\sqrt{\phi+1}}{4y^{3/2}\sqrt{\pi\mu}}\left(y+\frac{\phi\mu}{\phi+1}\right)\exp\left\{-\frac{\phi}{4}\left[\frac{(\phi+1)y}{\phi\mu}+\frac{\phi\mu}{(\phi+1)y}\right]\right\},$$

for all $y>0$, where $\mu>0$ is the mean parameter and $\phi>0$ is the precision parameter.

In order to carry out the model selection, different discrimination criterion methods based on log-likelihood function evaluated at the MLEs were considered. Let k be the number of parameters in the model and $\hat{\boldsymbol{\theta}}$ denote the MLE for the parameter vector $\boldsymbol{\theta}$. Then, the model discrimination criteria used here are: Akaike Information Criterion (AIC; Akaike 1974), Corrected AIC (AICc; Sugiura 1978), Bayesian or Schwarz Information Criterion (BIC; Schwarz 1978), Hannan-Quinn Information Criterion (HQIC; Hannan & Quinn 1979), and Consistent AIC (CAIC; Bozdogan 1987), which are computed, respectively, by

$$\begin{aligned}
\text{AIC}&=-2\ell\left(\hat{\boldsymbol{\theta}};\boldsymbol{y}\right)+2k,\\
\text{AICc}&=\text{AIC}+\frac{2k(k+1)}{(n-k-1)},\\
\text{BIC}&=-2\ell\left(\hat{\boldsymbol{\theta}};\boldsymbol{y}\right)+k\log(n),\\
\text{HQIC}&=-2\ell\left(\hat{\boldsymbol{\theta}};\boldsymbol{y}\right)+2k\log\left(\log(n)\right),\\
\text{CAIC}&=\text{AIC}+k\left[\log(n)-1\right],
\end{aligned}$$

where $\ell(.;\boldsymbol{y})$ is the log-likelihood function of the corresponding model and n is the sample size. According to these criteria, the best model is the one that provides the minimum values. The Kolmogorov-Smirnov test with confidence level $\alpha=0.05$ and Cox-Snell residuals were also considered for checking the goodness-of-fit of models to the uncensored and censored data, respectively (Cox & Snell 1968, Daniel 1990).

6.1. Cycles up to the Failure for Electrical Appliances

In this subsection, we reanalyzed the data set extracted from Lawless (2011), which consists of a number of cycles, divided by 1,000, up to the failure for 60 electrical appliances in a life test (see Table 3). Many authors have analyzed these uncensored data, including Reed (2011), Khan (2018) and Ramos et al. (2019). Such data are known to have a bathtub-shaped hazard rate function.

Table 4 displays the MLEs, standard errors (SEs) and 95% confidence intervals (95 % CIs) for the parameters μ and ϕ of the RWL model. Note that the estimated mean number of cycles to failure of an electrical appliance is 2.193 cycles. Furthermore, since $\hat{\phi} = 0.733$, the estimated hazard rate function is bathtub-shaped, that is, it is characterized by an increased number of failures (and thus, unavailability) in the initial period of electrical appliance usage after its commissioning, followed by a long span of normal use with a small and roughly constant number of failures, and finally, a period of a fast increasing number of failures occurring because of the age of the observed electrical appliance.

Table 3: Number of cycles, divided by 1,000, up to the failure for 60 electrical appliances in a life test.

0.014	0.034	0.059	0.061	0.069	0.080	0.123	0.142	0.165	0.210
0.381	0.464	0.479	0.556	0.574	0.839	0.917	0.969	0.991	1.064
1.088	1.091	1.174	1.270	1.275	1.355	1.397	1.477	1.578	1.649
1.702	1.893	1.932	2.001	2.161	2.292	2.326	2.337	2.628	2.785
2.811	2.886	2.993	3.122	3.248	3.715	3.790	3.857	3.912	4.100
4.106	4.116	4.315	4.510	4.580	5.267	5.299	5.583	6.065	9.701

Table 4: MLEs, SEs and 95% CIs for the parameters of the RWL distribution, considering the electrical appliances data.

Parameter	MLE	SE	95% CI
μ	2.193	0.272	(1.659; 2.727)
ϕ	0.733	0.136	(0.466; 1.001)

Table 5 gives the log-likelihood, AIC, AICc, BIC, HQIC and CAIC values, as well as the Kolmogorov-Smirnov (KS) test statistics and their p-values, for all four distributions considered. We can see that the RWL distribution offers a better fit to the electrical appliances data since it has the minimum values of these criteria. In addition, the KS test indicates that the electrical appliances data are a random sample from a RWL distribution with $\hat{\mu} = 2.193$ and $\hat{\phi} = 0.733$.

Table 5: Model selection criteria values and KS test (statistic and p-values) for the fitted probability distributions, considering the electrical appliances data.

Criterion	RWL	RG	RIG	RBS
Log-likelihood	**-105.774**	-107.012	-157.273	-118.912
AIC	**215.548**	218.024	318.546	241.824
AICc	**215.759**	218.235	318.756	242.035
BIC	**219.737**	222.213	322.734	246.013
HQIC	**217.187**	219.663	320.184	243.463
CAIC	**221.737**	224.213	324.734	248.013
KS	**0.072**	0.082	0.496	0.285
p-value	**0.907**	0.810	< 0.0001	< 0.001

Figure 5 presents the survival function adjusted by different probability distributions (RWL, RG, RIG and RBS distributions) superimposed to the estimated Kaplan-Meier survival curve. From this figure, it can be observed that the RWL distribution provides the better fit to the electrical appliances data. Therefore, from the proposed methodology, the data set related to the failure times of 60 electrical appliances can be well-described by the RWL distribution.

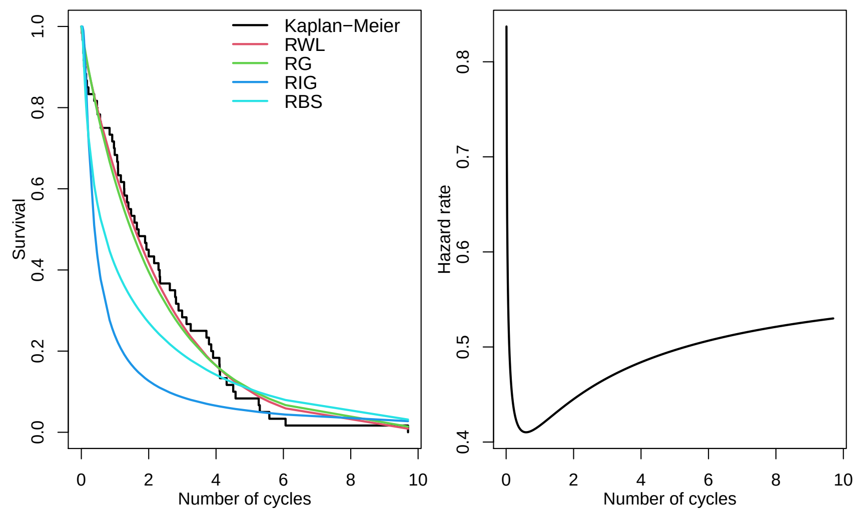

FIGURE 5: Left panel: Fitted survival functions superimposed to the estimated Kaplan-Meier survival curve, considering the electrical appliances data. Right panel: Estimated hazard rate function of the RWL distribution for these data.

6.2. Agricultural machine data

As a second application, in this subsection we reanalyzed the data related to the times up to corrective maintenance of an agricultural machine, presented by Ramos et al. (2019). This data set includes two censored observations, both in 13 days. Its analysis can be useful to correctly predict the next maintenance in order to reduce costs.

TABLE 6: Times up to corrective maintenance of an agricultural machine ("+" denotes censoring).

1	1	1	1	1	1	1	2	2	3	3	3
3	3	4	4	4	4	4	4	4	5	5	5
5	5	5	5	5	5	6	6	6	6	6	6
6	6	6	6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7	7	7	7
7	7	8	8	8	8	8	8	8	8	8	8
8	9	9	9	9	9	11	11	11	11	11	11
11		11	13	13+	13+	-	-	-	-	-	-

Table 7 shows the MLEs, SEs and 95% CIs for the parameters μ and ϕ of the RWL distribution. Notice that the estimated mean time to occur a fail in the agricultural machine is 6.404 days. Also, the fit of the RWL distribution suggests an increasing-shaped hazard rate function: $\hat{\phi} = 2.778$ (see Figure 6, right panel).

TABLE 7: MLEs, SEs and 95% CIs for the parameters of the RWL distribution, considering the agricultural machine data.

Parameter	MLE	SE	95% CI
μ	6.404	0.369	(5.680; 7.127)
ϕ	2.778	0.491	(1.816; 3.740)

Table 8 reports the results from different model discrimination/selection criteria, such as the log-likelihood, AIC, AICc, BIC, HQIC and CAIC, for the four considered probability distributions. From these results, we see that the RWL distribution provides slightly better description of the data compared to other candidate distributions, since it yields the lowest values in all criteria.

TABLE 8: Model selection criteria values for the fitted probability distributions, considering the agricultural machine data.

Criterion	RWL	RG	RIG	RBS
Log-likelihood	**-223.049**	-223.683	-248.159	-235.404
AIC	**450.098**	451.367	500.318	474.808
AICc	**450.237**	451.506	500.457	474.947
BIC	**455.075**	456.344	505.295	479.785
HQIC	**452.104**	453.373	502.324	476.814
CAIC	**457.075**	458.344	507.295	481.785

Figure 6 exhibits the survival functions superimposed to the estimated Kaplan-Meier survival curve (left panel), as well as the estimated hazard rate function (right panel). From this figure, it can be observed that the RWL distribution provides a good fit to the agricultural machine data.

After adjusting the RWL distribution to the agricultural machine data, we verified the goodness-of-fit of it through the Cox-Snell residuals (Cox & Snell 1968). The Cox-Snell residuals are defined by

$$e_i = -\log\left(\hat{S}(t_i)\right), \quad i = 1, 2, \ldots, n,$$

where $\hat{S}(t_i)$ is the fitted RWL survival function of the i-th lifetime. In this case, the Cox-Snell residuals e_i's are a censored random sample from the standard exponential distribution, if the RWL distribution is correctly specified.

Figure 7 presents the graph of Kaplan-Meier *versus* standard exponential survival, both fitted to the Cox-Snell residuals. By means of this figure, we can observe that most of the points are close to line, showing the goodness-of-fit of the proposed distribution to the agricultural machine data. Therefore, from the proposed methodology, the data set related to the failure times of agricultural machine can be well-described by the RWL distribution.

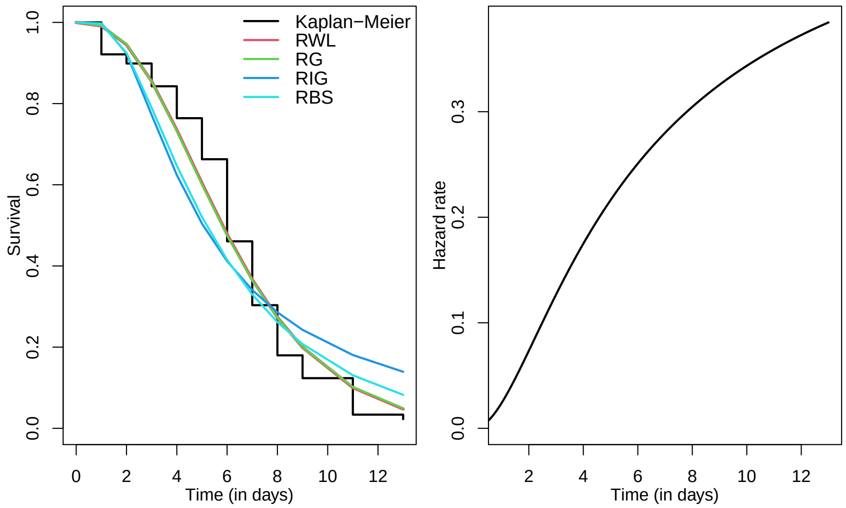

FIGURE 6: Left panel: Fitted survival functions superimposed to the estimated Kaplan-Meier survival function, considering the lifetimes of an agricultural machine. Right panel: Estimated hazard rate function of the RWL distribution for these data.

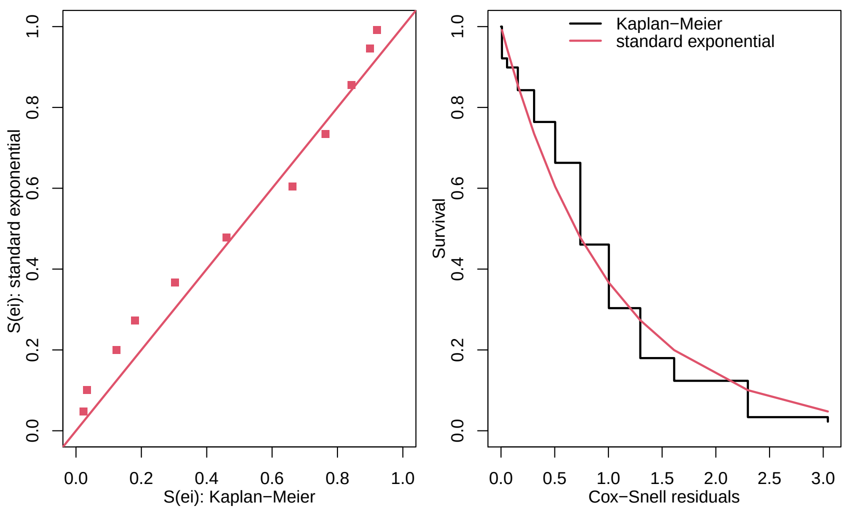

FIGURE 7: Left panel: Kaplan-Meier *versus* standard exponential survival, both fitted to the Cox-Snell residuals. Right panel: Estimated survival curves (Kaplan-Meier and standard exponential).

A preventive approach for this agricultural machine is given as follows. Through the quantile function of the RWL distribution given in Equation (6), we can get the number of days that are expected to have a certain percentage of failures. Table 9 displays different times of failure, assuming different percentages.

The results obtained from this table show that preventive maintenance can be performed assuming different percentages of failures. Thus, we recommend the agricultural enterprise to consider 4 days (25% of failures) after the last failure to perform maintenance in this agricultural machine.

Table 9: Days to perform preventive maintenance to agricultural machine by assuming different percentages of failures, based on the RWL distribution.

10%	25%	50%	75%	99%
2.55	3.88	5.82	8.29	16.87

7. Concluding Remarks

In this paper, we derived critical mathematical properties of the RWL distribution, which allow its application in many real problems. Under this parameterization, one of the parameters is given by the mean, whereas the other parameter can be interpreted as a precision parameter. The (classical) inferential method for the parameters was discussed under random censoring. An extensive Monte Carlo simulation study showed that the proposed estimators are consistent and return reasonable estimates for the parameters of the RWL distribution. The proposed methodology was used in two applications considering electrical appliances data and a data set related to the lifetimes of an agricultural machine, in which we observed that the RWL distribution returned better fit when compared to some well-known reparameterized models in the statistical literature.

There is a large number of possible extensions of this current work. For instance, we can easily formulate RWL regression models with varying precision (Santos-Neto et al. 2016, Bourguignon & Gallardo 2020). Another approach that is under investigation is the use of the RWL distribution in the context of frailty models, since the Laplace transform has a closed-form expression. In this setting, we believe that the RWL distribution can be an alternative frailty distribution to the traditional frailty distributions (Wienke 2010). The RWL distribution is also a promising model to be used in studies involving degradation and accelerated life test data, and should thus be investigated in future research (Meeker & Escobar 2014).

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