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Simultaneously Testing for Location and Scale Parameters of Two Multivariate Distributions

Prueba simultánea de ubicación y parámetros de escala de dos distribuciones multivariables

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Abstract

In this article, we propose nonparametric tests for simultaneously testing equality of location and scale parameters of two multivariate distributions by using nonparametric combination theory. Our approach is to combine the data depth based location and scale tests using combining function to construct a new data depth based test for testing both location and scale parameters. Based on this approach, we have proposed several tests. Fisher's permutation principle is used to obtain p-values of the proposed tests. Performance of proposed tests have been evaluated in terms of empirical power for symmetric and skewed multivariate distributions and compared to the existing test based on data depth. The proposed tests are also applied to a real-life data set for illustrative purpose.

***Key words*:** Combining function; Data depth; Permutation test; Two-sample test.

Resumen

En este artículo, proponemos pruebas no paramétricas para probar simultáneamente la igualdad de ubicación y los parámetros de escala de dos distribuciones multivariantes mediante la teoría de combinaciones no paramétricas. Nuestro enfoque es combinar las pruebas de escala y ubicación basadas en la profundidad de los datos utilizando la función de combinación para construir una nueva prueba basada en la profundidad de los datos para probar los parámetros de ubicación y escala. Con base en este enfoque, hemos propuesto varias pruebas. El principio de permutación de Fisher se usa para obtener valores p de las pruebas propuestas. El rendimiento de las pruebas propuestas se ha evaluado en términos de potencia empírica para

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distribuciones multivariadas simétricas y asimétricas y se comparó con la prueba existente basada en la profundidad de los datos. Las pruebas propuestas también se aplican a un conjunto de datos de la vida real con fines ilustrativos.

***Palabras clave*:** Función de combinación; Profundidad de datos; Prueba de permutación; Prueba de dos muestras.

1. Introduction

In real life, we come across with many ambiguous situations, where we have to take a decision whether the location and scale parameters of two multivariate distributions are equal or not. In such situations, testing of hypothesis is used to address such problems. Various multivariate two-sample tests for simultaneously testing location and scale parameters are available in the statistical literature. But most of these tests are based on the assumptions about underlying probability distributions, particularly assumption of multivariate normality. In reality, such assumption may not be satisfied and hence, we have to go for nonparametric multivariate two-sample testing procedures.

Based on notion of data depth, many nonparametric tests for multivariate locations and scales are available in literature. The word depth was first coined by Tukey (1975) for picturing data. Many data depth functions were proposed in literature for capturing different probabilistic properties of multivariate data. Among all of them, more popular choices of data depth functions are halfspace depth (Tukey 1975), Mahalanobis depth (Mahalanobis 1936), simplicial depth (Liu 1990), majority depth (Singh 1991) and projection depth (Donoho & Gasko 1992). The details about notion of data depth, data depth functions, depth versus depth plot (DD-plot), inference using data depth are provided by Liu, Parelius & Singh (1999). Zuo & Serfling (2000) described four desirable properties for the depth functions.

Liu & Singh (1993) proposed two-sample rank tests based on data depth for simultaneously testing shift in the location and change in the scale parameters. Rousson (2002) has proposed tests for testing both location and scale differences between two multivariate distributions. Li & Liu (2004) have proposed two depth-based nonparametric tests viz. T-based test and M-based test for multivariate location difference. These tests for locations are based on the depth versus depth plot (DD-plot) introduced by Liu et al. (1999). Dovoedo & Chakraborti (2015) have reported an extensive simulation study to evaluate the performance of T-based and M-based tests for well-known family of multivariate skewed as well as symmetric distributions and compared the performance of these tests for four popular affine-invariant depth functions. Chavan & Shirke (2016) have proposed two nonparametric tests for testing equality of location parameters of two multivariate distributions. These tests are extensions of the M-based test introduced by Li & Liu (2004). Li, Ban & Santiago (2011) have proposed two-sample nonparametric tests for comparing species assemblages. These tests can be considered as a natural generalization of the Kolmogorov-Smirnov (KS) and Cramer-Von Mises (CM) tests. Chenouri & Small (2012) proposed a family of nonparametric multivariate multisample tests based on depth rankings. Nonparametric tests for scale differences between two samples have been provided by Li & Liu (2016). Many of these tests use Fisher's permutation tests to calculate the p-value.

In literature, for a univariate data, many nonparametric tests are proposed by combining the permutation-based location test and scale test for simultaneously testing location and scale parameters. Based on squared ranks and squared contrary-ranks, Cucconi (1968) proposed a test for the location-scale problem. Lepage (1971) proposed a test which is the combination of the Wilcoxon test for location and the Ansari-Bradley test for scale. Podgor & Gastwirth (1994) devised a test statistic by taking a quadratic combination of the rank test for location and a rank test for scale. Neuhäuser (2000) modified Lepage L-test by replacing the Wilcoxon test for location with a location test proposed by Baumgartner, Weiß & Schindler (1998). Also, Murakami (2007) proposed a modification of the Lepage test. More recently, Park (2015) proposed several nonparametric simultaneous test procedures for location and scale parameters by using combining function. To the best of our knowledge, it appears that relatively less literature has been reported on combining location and scale tests for multivariate data

Sometimes assumption of normality is not satisfied for many complex multivariate datasets which are generated in various fields. In such situation, traditional parametric inferential procedures are not appropriate. Pesarin(2001) has proposed NPC (nonparametric combination) theory for a dependent tests. This theory is widely used in multi aspect testing. NPC divides a global null hypothesis into several partial null hypotheses and then use a Fisher's permutation principle to find the p-value of each of these partial null hypotheses. A suitable combining function is used to combine the p-values of partial null hypotheses. The null distribution of the choosen combining function is obtained by using the Fisher permutation principle to compute the global p-value. Based on this global p-value, decide whether to accept or reject the global null hypothesis. In the literature, less number of nonparametric multivariate two sample tests are proposed for simultaneously testing location and scale parameters of two multivariate distributions. Also, two sample tests based on the combination of nonparametric multivariate location test and scale test are appeared less in the literature. In the present article, we have used nonparametric combination (NPC) theory to develop nonparametric multivariate two-sample tests based on data depth for simultaneously testing location and scale parameters of two multivariate distributions. Data-depth based two-sample tests for location and scale are combined using an appropriate combining function to produce a new test for testing both location and scale parameters. Fisher's permutation test is used to compute the p-value of the tests. Monte Carlo simulations are used to obtain the empirical power of the proposed tests and performance of proposed tests is compared with the H-test provided by Chenouri & Small (2012). The rest of the article is organized as follows.

A review of combining functions is given in Section 2. In Section 3, we discuss the notion of data depth and some well-known data depth functions. Tests for testing differences between locations and scales based on the notion of data depth are discussed in Section 4. We describe the proposed tests in Section 5. Simulation results are reported in Section 6. Illustration with a real-life data is provided in Section 7 and concluding remarks are given in the last Section.

2. Combining Functions

In a meta-analysis, combining functions plays an important role while summarizing all the results obtained from various independent groups. Several combining functions are available in the literature, some of which are discussed in the following subsections.

2.1. Fisher's Combining Function

Let $p_i, i = 1, 2, \ldots, k$ denote the p-value of i^{th} hypothesis test. Then the Fisher's combining function (Fisher 1925) is denoted by F_c and is defined as,

$$F_c = -2\sum_{i=1}^{k} \log_e p_i.$$

If all the null hypotheses are true and p_i are independent then F_c follows a chi-square distribution with $2k$ degrees of freedom, where k is the number of tests being combined. Under the null hypothesis of each individual test, p-value follows continuous uniform distribution over interval $[0, 1]$. Therefore, the test statistic F_c follows chi-square distribution with $2k$ degrees of freedom. If all the p-values are small, then the quantity F_c is large. Smaller p-values indicate the rejection of individual test. As a result of this, test statistic F_c rejects the global null hypothesis.

2.2. Tippet Combining Function

The Tippet combining function (Tippett 1952) is denoted by T_c and is defined as,

$$T_c = \max_{1 \leq i \leq k} (1 - p_i).$$

If all the null hypotheses are true and p_i are independent, distribution of T_c behaves according to the largest (smallest) of k random values from the uniform distribution over $(0, 1)$. As smaller the p-value (p_i) of i^{th} individual test, the quantity $(1 - p_i)$ is large. Therefore, the test rejects the global null hypothesis for the larger value of T_c.

2.3. Liptak Combining Function

The Liptak combining function (Liptak 1958) is denoted by L_c and is defined as,

$$L_c = \sum_{i=1}^{k} \Phi^{-1}(1 - p_i),$$

where $\Phi(.)$ is the distribution function of standard normal distribution. As smaller the p-value (p_i) of i^{th} individual test, the quantity $\Phi^{-1}(1-p_i)$ is large. Therefore the test rejects the global null hypothesis for the larger value of L_c. If all the null hypotheses are true and p_i are independent then L_c follows the normal distribution with mean 0 and variance k.

In the next section, we discuss the notion of data depth.

3. Notion of Data Depth

Let $(X_1, X_2, \dots, X_m)$ be a data set (cloud), where each $X_i \in \mathbb{R}^p$, $i = 1, 2, \dots, m$ is assumed to follow a continuous distribution with cumulative distribution function (CDF) $F(.)$. Let $D(x, F)$ be the depth of a point x with respect to $F(.)$. A data depth is a non-negative function defined from $\mathbb{R}^p$ to $[0, \infty)$. The notion of data depth can be used to obtain the location of a given data points with respect to a data cloud. It measures the centrality of a given data point with respect to a given distribution $F(.)$ or data cloud. Data depth gives a natural center-outward ranking to a data points with respect to data cloud. Such rankings were used for testing difference in the location and scale parameters of two or more multivariate distributions, constructing nonparametric control charts, outliers detection, and classification problems etc.

In the following, we describe three well-known depth functions. However, we use simplicial and Tukey's halfspace depth functions for our discussion through out this article.

- **Simplicial Depth**

The simplicial depth (Liu 1990) for any point $x \in \mathbb{R}^p$ with respect to $F(.)$ on $\mathbb{R}^p$ is denoted by $SD(x, F)$ and is given by,

$$SD(x, F) = P_F(s[X_1, X_2, \dots, X_{p+1}] \ni x),$$

where, $X_1, X_2, \dots, X_{p+1}$ are independent and identically distributed observations from F and $s[X_1, X_2, \dots, X_{p+1}]$ is a closed simplex formed by $X_1, X_2, \dots, X_{p+1}$. The sample version of simplicial depth can be obtained by replacing F by F_m in this expression. That is,

$$SD(x, F_m) = \binom{m}{p+1}^{-1} \sum_* I(x \epsilon s[X_{i1}, X_{i2}, \dots, X_{ip+1}]),$$

where $(*)$ runs over all possible subsets of $X_1, X_2, \dots, X_m$ of size $(p+1)$ and $I(.)$ is an usual indicator function.

- **Tukey's Halfspace Depth**

The Tukey's halfspace depth (Tukey 1975) for any point $x \in \mathbb{R}^p$ with respect to $F(.)$ on $\mathbb{R}^p$ is denoted by $HSD(x, F)$ and is given by,

$$HSD(x,F) = \inf_H\{P(H) : H \text{ is a closed halfspace containing } x\},$$

where $P(.)$ is a probability. The sample version of $HSD(x,F)$ is obtained by replacing F by F_m. It is a smallest fraction of the data points contained in a closed halfspace which contains x. That is,

$$HSD(x,F_m) = \frac{\min_{||l||=1}\#\{i : l'x_i \geq l'x\}}{m}.$$

If $p = 1$ then $HSD(x,F) = min\{F(x), 1 - F(x^-)\}$.

- **Mahalanobis Depth**

The Mahalanobis depth (Mahalanobis 1936) for any point $x \in \mathbb{R}^p$ with respect to $F(.)$ on $\mathbb{R}^p$ is denoted by $MD(x,F)$ and is given by,

$$MD(x,F) = \tfrac{1}{1+(x-\mu)^{'}\Sigma^{-1}(x-\mu)},$$

where μ and Σ are the location parameter (center) and the variance-covariance matrix (dispersion matrix) of $F(.)$. The quantity $(x-\mu)^{'}\Sigma^{-1}(x-\mu)$ is a Mahalanobis distance of a point x from μ. That is Mahalanobis depth is defined by using Mahalanobis distance. The sample version of Mahalanobis depth can be obtained by replacing μ and Σ by $\bar{X}$ (sample mean) and S (sample variance-covariance matrix).

In the next section, we discuss tests for testing differences between location parameters and scale parameters based on the notion of data depth.

4. Tests Based on Data Depth

Let $(X_1, X_2, \ldots, X_m)$ and $(Y_1, Y_2, \ldots, Y_n)$ be two independent random samples from two continuous distributions $F(.)$ and $G(.)$ respectively, where X_i, $Y_j \in \mathbb{R}^p$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Let $D(x,F)$ and $D(x,G)$ be the depths of a point $x \in Z$ with respect to $F(.)$ and $G(.)$ respectively, where $Z = X \cup Y$. A set containing such points is defined as,

$$DD(F,G) = \{(D(x,F), D(x,G)), \quad \forall\ x \in Z\}.$$

The empirical version of $DD(F,G)$ based on the above described two random samples is given by,

$$DD(F_m,G_n) = \{(D(x,F_m), D(x,G_n)), \quad \forall\ x \in Z\}.$$

DD plot is a scatter plot, which is the plot of points in the set $DD(F_m, G_n)$.

The DD plot can be used for comparing two multivariate samples by graphically. The difference between locations or scales or skewness or kurtosis is associated with different pictures observed on the DD plots. If F and G are identical then the points should fall on a 45^0 line segment on the empirical DD Plot. This is illustrated in Figure 1(a), which is the DD plot of two multivariate samples drawn from the bivariate normal distribution with mean vector $\mu = \underline{0}$ and variance-covariance matrix I_2, where I_2 is the identity matrix of order two. The divergence of F from G will indicate divergence of points from 45^0 line segment and Figure 1(b), Figure 2(a), Figure 2(b) and Figure 3 reveal different pictures of DD plot that indicate the location differences, large location differences, scale differences and skewness differences (both location and scale differences) respectively. The DD plot in Figure 1(b) has a leaf-shaped picture with the cusp lying on the diagonal line towards the upper right corner and the leaf steam at the lower left corner point $(0, 0)$, when there is a shift in location parameters of two multivariate samples. In each of these Figures, we plot DD plot of DG against DF where F and G have chosen appropriately, where DF and DG are the depth of the points with respect to F and G respectively. We use simplicial depth as a depth function to plot the DD plot in Figure 1-3. The DD plots have been plotted using 'depth' package available in R.

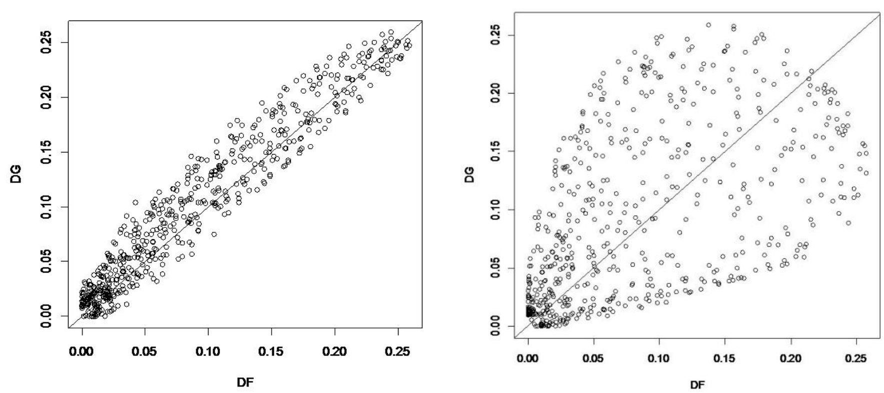

Figure 1: DD plots of (a) identical distributions and (b) location shift.

In the following subsections, we review four tests for testing equality of locations and a test for testing scale differences between two multivariate distributions.

4.1. Nonparametric Tests for Location Differences Between Two Multivariate Distributions

Li & Liu (2004) proposed two tests viz. $T-$based and $M-$based tests for testing location differences between two multivariate distributions based on DD-plot.

- $T-$**based test**

In the presence of location shift in two distribution, the DD plot has a leaf-shaped picture (Figure 1(b), Figure 2(a)) with the leaf stem anchoring at the lower

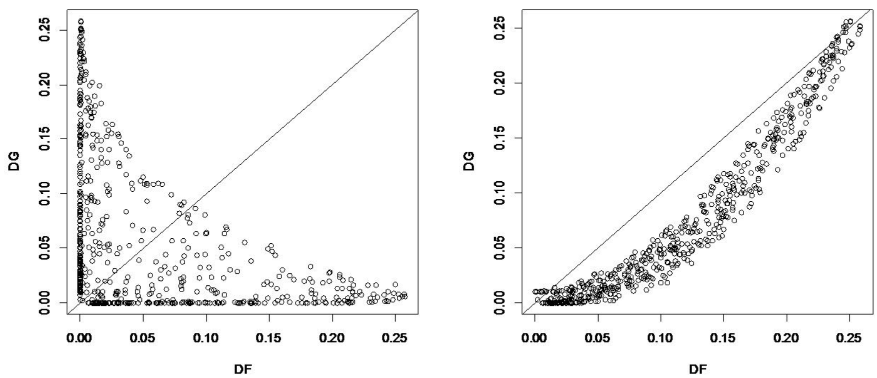

FIGURE 2: DD plots of (a) large location shift and (b) scale increase.

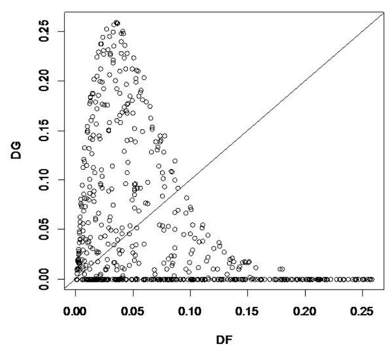

FIGURE 3: DD plot of skewness difference.

left corner point $(0,0)$ and the cusp lying on the diagonal line pointing towards the upper right corner. On the basis of this observation, Li & Liu (2004) constructed the test statistic which is the distance between the origin $(0,0)$ and the cusp point. Li & Liu (2004) suggested the procedure to calculate the distance between the cusp point and the origin $(0,0)$. Smaller the distance indicates the larger shift in location. The *p*-value of the test is obtained by using the Fisher's permutation test.

- **$M-$based test**

In the literature of data depth, the point having largest depth is called as the location parameter or the deepest point. Therefore if the two distributions F and G are identical then they should have the same deepest point. In case of location shift, the deepest point with respect to the distribution F would not be the deepest point with respect to the distribution G. In fact, the deepest point of F will have a smaller depth value with respect to G. M-based test statistic due to Li & Liu (2004) is given by,

$$M = min\{D(v, F_m), D(u, G_n)\},$$

where v is the deepest point of $X \cup Y$ corresponding to G_n, and u is the deepest point of $X \cup Y$ corresponding to F_m. Here smaller the value of M, stronger the evidence against H_0. The p-value of the test is determined by using the Fisher's permutation test.

Li et al. (2011) have proposed two depth based Kolmogorov-Smirnov (KS) and Cramer-Von Mises (CM) type tests for comparing species assemblages. The test statistics are defined as follows,

- **KS type test statistic**

$$T_{KS} = \textstyle\sum_{i=1}^{m+n} |D(z_i, F_m) - D(z_i, G_n)|$$

- **CM type test statistic**

$$T_{CM} = \textstyle\sum_{i=1}^{m+n} (D(z_i, F_m) - D(z_i, G_n))^2,$$

where, z_i is the i^{th} point in $X \cup Y$. These tests reject H_0 for large values of T_{KS} and T_{CM}. Larger the value of T_{KS} and T_{CM}, stronger is the evidence against H_0 and it represents larger location shift between the two distributions. The p-value of these tests are also obtained by using the Fisher's permutation test.

4.2. Nonparametric Test for Testing Equality of Scale Parameters of Two Multivariate Distributions

Li & Liu (2016) proposed a test for testing scale differences between two multivariate distributions and the test statistic is defined as,

$$S = \sum_{z \in X \cup Y} (D(z, F_m) - D(z, G_n)),$$

where z is the point in $X \cup Y$. Larger the value of S, stronger the evidence against H_0. The p-value of this test is also obtained by using Fisher's permutation test.

In the following section, we propose tests for simultaneously testing location and scale parameters of two multivariate distributions using NPC theory.

5. Proposed Tests

Let $(X_1, X_2, \ldots, X_m)$ and $(Y_1, Y_2, \ldots, Y_n)$ be the two independent random samples of size m and n from two continuous distribution with cumulative distribution functions $F(.)$ and $G(.)$ respectively, where each $X_i, Y_j \in \mathbb{R}^p$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. We wish to test the null hypothesis,

$$H_0 : F(x) = G(x) \quad \forall\ x = (x_1, x_2, \dots, x_p)^T,$$

against an alternative hypothesis

$$H_1 : F(x) = G(\tfrac{x-\mu}{\sigma}) \quad \forall\ x = (x_1, x_2, \dots, x_p)^T,$$

where, $(\frac{x-\mu}{\sigma}) = \left(\frac{(x_1-\mu_1)}{\sigma_1}, \frac{(x_2-\mu_2)}{\sigma_2}, \dots, \frac{(x_p-\mu_p)}{\sigma_p}\right)^T$, $\mu = (\mu_1, \mu_2, \dots, \mu_p)^T$, $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_p)^T$ and σ is a diagonal matrix.

It is equivalent to test

$$H_0 : \{\mu = 0\} \cap \{\sigma = 1\}, \tag{1}$$

against

$$H_1 : \{\mu \neq 0\} \cup \{\sigma \neq 1\}.$$

We need a simultaneous test procedure for testing both location and scale parameters to test null hypothesis in (1). Pesarin (2001) has developed a theory of NPC of dependent tests. Such theory is used to develop a nonparametric test for two-sample locations and/or scales problem. NPC assumes a global null hypothesis of both location and scale parameters of two multivariate distributions are same. This global null hypothesis is divided into two partial null hypotheses, one for location problem and other for scale problem. NPC then uses a Fisher's permutation test to obtain the single p-value of global null hypothesis and partial p-values for each partial null-hypothesis. Based on NPC theory, Park (2015) provided several nonparametric simultaneous test procedures for the univariate two-sample location-scale problem using combining functions.

We use the same NPC theory for developing tests for simultaneously testing hypothesis given in (1). We use separate test statistic for testing the sub-null hypotheses $H_{01} : \mu = 0$ and $H_{02} : \sigma = 1$ and then combine the p-values of these two separate tests to obtain the global p-value of the test. Suppose the statistic T^L is used for location test and T^S is used for scale test and p^L and p^S be the p-values of the location test and scale test respectively. Then for obtaining the global p-value for simultaneously testing (1), we use an appropriate combining function to combine the p-values p^L and p^S.

The null distribution of the selected combined function is obtained by using Fisher's permutation principle to compute the global p-value of the location-scale test. An algorithm for obtaining global p-value is given below.

Algorithm for Obtaining Global p-value

The following algorithm is required to get the global p-value of the location-scale test by using NPC theory.

Algorithm

1: Divide the global null hypothesis that is location-scale problem in the partial null hypotheses. That is, $H_{01} : \mu = 0$ and $H_{02} : \sigma = 1$.

2: Select an appropriate test statistic for each partial null hypotheses H_{01} and H_{02}. That is T^L and T^S, which are sensitive to the alternative hypothesis.

3: Calculate its value for observed data, which is denoted by $(T^L_{b=0},\ T^S_{b=0})$.

4: Take B permutations of the original samples and calculate the value of each test statistic for permuted data, $(T^L_b,\ T^S_b), b \in \{1, 2, \ldots, B\}$.

5: Compute the permutation p-values for partial null hypotheses that is partial p-values, $(p^L_{b=0},\ p^S_{b=0})$. If we reject null hypothesis for larger (or smaller) values of test statistic then we use following formula

$$p^L_{b=0} = \frac{[1+\sum I(T^L_{b\neq 0} \geq T^L_{b=0})]}{(B+1)}$$

$$\left(or\ p^L_{b=0} = \frac{[1+\sum I(T^L_{b\neq 0} \leq T^L_{b=0})]}{(B+1)} \right). \tag{2}$$

Similarly, we calculate $p^S_{b=0}$.

6: Compute partial pseudo p-value $(p^L_b,\ p^S_b)$ for each permutation. If we reject null hypothesis for larger (or smaller) values of test statistic then we use following formula

$$p^L_k = \frac{[1+\sum I(T^L_{b\neq k} \geq T^L_{b=k})]}{(B+1)}, \qquad k \in \{1, 2, \ldots, B\}$$

$$\left(or\ p^L_k = \frac{[1+\sum I(T^L_{b\neq k} \leq T^L_{b=k})]}{(B+1)}, \qquad k \in \{1, 2, \ldots, B\} \right). \tag{3}$$

Similarly, we define p^S_k for scale test.

7: Use an appropriate combining function to combine p-values to get a single global test statistic. We get global test statistic as,

- For Fisher's combining function

$$T^{global}_b = -2(log_e p^L_b + log_e p^S_b). \tag{4}$$

- For Tippet combining function

$$T^{global}_b = max\{(1 - p^L_b), (1 - p^S_b)\}. \tag{5}$$

- For Liptak combining function

$$T^{global}_b = \Phi^{-1}(1 - p^L_b) + \Phi^{-1}(1 - p^S_b). \tag{6}$$

8: By combining the p-values, a final vector of length $(B + 1)$ is produced. The first element $T^{global}_{b=0}$ summarizes the partial p-values $(p^L_{b=0},\ p^S_{b=0})$ observed on the initial data sets and the remaining elements $T^{global}_b, b \in \{1, 2, \ldots, B\}$ are produced by combining the corresponding pseudo p-values.

9: Similarly to the partial p-values, calculate a p-value of the global test by using the following equation

$$p^{global} = \frac{[1 + \sum I(T_{b\neq 0}^{global} \geq T_{b=0}^{global})]}{(B+1)}$$

10: Take decision about acceptance or rejection of the global null hypothesis (1) by using global p-value.

In the next section, we evaluate the performance of the proposed tests and data depth based H-test proposed by Chenouri & Small (2012) in terms of the empirical power.

6. Simulation Study

The performance of the proposed tests is investigated through Monte Carlo simulation experiment in terms of empirical power for different bivariate symmetric (normal and t) and bivariate skewed (normal and t) distributions, which are listed in Table 1. The parameter μ represents the shift in the location parameter, σ represents the change in scale parameter, a represents the shape parameter (or skewness parameter) and v represents the degrees of freedom. The number of observations generated from each of F and G are taken to be $m = n = 50$ and $m = n = 100$ respectively. The p-value of the test is obtained by permuting original samples 500 times and the power is obtained by the proportion of times the p-values are less than or equal to the nominal level of significance α. An empirical power of the proposed tests is compared to the H-test. All the results in the Table 3 to Table 10 are reported for 1000 Monte Carlo simulations and R-software is used for simulation studies.

We use KS-type, CM-type (Li et al. 2011), T-based and M-based tests (Li & Liu 2004) for a location problem and S-test (Li & Liu 2016) for a scale problem. Simplicial and Tukey's halfspace depth functions are used in our study to compute the depth of an observation. The Fisher and Tippet combining functions are used to combine the p-values of location test and scale test. The results are reported for all combinations of various values of $\mu = b * (1, 1)$ and $\sigma = c * (1, 1)$ which are given in Table 2. A distribution F always corresponds to value of $\mu = (0, 0)$ and $\sigma = (1, 1)$ and the distribution G corresponds to any value of μ and σ in the Table 2.

TABLE 1: Distributions used in the simulation study.

Distribution	Under H_0	Under H_1
Symmetric normal	$N_2(\mu = (0,0), \sigma = (1,1))$	$N_2(\mu, \sigma)$
Symmetric t	$t_2(\mu = (0,0), \sigma = (1,1))$	$t_2(\mu, \sigma)$
Skewed normal	$SN_2(\mu = (0,0), \sigma = (1,1), a = (10,4)^T)$	$SN_2(\mu, \sigma, a = (10,4)^T)$
Skewed t	$ST_2(\mu = (0,0), \sigma = (1,1), a = (10,4)^T, v = 3)$	$ST_2(\mu, \sigma, a = (10,4)^T, v = 3)$

Table 2: Values of b and c.

b	0.0	0.2	0.4	0.6	0.8	1.0
c	1.0	1.2	1.4	1.6	1.8	2.0

Simulation results with sample sizes $m = n = 50$ and $m = n = 100$ for simplicial depth are reported in Table 3 to Table 6 and for Tukey's halfspace depth, results are reported in Table 7 to Table 10. In the Table 3 to Table 10, T_1, T_2, T_3 and T_4 indicates proposed test using the combination of KS and S tests, CM and S tests, T-based and S tests and M-based and S tests respectively. In the following, we have given conclusions which are obtained for a sample size $m = n = 100$. The conclusions reached with the sample sizes $m = n = 100$ are similar to those reached in the $m = n = 50$. The size of the proposed tests is obtained and it is shown in the first row of Table of each simulation scenario.

6.1. Simulation Results for Symmetric Normal Distribution

In this case, the proposed tests have greater empirical power as compared to the H-test for both depth functions on a shift by shift increase in the location parameter as well as in the scale parameter. The tests T_1 and T_2 perform better than the T_3 and T_4 tests for both Fisher's and Tippet combining functions. Note that, the proposed tests with Fisher's combining function work better than that of the Tippet combining function. The performance of the proposed tests with simplicial depth is similar to the halfspace depth.

6.2. Simulation Results for Symmetric t Distribution

Also, in this case, the proposed tests perform better than the H-test. The tests T_2, T_3 and T_4 are more powerful than T_1 test for both Fisher's and Tippet combining functions with simplicial depth. But, for halfspace depth, all the T_1, T_2, T_3 and T_4 tests perform equally well. Also, the proposed tests with Fisher's combining function have greater power than that of the Liptak combining function for both simplicial depth and halfspace deph.

6.3. Simulation Results for Skewed Normal and Skewed t Distributions

In this case, the proposed tests with Fisher's and Tippet combining functions have greater empirical power as compared to the H-test for a small shift in the location parameteras well as small change in the scale parameter. For a large shift, proposed tests and H-test perform equally well for both combining functions with simplcial and halfspace depths. The tests T_1 and T_2 are more powerful than the T_3 and T_4 tests for a small shift in location parameter and a small change in scale parameter with simplcial and halfspace depths.

The performance of the proposed tests for skewed t distribution is same as in the case of skewed normal distribution.

Table 3: Simulated powers of proposed tests for normal distribution with sample sizes $m = n = 50$ and $m = n = 100$.

		$m=n=50$									$m=n=100$								
		Fisher				Tippet					Fisher				Tippet				
c	b	T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	H-test	T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	H-test
1.0	0.0	0.052	0.055	0.053	0.050	0.049	0.053	0.050	0.053	0.048	0.048	0.045	0.053	0.048	0.049	0.047	0.048	0.048	0.051
	0.2	0.115	0.148	0.108	0.116	0.129	0.153	0.103	0.127	0.140	0.190	0.261	0.188	0.209	0.217	0.293	0.183	0.207	0.168
	0.4	0.370	0.514	0.348	0.362	0.419	0.557	0.342	0.373	0.439	0.721	0.867	0.703	0.715	0.784	0.893	0.719	0.743	0.708
	0.6	0.765	0.912	0.734	0.749	0.827	0.933	0.758	0.783	0.829	0.988	0.999	0.985	0.986	0.997	1.000	0.989	0.989	0.992
	0.8	0.970	0.995	0.948	0.956	0.984	0.998	0.959	0.971	0.977	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	0.999	0.999	1.000	1.000	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.0	0.161	0.149	0.142	0.146	0.150	0.146	0.143	0.148	0.068	0.272	0.259	0.240	0.241	0.240	0.238	0.235	0.238	0.162
	0.2	0.238	0.274	0.216	0.222	0.203	0.238	0.191	0.193	0.156	0.419	0.479	0.382	0.405	0.360	0.416	0.326	0.339	0.284
	0.4	0.495	0.608	0.451	0.480	0.469	0.579	0.396	0.422	0.441	0.837	0.919	0.792	0.823	0.824	0.903	0.730	0.764	0.741
	0.6	0.793	0.909	0.766	0.789	0.809	0.917	0.745	0.766	0.798	0.988	0.998	0.984	0.980	0.990	0.997	0.980	0.973	0.979
	0.8	0.965	0.993	0.951	0.956	0.970	0.995	0.943	0.950	0.968	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	0.996	1.000	0.998	0.997	0.997	1.000	0.997	0.996	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.4	0.0	0.354	0.336	0.323	0.322	0.334	0.326	0.320	0.324	0.135	0.611	0.609	0.553	0.554	0.591	0.584	0.580	0.580	0.412
	0.2	0.422	0.439	0.384	0.389	0.368	0.381	0.345	0.346	0.206	0.712	0.744	0.651	0.678	0.636	0.653	0.592	0.611	0.522
	0.4	0.619	0.706	0.574	0.597	0.542	0.614	0.469	0.490	0.458	0.929	0.957	0.880	0.895	0.885	0.932	0.810	0.821	0.813
	0.6	0.854	0.931	0.812	0.828	0.828	0.908	0.734	0.757	0.781	0.995	1.000	0.992	0.991	0.995	1.000	0.985	0.980	0.986
	0.8	0.977	0.999	0.961	0.963	0.975	0.996	0.941	0.946	0.960	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000
	1.0	0.998	1.000	0.998	0.998	0.996	1.000	0.998	0.994	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.6	0.0	0.554	0.551	0.518	0.522	0.549	0.547	0.543	0.541	0.220	0.854	0.853	0.827	0.824	0.857	0.852	0.849	0.850	0.690
	0.2	0.618	0.630	0.580	0.588	0.568	0.569	0.548	0.549	0.285	0.892	0.903	0.869	0.858	0.857	0.862	0.839	0.840	0.745
	0.4	0.754	0.803	0.704	0.712	0.668	0.718	0.603	0.620	0.504	0.977	0.986	0.949	0.964	0.957	0.970	0.914	0.928	0.900
	0.6	0.926	0.968	0.885	0.900	0.884	0.937	0.799	0.833	0.821	0.998	1.000	0.996	0.997	0.995	0.999	0.983	0.985	0.987
	0.8	0.977	0.995	0.971	0.969	0.970	0.990	0.940	0.943	0.952	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	0.998	1.000	0.997	0.999	0.998	1.000	0.992	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.8	0.0	0.730	0.714	0.693	0.695	0.730	0.726	0.724	0.725	0.351	0.957	0.959	0.949	0.945	0.963	0.960	0.960	0.961	0.852
	0.2	0.766	0.778	0.731	0.739	0.738	0.742	0.720	0.733	0.397	0.972	0.975	0.957	0.965	0.962	0.963	0.956	0.959	0.894
	0.4	0.865	0.895	0.824	0.837	0.797	0.830	0.746	0.762	0.595	0.994	0.997	0.989	0.990	0.983	0.989	0.969	0.974	0.955
	0.6	0.946	0.974	0.916	0.939	0.907	0.937	0.852	0.873	0.823	0.999	1.000	0.999	0.999	0.998	1.000	0.992	0.996	0.993
	0.8	0.989	0.998	0.978	0.982	0.981	0.993	0.948	0.952	0.952	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	0.995	0.998	0.999	1.000	0.989	0.990	0.992	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.0	0.0	0.868	0.863	0.837	0.836	0.866	0.863	0.862	0.862	0.473	0.990	0.991	0.984	0.984	0.992	0.992	0.992	0.992	0.938
	0.2	0.876	0.875	0.849	0.844	0.843	0.846	0.831	0.834	0.517	0.997	0.998	0.998	0.994	0.995	0.995	0.994	0.994	0.959
	0.4	0.924	0.945	0.888	0.911	0.881	0.895	0.847	0.863	0.654	0.998	1.000	0.996	0.997	0.996	0.996	0.992	0.994	0.984
	0.6	0.972	0.986	0.949	0.958	0.941	0.961	0.894	0.907	0.818	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.995
	0.8	0.994	0.999	0.985	0.988	0.984	0.993	0.962	0.962	0.960	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	0.996	0.998	0.999	1.000	0.990	0.990	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

TABLE 4: Simulated powers of proposed tests for t distribution with sample sizes $m = n = 50$ and $m = n = 100$.

c	b	$m=n=50$									$m=n=100$								
		Fisher				Tippet				H-test	Fisher				Tippet				H-test
		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	
1.0	0.0	0.060	0.055	0.047	0.048	0.051	0.046	0.045	0.053	0.049	0.048	0.050	0.051	0.056	0.049	0.052	0.054	0.053	0.050
	0.2	0.076	0.099	0.099	0.096	0.085	0.107	0.103	0.101	0.114	0.133	0.183	0.167	0.181	0.152	0.194	0.172	0.174	0.150
	0.4	0.219	0.323	0.280	0.292	0.248	0.355	0.294	0.308	0.415	0.448	0.610	0.623	0.602	0.526	0.672	0.633	0.629	0.567
	0.6	0.507	0.692	0.654	0.627	0.605	0.754	0.678	0.664	0.779	0.874	0.956	0.955	0.949	0.920	0.973	0.961	0.962	0.952
	0.8	0.793	0.927	0.901	0.870	0.852	0.947	0.919	0.903	0.959	0.989	1.000	0.999	0.999	0.993	1.000	0.999	0.999	0.998
	1.0	0.943	0.989	0.984	0.975	0.969	0.995	0.986	0.984	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.0	0.116	0.118	0.114	0.109	0.103	0.102	0.103	0.104	0.076	0.186	0.185	0.156	0.161	0.160	0.155	0.154	0.154	0.123
	0.2	0.164	0.168	0.160	0.163	0.138	0.148	0.141	0.146	0.132	0.255	0.291	0.275	0.294	0.226	0.259	0.235	0.246	0.196
	0.4	0.282	0.381	0.357	0.355	0.261	0.362	0.307	0.304	0.380	0.563	0.691	0.671	0.687	0.547	0.670	0.624	0.633	0.566
	0.6	0.533	0.702	0.658	0.646	0.550	0.707	0.627	0.615	0.755	0.886	0.959	0.955	0.949	0.906	0.963	0.945	0.941	0.930
	0.8	0.781	0.916	0.889	0.855	0.816	0.923	0.878	0.864	0.944	0.990	0.998	1.000	0.996	0.995	0.999	1.000	0.997	0.999
	1.0	0.932	0.986	0.974	0.967	0.944	0.990	0.979	0.973	0.992	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.4	0.0	0.225	0.219	0.196	0.216	0.205	0.198	0.200	0.202	0.112	0.386	0.378	0.340	0.340	0.359	0.353	0.349	0.348	0.304
	0.2	0.301	0.316	0.276	0.291	0.267	0.277	0.257	0.267	0.191	0.483	0.514	0.474	0.489	0.425	0.440	0.422	0.426	0.398
	0.4	0.406	0.491	0.444	0.455	0.345	0.417	0.358	0.379	0.416	0.722	0.803	0.779	0.804	0.671	0.748	0.691	0.713	0.697
	0.6	0.604	0.740	0.696	0.694	0.571	0.699	0.628	0.620	0.724	0.926	0.967	0.962	0.965	0.921	0.962	0.946	0.942	0.946
	0.8	0.817	0.927	0.895	0.890	0.819	0.919	0.869	0.862	0.940	0.991	0.998	0.997	0.998	0.992	0.999	0.997	0.997	0.996
	1.0	0.934	0.987	0.978	0.971	0.944	0.988	0.971	0.964	0.990	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.6	0.0	0.389	0.381	0.334	0.351	0.356	0.349	0.342	0.350	0.209	0.614	0.607	0.556	0.564	0.591	0.586	0.582	0.584	0.507
	0.2	0.411	0.419	0.379	0.394	0.363	0.365	0.337	0.353	0.259	0.661	0.676	0.641	0.654	0.619	0.619	0.599	0.610	0.565
	0.4	0.540	0.603	0.548	0.567	0.459	0.516	0.469	0.482	0.473	0.829	0.879	0.852	0.865	0.771	0.825	0.776	0.783	0.790
	0.6	0.701	0.798	0.750	0.754	0.635	0.746	0.663	0.660	0.754	0.950	0.980	0.979	0.976	0.933	0.965	0.948	0.948	0.956
	0.8	0.841	0.928	0.900	0.886	0.823	0.913	0.862	0.853	0.926	0.994	0.999	0.998	0.999	0.992	0.997	0.993	0.997	0.996
	1.0	0.944	0.987	0.983	0.972	0.941	0.986	0.974	0.963	0.988	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.8	0.0	0.491	0.486	0.447	0.454	0.470	0.464	0.459	0.462	0.289	0.780	0.778	0.725	0.728	0.770	0.763	0.754	0.759	0.682
	0.2	0.564	0.575	0.531	0.541	0.523	0.527	0.505	0.514	0.362	0.841	0.849	0.796	0.805	0.798	0.797	0.783	0.788	0.759
	0.4	0.627	0.668	0.646	0.636	0.544	0.582	0.553	0.551	0.512	0.903	0.924	0.900	0.915	0.844	0.868	0.851	0.853	0.870
	0.6	0.762	0.839	0.794	0.804	0.679	0.773	0.696	0.711	0.777	0.976	0.994	0.990	0.990	0.959	0.980	0.970	0.971	0.979
	0.8	0.868	0.950	0.928	0.922	0.836	0.924	0.874	0.866	0.930	0.997	1.000	0.999	0.999	0.995	1.000	0.997	0.997	0.998
	1.0	0.953	0.986	0.981	0.976	0.941	0.982	0.964	0.959	0.987	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.0	0.0	0.648	0.626	0.592	0.608	0.621	0.611	0.608	0.615	0.393	0.905	0.905	0.879	0.875	0.900	0.897	0.891	0.890	0.853
	0.2	0.663	0.669	0.630	0.640	0.628	0.621	0.619	0.617	0.433	0.914	0.923	0.889	0.899	0.901	0.900	0.886	0.888	0.867
	0.4	0.721	0.751	0.709	0.732	0.639	0.657	0.628	0.645	0.580	0.959	0.973	0.963	0.964	0.931	0.940	0.925	0.930	0.937
	0.6	0.818	0.871	0.830	0.849	0.746	0.804	0.739	0.752	0.784	0.990	0.994	0.994	0.994	0.978	0.988	0.974	0.978	0.991
	0.8	0.912	0.964	0.940	0.943	0.875	0.929	0.885	0.891	0.938	1.000	1.000	1.000	0.999	0.998	1.000	0.998	0.997	0.999
	1.0	0.966	0.989	0.984	0.985	0.957	0.984	0.965	0.970	0.988	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 5: Simulated powers of proposed tests for skewed normal distribution with sample sizes $m = n = 50$ and $m = n = 100$.

c	b	$m=n=50$									$m=n=100$								
		Fisher				Tippet				H-test	Fisher				Tippet				H-test
		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	
1.0	0.0	0.052	0.056	0.050	0.053	0.047	0.048	0.050	0.053	0.043	0.051	0.048	0.054	0.053	0.054	0.053	0.057	0.057	0.047
	0.2	0.286	0.328	0.135	0.184	0.417	0.450	0.187	0.254	0.282	0.728	0.762	0.335	0.407	0.855	0.877	0.448	0.522	0.605
	0.4	0.961	0.960	0.693	0.790	0.984	0.984	0.785	0.858	0.866	1.000	1.000	0.979	0.985	1.000	1.000	0.990	0.993	0.999
	0.6	1.000	1.000	0.992	0.996	1.000	1.000	0.997	0.998	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.0	0.142	0.144	0.132	0.129	0.121	0.123	0.120	0.122	0.077	0.214	0.221	0.192	0.203	0.180	0.189	0.175	0.176	0.134
	0.2	0.409	0.536	0.279	0.316	0.529	0.621	0.318	0.375	0.346	0.841	0.895	0.521	0.572	0.911	0.942	0.599	0.656	0.647
	0.4	0.980	0.984	0.798	0.872	0.991	0.994	0.852	0.917	0.902	1.000	1.000	0.990	0.994	1.000	1.000	0.997	0.998	1.000
	0.6	1.000	1.000	0.991	0.998	1.000	1.000	0.995	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.4	0.0	0.288	0.310	0.257	0.275	0.244	0.261	0.235	0.253	0.129	0.478	0.533	0.439	0.445	0.425	0.446	0.401	0.394	0.317
	0.2	0.582	0.704	0.400	0.480	0.625	0.741	0.410	0.478	0.466	0.937	0.975	0.748	0.791	0.950	0.981	0.763	0.808	0.771
	0.4	0.989	0.991	0.851	0.911	0.995	0.997	0.893	0.937	0.925	1.000	1.000	0.996	0.999	1.000	1.000	0.998	0.999	1.000
	0.6	1.000	1.000	0.995	0.998	1.000	1.000	0.995	0.998	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.6	0.0	0.452	0.495	0.408	0.430	0.397	0.423	0.373	0.389	0.226	0.741	0.793	0.692	0.709	0.672	0.712	0.637	0.652	0.546
	0.2	0.723	0.852	0.542	0.593	0.734	0.847	0.511	0.558	0.562	0.984	0.996	0.894	0.908	0.982	0.997	0.876	0.886	0.894
	0.4	0.991	0.998	0.913	0.938	0.996	0.999	0.932	0.952	0.952	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000
	0.6	1.000	1.000	0.998	0.999	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.8	0.0	0.610	0.654	0.543	0.577	0.525	0.567	0.480	0.500	0.319	0.909	0.938	0.875	0.883	0.858	0.905	0.828	0.838	0.755
	0.2	0.796	0.905	0.676	0.715	0.788	0.909	0.615	0.660	0.658	0.992	0.998	0.950	0.965	0.991	0.997	0.928	0.944	0.947
	0.4	0.998	1.000	0.954	0.961	0.999	1.000	0.958	0.963	0.969	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.6	1.000	1.000	0.999	0.999	1.000	1.000	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.0	0.0	0.773	0.812	0.719	0.738	0.690	0.739	0.652	0.660	0.462	0.977	0.990	0.958	0.965	0.953	0.974	0.936	0.945	0.885
	0.2	0.894	0.961	0.793	0.819	0.865	0.953	0.700	0.744	0.726	1.000	1.000	0.987	0.990	0.998	1.000	0.971	0.974	0.984
	0.4	0.995	1.000	0.967	0.983	0.997	1.000	0.966	0.981	0.971	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.6	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 6: Simulated powers of proposed tests for skewed t distribution with sample sizes $m = n = 50$ and $m = n = 100$.

c	b	$m=n=50$									$m=n=100$								
		Fisher				Tippet				H-test	Fisher				Tippet				H-test
		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	
1.0	0.0	0.053	0.045	0.051	0.053	0.048	0.045	0.051	0.048	0.053	0.051	0.047	0.048	0.048	0.049	0.048	0.051	0.047	0.048
	0.2	0.194	0.207	0.092	0.118	0.339	0.334	0.149	0.191	0.267	0.583	0.555	0.190	0.237	0.766	0.742	0.276	0.346	0.604
	0.4	0.888	0.831	0.437	0.611	0.962	0.937	0.578	0.716	0.809	1.000	1.000	0.848	0.899	1.000	1.000	0.915	0.945	0.997
	0.6	0.999	0.996	0.855	0.942	1.000	0.998	0.920	0.969	0.992	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.987	0.998	1.000	1.000	0.993	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.0	0.090	0.091	0.084	0.082	0.079	0.084	0.081	0.079	0.066	0.143	0.150	0.126	0.131	0.116	0.122	0.111	0.115	0.091
	0.2	0.259	0.299	0.143	0.174	0.389	0.429	0.192	0.241	0.309	0.676	0.704	0.337	0.384	0.811	0.828	0.436	0.491	0.597
	0.4	0.915	0.880	0.548	0.687	0.966	0.958	0.661	0.782	0.831	1.000	0.999	0.902	0.947	1.000	1.000	0.952	0.970	0.997
	0.6	1.000	0.997	0.895	0.960	1.000	1.000	0.943	0.977	0.988	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.987	0.998	1.000	1.000	0.994	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.4	0.0	0.151	0.161	0.137	0.140	0.129	0.136	0.132	0.126	0.106	0.245	0.271	0.225	0.243	0.206	0.230	0.208	0.207	0.163
	0.2	0.341	0.440	0.231	0.271	0.475	0.550	0.283	0.335	0.370	0.756	0.826	0.508	0.551	0.862	0.908	0.600	0.640	0.644
	0.4	0.922	0.920	0.641	0.728	0.975	0.974	0.737	0.824	0.850	1.000	1.000	0.946	0.965	1.000	1.000	0.975	0.985	0.994
	0.6	0.998	0.999	0.920	0.973	1.000	1.000	0.958	0.989	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.995	0.999	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.6	0.0	0.250	0.268	0.230	0.227	0.199	0.222	0.197	0.192	0.166	0.444	0.496	0.419	0.439	0.376	0.422	0.362	0.378	0.321
	0.2	0.434	0.543	0.314	0.354	0.535	0.619	0.349	0.401	0.457	0.840	0.903	0.647	0.681	0.902	0.949	0.698	0.733	0.734
	0.4	0.939	0.944	0.703	0.779	0.979	0.977	0.780	0.858	0.886	1.000	1.000	0.971	0.976	1.000	1.000	0.986	0.990	0.998
	0.6	1.000	1.000	0.956	0.976	1.000	1.000	0.976	0.988	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.8	0.0	0.332	0.377	0.308	0.317	0.280	0.317	0.264	0.270	0.221	0.578	0.658	0.572	0.592	0.506	0.564	0.479	0.499	0.468
	0.2	0.487	0.635	0.378	0.426	0.558	0.697	0.389	0.465	0.506	0.917	0.955	0.765	0.797	0.944	0.977	0.794	0.828	0.833
	0.4	0.955	0.966	0.756	0.817	0.983	0.987	0.827	0.875	0.906	1.000	1.000	0.986	0.993	1.000	1.000	0.993	0.998	0.999
	0.6	0.998	0.999	0.953	0.978	0.999	1.000	0.970	0.989	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.996	1.000	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.0	0.0	0.427	0.486	0.429	0.422	0.358	0.414	0.353	0.362	0.315	0.722	0.790	0.706	0.728	0.629	0.708	0.618	0.629	0.625
	0.2	0.577	0.718	0.485	0.528	0.636	0.763	0.498	0.542	0.595	0.944	0.975	0.859	0.866	0.961	0.985	0.858	0.871	0.886
	0.4	0.954	0.970	0.799	0.846	0.979	0.991	0.852	0.890	0.921	1.000	1.000	0.994	0.996	1.000	1.000	0.997	0.999	0.999
	0.6	0.998	0.999	0.968	0.982	1.000	1.000	0.984	0.992	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.997	1.000	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

TABLE 7: Simulated powers of proposed tests for normal distribution with sample sizes $m = n = 50$ and $m = n = 100$.

		$m=n=50$									$m=n=100$								
		Fisher				Tippet					Fisher				Tippet				
c	b	T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	H-test	T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	H-test
1.0	0.0	0.050	0.044	0.045	0.050	0.044	0.047	0.046	0.047	0.047	0.056	0.053	0.057	0.058	0.055	0.057	0.057	0.057	0.055
	0.2	0.117	0.145	0.111	0.120	0.113	0.158	0.107	0.097	0.066	0.211	0.257	0.216	0.194	0.219	0.287	0.199	0.177	0.066
	0.4	0.388	0.502	0.378	0.375	0.422	0.559	0.380	0.362	0.250	0.768	0.865	0.716	0.738	0.821	0.903	0.737	0.749	0.328
	0.6	0.772	0.876	0.745	0.757	0.816	0.917	0.746	0.757	0.651	0.991	0.999	0.979	0.984	0.995	0.998	0.982	0.985	0.883
	0.8	0.975	0.993	0.958	0.968	0.983	0.997	0.961	0.971	0.945	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
	1.0	0.999	1.000	0.997	0.996	0.999	1.000	0.997	0.996	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.0	0.159	0.160	0.155	0.155	0.151	0.141	0.152	0.151	0.120	0.249	0.252	0.231	0.228	0.239	0.236	0.237	0.242	0.191
	0.2	0.226	0.249	0.211	0.214	0.189	0.214	0.181	0.174	0.137	0.438	0.490	0.405	0.424	0.364	0.423	0.329	0.336	0.220
	0.4	0.496	0.582	0.469	0.496	0.438	0.564	0.385	0.407	0.271	0.833	0.898	0.797	0.816	0.799	0.894	0.714	0.737	0.430
	0.6	0.802	0.902	0.788	0.793	0.787	0.909	0.729	0.732	0.632	0.993	1.000	0.987	0.991	0.990	0.999	0.974	0.979	0.867
	0.8	0.966	0.991	0.953	0.963	0.968	0.994	0.941	0.951	0.922	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997
	1.0	0.996	1.000	0.994	0.997	0.997	1.000	0.994	0.996	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.4	0.0	0.345	0.346	0.318	0.323	0.337	0.323	0.334	0.336	0.270	0.564	0.583	0.548	0.552	0.574	0.572	0.578	0.580	0.510
	0.2	0.421	0.442	0.411	0.412	0.379	0.392	0.369	0.378	0.306	0.719	0.754	0.682	0.696	0.650	0.683	0.623	0.642	0.553
	0.4	0.657	0.725	0.621	0.635	0.547	0.645	0.512	0.512	0.448	0.935	0.959	0.909	0.918	0.876	0.934	0.834	0.846	0.710
	0.6	0.868	0.926	0.841	0.858	0.810	0.906	0.753	0.754	0.703	0.998	1.000	0.991	0.996	0.994	1.000	0.979	0.985	0.920
	0.8	0.972	0.992	0.964	0.970	0.956	0.991	0.932	0.943	0.920	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	0.997
	1.0	0.999	1.000	0.996	0.996	0.997	1.000	0.994	0.994	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.6	0.0	0.550	0.576	0.533	0.555	0.566	0.553	0.570	0.578	0.491	0.849	0.856	0.842	0.839	0.864	0.857	0.862	0.865	0.805
	0.2	0.630	0.654	0.606	0.619	0.584	0.591	0.579	0.587	0.518	0.889	0.909	0.869	0.876	0.858	0.870	0.850	0.853	0.798
	0.4	0.741	0.798	0.719	0.725	0.650	0.721	0.611	0.617	0.562	0.980	0.986	0.963	0.971	0.945	0.969	0.917	0.923	0.862
	0.6	0.915	0.958	0.894	0.898	0.855	0.933	0.790	0.799	0.770	0.999	1.000	0.996	0.996	0.994	1.000	0.988	0.989	0.967
	0.8	0.984	0.997	0.970	0.980	0.972	0.997	0.929	0.944	0.943	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	0.999
	1.0	0.999	1.000	0.997	0.999	0.998	1.000	0.991	0.993	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.8	0.0	0.707	0.716	0.682	0.689	0.713	0.707	0.716	0.723	0.635	0.951	0.956	0.941	0.943	0.957	0.951	0.957	0.957	0.931
	0.2	0.771	0.794	0.759	0.763	0.745	0.749	0.747	0.752	0.682	0.964	0.976	0.961	0.966	0.966	0.965	0.959	0.963	0.933
	0.4	0.852	0.882	0.828	0.840	0.784	0.817	0.752	0.761	0.723	0.994	0.997	0.990	0.992	0.981	0.989	0.976	0.969	0.956
	0.6	0.951	0.976	0.922	0.938	0.902	0.943	0.842	0.856	0.855	1.000	1.000	1.000	1.000	1.000	1.000	0.995	0.998	0.991
	0.8	0.991	0.997	0.983	0.986	0.971	0.990	0.945	0.951	0.948	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	0.998	1.000	1.000	0.997	0.996	1.000	0.993	0.990	0.992	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.0	0.0	0.831	0.850	0.817	0.824	0.854	0.846	0.855	0.859	0.775	0.991	0.995	0.988	0.991	0.992	0.992	0.992	0.993	0.981
	0.2	0.865	0.879	0.859	0.856	0.849	0.854	0.852	0.853	0.800	0.991	0.992	0.986	0.989	0.991	0.990	0.992	0.990	0.979
	0.4	0.913	0.933	0.902	0.908	0.870	0.897	0.858	0.864	0.834	0.999	1.000	0.996	0.998	0.996	0.999	0.994	0.993	0.991
	0.6	0.969	0.983	0.957	0.961	0.930	0.960	0.900	0.904	0.903	1.000	1.000	1.000	0.999	0.999	1.000	0.999	0.997	0.997
	0.8	0.995	0.999	0.990	0.992	0.982	0.991	0.960	0.967	0.967	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	0.999	1.000	0.997	1.000	0.997	1.000	0.991	0.990	0.992	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

TABLE 8: Simulated powers of proposed tests for t distribution with sample sizes $m = n = 50$ and $m = n = 100$.

c	b	$m=n=50$									$m=n=100$								
		Fisher				Tippet				H-test	Fisher				Tippet				H-test
		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	
1.0	0.0	0.047	0.046	0.049	0.048	0.049	0.046	0.042	0.047	0.052	0.058	0.055	0.048	0.052	0.050	0.050	0.046	0.047	0.050
	0.2	0.084	0.094	0.094	0.092	0.092	0.102	0.090	0.078	0.057	0.158	0.172	0.180	0.168	0.167	0.185	0.174	0.156	0.065
	0.4	0.280	0.309	0.327	0.309	0.304	0.345	0.312	0.278	0.170	0.596	0.603	0.614	0.618	0.641	0.662	0.626	0.617	0.215
	0.6	0.576	0.620	0.656	0.626	0.635	0.691	0.669	0.619	0.452	0.960	0.955	0.952	0.954	0.971	0.979	0.962	0.964	0.643
	0.8	0.859	0.884	0.895	0.894	0.901	0.920	0.900	0.891	0.800	0.998	0.998	0.998	0.999	0.999	1.000	0.998	0.999	0.954
	1.0	0.974	0.982	0.986	0.980	0.982	0.989	0.988	0.981	0.962	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
1.2	0.0	0.130	0.128	0.115	0.117	0.110	0.110	0.108	0.112	0.086	0.168	0.180	0.150	0.156	0.160	0.162	0.160	0.163	0.131
	0.2	0.168	0.174	0.167	0.168	0.139	0.151	0.141	0.129	0.106	0.312	0.319	0.316	0.329	0.267	0.269	0.264	0.260	0.156
	0.4	0.340	0.366	0.372	0.365	0.287	0.332	0.303	0.281	0.197	0.664	0.681	0.702	0.690	0.647	0.666	0.648	0.625	0.271
	0.6	0.621	0.656	0.696	0.673	0.609	0.671	0.643	0.606	0.457	0.943	0.945	0.954	0.953	0.949	0.954	0.947	0.944	0.658
	0.8	0.849	0.884	0.899	0.874	0.849	0.896	0.881	0.858	0.757	0.997	0.998	0.999	1.000	0.998	1.000	0.997	0.999	0.949
	1.0	0.965	0.981	0.985	0.984	0.975	0.986	0.981	0.980	0.948	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998
1.4	0.0	0.232	0.230	0.221	0.217	0.215	0.207	0.215	0.213	0.189	0.368	0.375	0.348	0.345	0.361	0.355	0.360	0.363	0.323
	0.2	0.291	0.299	0.272	0.285	0.241	0.249	0.245	0.249	0.206	0.489	0.506	0.478	0.491	0.432	0.428	0.419	0.422	0.344
	0.4	0.445	0.453	0.468	0.457	0.363	0.400	0.376	0.357	0.285	0.781	0.790	0.789	0.796	0.710	0.731	0.690	0.681	0.463
	0.6	0.681	0.719	0.746	0.710	0.612	0.687	0.663	0.613	0.491	0.966	0.965	0.970	0.970	0.953	0.958	0.943	0.948	0.733
	0.8	0.865	0.895	0.914	0.902	0.839	0.890	0.864	0.849	0.754	0.997	0.999	0.999	0.998	0.998	0.999	0.998	0.998	0.945
	1.0	0.964	0.982	0.983	0.981	0.964	0.980	0.972	0.968	0.940	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998
1.6	0.0	0.355	0.364	0.331	0.346	0.343	0.336	0.348	0.347	0.305	0.608	0.612	0.570	0.572	0.601	0.589	0.591	0.597	0.574
	0.2	0.424	0.441	0.407	0.403	0.379	0.390	0.386	0.381	0.335	0.687	0.694	0.674	0.669	0.636	0.630	0.625	0.631	0.583
	0.4	0.538	0.558	0.555	0.545	0.449	0.476	0.460	0.447	0.393	0.849	0.855	0.848	0.858	0.778	0.787	0.754	0.760	0.645
	0.6	0.726	0.747	0.767	0.752	0.667	0.702	0.666	0.645	0.566	0.981	0.980	0.980	0.981	0.967	0.968	0.953	0.953	0.828
	0.8	0.895	0.910	0.922	0.915	0.860	0.896	0.872	0.854	0.793	0.999	0.999	0.999	0.997	0.999	0.999	0.997	0.996	0.968
	1.0	0.963	0.975	0.978	0.976	0.965	0.971	0.963	0.953	0.931	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996
1.8	0.0	0.495	0.487	0.454	0.462	0.477	0.462	0.469	0.476	0.434	0.784	0.788	0.741	0.745	0.783	0.778	0.779	0.785	0.758
	0.2	0.546	0.549	0.518	0.532	0.500	0.490	0.497	0.495	0.467	0.827	0.838	0.806	0.814	0.790	0.785	0.784	0.783	0.765
	0.4	0.632	0.657	0.643	0.645	0.540	0.557	0.542	0.537	0.502	0.931	0.940	0.919	0.935	0.883	0.887	0.864	0.876	0.809
	0.6	0.800	0.826	0.829	0.825	0.710	0.750	0.706	0.681	0.670	0.993	0.992	0.989	0.990	0.976	0.981	0.964	0.964	0.899
	0.8	0.919	0.934	0.936	0.928	0.870	0.908	0.875	0.863	0.833	0.998	0.999	0.998	0.999	0.997	0.998	0.997	0.997	0.973
	1.0	0.975	0.983	0.987	0.986	0.962	0.978	0.972	0.965	0.948	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.0	0.0	0.598	0.603	0.554	0.569	0.602	0.584	0.601	0.601	0.556	0.889	0.895	0.865	0.868	0.889	0.885	0.891	0.889	0.877
	0.2	0.663	0.669	0.617	0.634	0.628	0.615	0.614	0.626	0.581	0.920	0.922	0.905	0.902	0.899	0.895	0.896	0.891	0.883
	0.4	0.742	0.750	0.733	0.742	0.662	0.685	0.651	0.653	0.651	0.969	0.968	0.962	0.970	0.936	0.939	0.918	0.928	0.910
	0.6	0.849	0.861	0.859	0.860	0.752	0.790	0.758	0.751	0.741	0.991	0.991	0.994	0.994	0.983	0.983	0.976	0.979	0.950
	0.8	0.937	0.948	0.950	0.949	0.894	0.915	0.900	0.889	0.875	0.999	1.000	0.999	1.000	0.998	0.998	0.998	0.998	0.989
	1.0	0.980	0.985	0.988	0.987	0.958	0.979	0.972	0.961	0.955	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999

Table 9: Simulated powers of proposed tests for skewed normal distribution with sample sizes $m = n = 50$ and $m = n = 100$.

c	b	$m=n=50$									$m=n=100$								
		Fisher				Tippet				H-test	Fisher				Tippet				H-test
		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	
1.0	0.0	0.047	0.049	0.052	0.047	0.047	0.048	0.050	0.046	0.047	0.052	0.057	0.047	0.059	0.048	0.047	0.043	0.053	0.050
	0.2	0.229	0.332	0.156	0.180	0.336	0.473	0.213	0.231	0.302	0.587	0.726	0.357	0.388	0.730	0.855	0.461	0.477	0.522
	0.4	0.920	0.958	0.714	0.784	0.967	0.988	0.783	0.834	0.890	1.000	1.000	0.976	0.987	1.000	1.000	0.990	0.996	0.998
	0.6	1.000	1.000	0.988	0.996	1.000	1.000	0.995	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.0	0.110	0.117	0.109	0.113	0.105	0.100	0.105	0.111	0.076	0.196	0.203	0.179	0.196	0.173	0.172	0.168	0.179	0.132
	0.2	0.348	0.509	0.281	0.302	0.426	0.605	0.314	0.309	0.267	0.752	0.883	0.562	0.599	0.819	0.928	0.622	0.641	0.412
	0.4	0.958	0.984	0.812	0.879	0.979	0.994	0.857	0.908	0.904	1.000	1.000	0.993	0.992	1.000	1.000	0.995	0.996	0.997
	0.6	1.000	1.000	0.994	0.997	1.000	1.000	0.998	0.997	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.4	0.0	0.254	0.275	0.246	0.246	0.217	0.229	0.217	0.215	0.169	0.463	0.505	0.446	0.455	0.419	0.448	0.403	0.419	0.327
	0.2	0.500	0.670	0.427	0.440	0.524	0.717	0.407	0.415	0.328	0.881	0.961	0.775	0.784	0.899	0.975	0.778	0.779	0.538
	0.4	0.973	0.992	0.877	0.922	0.984	0.995	0.903	0.928	0.923	1.000	1.000	0.998	0.999	1.000	1.000	0.999	0.999	0.999
	0.6	1.000	1.000	0.996	1.000	1.000	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.6	0.0	0.460	0.484	0.430	0.446	0.392	0.422	0.385	0.398	0.319	0.756	0.803	0.729	0.735	0.690	0.717	0.667	0.677	0.597
	0.2	0.685	0.841	0.590	0.619	0.655	0.840	0.522	0.534	0.462	0.963	0.994	0.907	0.919	0.957	0.996	0.885	0.897	0.743
	0.4	0.984	0.997	0.928	0.954	0.987	1.000	0.938	0.954	0.949	1.000	1.000	0.999	1.000	1.000	1.000	0.999	1.000	1.000
	0.6	1.000	1.000	0.997	0.999	1.000	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.8	0.0	0.623	0.667	0.601	0.612	0.569	0.590	0.547	0.561	0.481	0.906	0.940	0.888	0.895	0.855	0.890	0.835	0.842	0.793
	0.2	0.791	0.915	0.705	0.735	0.742	0.903	0.616	0.641	0.600	0.993	0.999	0.964	0.967	0.982	0.998	0.935	0.937	0.870
	0.4	0.988	0.998	0.946	0.963	0.989	1.000	0.948	0.962	0.966	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.6	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.0	0.0	0.754	0.801	0.728	0.738	0.675	0.723	0.654	0.666	0.595	0.974	0.982	0.965	0.967	0.946	0.966	0.934	0.938	0.899
	0.2	0.874	0.952	0.809	0.841	0.815	0.944	0.718	0.738	0.723	0.999	0.999	0.989	0.993	0.996	1.000	0.972	0.979	0.953
	0.4	0.996	1.000	0.977	0.982	0.995	1.000	0.966	0.975	0.983	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

TABLE 10: Simulated powers of proposed tests for skewed t distribution with sample sizes $m = n = 50$ and $m = n = 100$.

c	b	$m=n=50$									$m=n=100$								
		Fisher				Tippet				H-test	Fisher				Tippet				H-test
		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4		T_1	T_2	T_3	T_4	T_1	T_2	T_3	T_4	
1.0	0.0	0.037	0.042	0.044	0.044	0.044	0.046	0.042	0.044	0.047	0.054	0.047	0.052	0.044	0.047	0.046	0.049	0.045	0.044
	0.2	0.143	0.195	0.089	0.104	0.251	0.336	0.126	0.141	0.327	0.428	0.514	0.200	0.242	0.597	0.703	0.290	0.317	0.580
	0.4	0.818	0.836	0.444	0.588	0.912	0.934	0.572	0.672	0.890	0.997	0.998	0.835	0.901	0.999	1.000	0.905	0.942	0.997
	0.6	0.998	0.996	0.845	0.957	1.000	1.000	0.909	0.974	0.997	1.000	1.000	0.995	0.999	1.000	1.000	0.997	0.999	1.000
	0.8	1.000	1.000	0.994	1.000	1.000	1.000	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.0	0.093	0.094	0.090	0.090	0.076	0.083	0.090	0.080	0.061	0.121	0.126	0.124	0.120	0.108	0.109	0.105	0.105	0.075
	0.2	0.205	0.281	0.152	0.164	0.311	0.409	0.194	0.206	0.272	0.526	0.640	0.349	0.353	0.673	0.778	0.440	0.453	0.454
	0.4	0.855	0.876	0.535	0.665	0.928	0.952	0.645	0.741	0.870	0.999	0.999	0.914	0.944	1.000	1.000	0.967	0.973	0.994
	0.6	0.998	0.997	0.915	0.966	0.999	1.000	0.950	0.980	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.992	0.998	1.000	1.000	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.4	0.0	0.168	0.175	0.167	0.170	0.151	0.150	0.149	0.150	0.110	0.270	0.287	0.257	0.261	0.221	0.246	0.226	0.222	0.152
	0.2	0.265	0.388	0.229	0.254	0.356	0.510	0.265	0.272	0.278	0.688	0.783	0.529	0.539	0.785	0.880	0.601	0.611	0.428
	0.4	0.865	0.912	0.638	0.739	0.945	0.970	0.727	0.805	0.859	0.999	0.999	0.946	0.970	1.000	1.000	0.973	0.982	0.990
	0.6	0.997	0.998	0.937	0.974	0.999	0.999	0.963	0.984	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.994	0.999	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.6	0.0	0.249	0.264	0.245	0.247	0.201	0.225	0.204	0.195	0.159	0.793	0.880	0.656	0.683	0.855	0.934	0.714	0.722	0.441
	0.2	0.380	0.508	0.337	0.351	0.450	0.603	0.349	0.357	0.306	0.965	0.990	0.905	0.915	0.961	0.992	0.883	0.894	0.746
	0.4	0.884	0.936	0.705	0.787	0.944	0.977	0.774	0.834	0.866	1.000	1.000	0.977	0.985	1.000	1.000	0.988	0.991	0.991
	0.6	0.999	0.999	0.947	0.980	1.000	1.000	0.967	0.989	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.991	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.8	0.0	0.335	0.359	0.327	0.330	0.293	0.301	0.289	0.277	0.225	0.610	0.648	0.605	0.603	0.529	0.559	0.520	0.522	0.394
	0.2	0.440	0.584	0.424	0.436	0.500	0.661	0.418	0.412	0.334	0.866	0.933	0.765	0.782	0.895	0.951	0.789	0.795	0.508
	0.4	0.905	0.952	0.757	0.823	0.951	0.979	0.820	0.857	0.880	0.999	1.000	0.987	0.992	1.000	1.000	0.996	0.997	0.995
	0.6	0.997	0.998	0.963	0.989	0.999	1.000	0.979	0.994	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.997	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2.0	0.0	0.445	0.485	0.425	0.435	0.364	0.397	0.355	0.359	0.309	0.756	0.769	0.733	0.736	0.653	0.684	0.637	0.633	0.516
	0.2	0.559	0.687	0.512	0.526	0.576	0.745	0.486	0.477	0.389	0.924	0.970	0.861	0.873	0.933	0.977	0.857	0.873	0.620
	0.4	0.947	0.978	0.832	0.874	0.968	0.991	0.873	0.899	0.901	1.000	1.000	0.994	0.997	1.000	1.000	0.996	0.998	0.996
	0.6	1.000	0.999	0.969	0.989	1.000	1.000	0.981	0.994	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.8	1.000	1.000	0.998	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

7. Illustration With Real Life Data

The proposed tests are applied to turtle data set (Jolicoeur & Mosimann 1960) for assessing its performance for practical situations. This data set contains three measures of 48 turtles, 24 each of female and male turtles. The three variables are the carapace length, carapace width and carapace height of a turtle. The location and scale parameters consist of values of carapace length, carapace width and carapace height in the respective populations.

We wish to test whether the location and scale parameters of these two populations are equal or not. Multivariate Shapiro-Wilk normality test for female and male turtle populations gives p-values 0.01551 and 0.1626 respectively. Therefore, carapace length, carapace width and carapace height of female population do not follow the trivariate normal distribution. In this situation, tests based on assumption of multivariate normality are not useful. Therefore, we use proposed tests for testing the equality of the location and scale parameters of male and female turtle populations.

Based on 100000 permutations and using simplicial depth, the p-values of the proposed tests and H-test are estimated and provided in Table 11.

Table 11: p-values of the proposed tests and H-test.

Combining function	Equation number	Test	p-value
Fisher	4	T_1	0.0480
		T_2	0.0014
		T_3	0.0008
		T_4	0.0449
Tippet	5	T_1	0.0438
		T_2	0.0012
		T_3	0.0018
		T_4	0.0378
		H-test	0.0040

From Table 11, all the p-values of the proposed tests show that the location and scale parameters of male and female turtle populations are not equal.

8. Concluding Remarks

In the present article, we have used nonparametric combination (NPC) theory to develop the tests for simultaneously testing the location and scale parameters of two multivariate distributions. These tests are nonparametric and they have better performance in terms of empirical power for bivariate normal, bivariate t, skewed normal and skewed t distributions. The proposed tests perform well for both combining functions with simplicial and halfspace depth. NPC theory is also useful for developing a test for simultaneously testing the location and scale parameters of more than two multivariate distributions. A similar study can be attempted by using other depth functions such as Mahalanobis depth, Spatial depth etc.

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