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Bayesian Estimation of the Limiting Availability in a Repairable One-Unit System

Estimación Bayesiana de la disponibilidad límite en un sistema reparable uni-componente

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Abstract

This work presents a Bayesian approach for estimating the limiting availability of an one-unit repairable system. A Bayesian analysis is developed considering an informative prior and a less informative prior distribution, respectively. Simulations are presented to study the performance of the Bayesian solutions. The maximum likelihood method is also revisited. Finally, a case study is considered, the Bayesian methodology is applied to estimate the limiting availability of a palletizer, which is used in the packaging of glass bottles. Extensions to a coherent system are also discussed.

***Key words*:** Conjugate analysis; Coherent system; Exponential and Gamma distributions; Generalized Beta distribution.

Resumen

En este trabajo se presenta un enfoque bayesiano para estimar la disponibilidad límite de un sistema reparable uni-componente. Un análisis bayesiano es desarrollado considerando distribuciones a priori informativa y poco informativa, respectivamnete. Simulaciones son presentadas para estudiar el desempeño de las soluciones bayesianas. El método de máxima verosimilitud también es reconsiderado. Finalmente, un caso de estudio es considerado, la metodología bayesiana es aplicada para estimar la disponibilidad límite de un paletizador. el cual es usado en el embalaje de botellas de vidrio. Extensiones a un sistema coherente, también son discutidas.

***Palabras clave*:** Análisis conjugado; Distribuciones Exponencial y Gamma; Distribución Beta generalizada; Sistema coherente.

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1. Introduction

Consider a component that is placed in operation at time $t = 0$. At any time if the component fails, it is repaired or replaced by another component, starting back to work the same way as when the component was new, thus we have a sequence of independent and identically distributed lifetime variables. When a component fails, this is off for a certain period of time, which is called repair time of the component. It is assumed that repair times are mutually independent and independent of future lifetimes, i.e., the repair or replacement of the component does not affect the future performance of the component. Such a situation can be modelled by an alternating renewal process.

Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ two sequences of non-negative independent random variables so that the sequences are mutually independent. X has cumulative distribution function given by $F_X(\cdot)$, with mean μ_X and variance $\sigma_X^2 < \infty$, and Y has cumulative distribution function $F_Y(\cdot)$, with mean μ_Y and variance $\sigma_Y^2 < \infty$. If

$$S_n = \begin{cases} \sum_{i=1}^{n}(X_i + Y_i), \text{and } n \geq 1 \\ 0, \text{ and } n = 0 \end{cases}$$

and $S'_{n+1} = S_n + X_{n+1}$, the sequence $\{X_n, Y_n\}_{n\in\mathbb{N}}$ defines a stochastic process $\{R(t) : t \in [0, \infty)\}$ known as alternating renewal process where

$$R(t) = \begin{cases} 1, & \exists\, n \in \mathbb{N} \cup \{0\} : S_n \leq t < S'_{n+1} \\ 0, & \text{otherwise.} \end{cases}$$

If $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are sequences between failures times and repair times, respectively, then S_n corresponds to the time to repair (or replacement) of the n-th failure, and thus S'_m is the time until the m-th failure. Thus, the variable $R(t)$ takes the value 1 if the component is working at the time instant $t > 0$ and takes the value 0 if the component is not operating on $t > 0$.

The point availability of a repairable system $A(t)$ is the probability that the system is operating at time t, i.e.,

$$A(t) = \mathbb{P}(R(t) = 1). \tag{1}$$

Certainly it is desired to find the expression of $A(t)$, but this is too hard except for a few simple cases (Sarkar & Chandhuri 1999). In practice, there is interest in the steady system availability, say A.

A random variable is called lattice, if it only takes on integral multiples of some nonnegative number d. The largest d having this property is said to be the period of the random variable. If a random variable is lattice, we say that its cumulative function distribution is lattice, see, for example, Ross (1996). If the distribution $F_T(\cdot)$ of $T_i = X_i + Y_i$ is not lattice, then the limit availability is given by Barlow & Proschan (1996).

$$A = \lim_{t\to\infty} A(t) = \frac{\mu_X}{\mu_X + \mu_Y}. \tag{2}$$

The term availability here refers to the ability of a component to operate over a certain period of time. The term operate refers to the fact that item is operating or able to operate if required. This result is very important because it not only allows an easy calculation of the limit availability, but also in turn bring the availability of a component or system in the long term, i.e., in a stationary state. Examples of these are the case of mineral conveyor belts in large mining, in a series arrangement, industrial washing systems in a parallel arrangement and the so-called Kanban system (card in Japanese), which is a linear system of production cells (Marsan, Balbo, Conte, Donatelli & Franceschinis 1995). A clever deduction of Equation (2), via renewal theory, can be found in Vásquez (2006).

Estimation of the point and limiting availabilities has been discussed by many authors using Bayesian and classical methods. Thompson & Palicio (1975) present a numerical procedure for computing Bayes credibility interval and the confidence interval estimation is considered by Mi (1991), in a series system case. Moreover, Baxter & Li (1994) and Baxter & Li (1996), focus on making the estimation of availability and limiting availability using a nonparametric approach, invoking the product-limit estimator (Kaplan & Meier 1958) when the data are subject to right censorship. More recently, Abraham & Balakrishna (2000) estimate the limiting availability for a system with failure and repair times from a stationary bivariate sequence. The limiting availability with non-identical failure and repair times distributions is considered by Mi (2006). On the other hand, Ananda (1999) considers the estimation of the long-run availability of a parallel system having several independent renewable components with exponentially distributed failure and repair times.The statistical inference about the steady state availability, with particular Gamma lifetime and repair time is considered by Lu & Mi (2011). The k-out-of-n system is studied by Mishra & Jain (2013), with exponential failure time and different distributions of the repair time. The interval estimation (computation) of the reliability (availability) is the subject of Huang & Mi (2013) and Mathew & Balakrishna (2014). An interesting application of the estimation of the availability of wind farm electric system can be found in Sobolewski (2016). All these works are addressed under a standard approach. Furthermore a general model for repair models is defined in Sethuraman & Hollander (2009) under a nonparametric Bayes setting.

This work proposes a Bayesian estimation method for the limiting availability of an one-unit system. To the best of our knowledge, this approach has not been studied in the literature yet. In Section 2 we revisited the methodology of maximum likelihood estimation. The Bayesian approach is presented in Section 3. In both sections, simulation studies are carried out in order to analyze the behavior of the estimators. We focussed our work on one-unit systems, with exponential (respectively gamma) distributions for failure and repair times. The exponential distribution is a standard referent in reliability. In particular, it is used when a certain component does not wear out over time. On the other hand, the Gamma distribution takes into account a larger family of models and allows to include the possibility that the failure rate varies as a function of time.The Bayesian methodology, presented in this paper, is applied considering these distributions (failure/repair times), since they allow us to develop a conjugate analysis, having as a priori distribution the generalized Beta distribution. The analytical development can be carried out completely. However, the use of other distributions should make use of methods of the Markov Chain Monte Carlo type. In addition, this involves certain aspects that could be useful, from the educational statistical point of view. A case study is considered in Section 4, the Bayesian methodology is applied to estimate the limiting availability of a palletizer, which is used in the packaging of glass bottles. This machine (manually operated) is commonly used in a certain Chilean factory of glass bottles or similar. Finally, a discussion and extensions are presented in Section 5.

2. Maximum Likelihood Estimation

In this section, we revisit the maximum likelihood estimation of the limiting availability. Also, we estimate the variance of the estimator using a Taylor expansion, in concordance with Baxter & Li (1996).

Consider X the failure time with mean μ_X and Y the repairable time with mean μ_Y.

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random sample of size n of the failure times and $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ a random sample of size m of the repairable times. If the samples are independent, then the complete likelihood function is the product of the marginal likelihoods, i.e.,

$$L(\mu_X, \mu_Y; \boldsymbol{x}, \boldsymbol{y}) = L(\mu_X; \boldsymbol{x}) \times L(\mu_Y; \boldsymbol{y}). \tag{3}$$

By the invariance principle (Zacks 1971) the maximum likelihood estimator of the limiting availability A is given by

$$\widehat{A}_{mle} = \frac{\widehat{\mu}_X}{\widehat{\mu}_X + \widehat{\mu}_Y}. \tag{4}$$

Following Baxter & Li (1996), for the calculus of the variance of the estimator, we consider a Taylor expansion of second order of the function $f(u, v) = u/(u + v)$, $(u, v) \in (0, \infty) \times (0, \infty)$, around the point (a, b), i.e.,

$$\begin{aligned} f(u,v) &\approx f(a,b) + \frac{\partial f(a,b)}{\partial u}(u-a) + \frac{\partial f(a,b)}{\partial v}(v-b) \\ &\quad + \frac{1}{2}\left[\frac{\partial^2 f(a_1,b_1)}{\partial u^2}(u-a)^2 + 2\frac{\partial^2 f(a_1,b_1)}{\partial u \partial v}(u-a)(v-b) + \frac{\partial^2 f(a_1,b_1)}{\partial v^2}(v-b)^2\right] \\ &= \frac{a}{a+b} + \frac{b}{(a+b)^2}(u-a) - \frac{a}{(a+b)^2}(v-b) \\ &\quad - \frac{b_1}{(a_1+b_1)^3}(u-a)^2 + \frac{a_1-b_1}{(a_1+b_1)^3}(u-a)(v-b) + \frac{a_1}{(a_1+b_1)^3}(v-b)^2, \end{aligned} \tag{5}$$

where (a_1, b_1) is a point between (a, b) and (u, v).

Now, by using the consistency property of the MLE for (μ_X, μ_Y), the variance of the estimator of the limiting availability can be approximated by

$$Var(\widehat{A}_{mle}) \approx \frac{\mu_Y^2}{(\mu_X+\mu_Y)^4} Var(\widehat{\mu}_X) + \frac{\mu_X^2}{(\mu_X+\mu_Y)^4} Var(\widehat{\mu_Y}), \tag{6}$$

then

$$\widehat{Var}(\widehat{A}_{mle}) \approx \frac{\widehat{\mu_Y}^2}{(\widehat{\mu_X}+\widehat{\mu_Y})^4} \widehat{Var}(\widehat{\mu}_X) + \frac{\widehat{\mu_X}^2}{(\widehat{\mu_X}+\widehat{\mu_Y})^4} \widehat{Var}(\widehat{\mu}_Y). \tag{7}$$

Approximate confidence intervals can be obtained using the estimator and approximation given in Equation 7.

2.1. Simulation Study

In the simulation study, we consider two repairable systems. In the first system, failure and repair times follow exponential distributions. In the second, failure time follows a gamma distribution and repair time an exponential law. In this process, we use R software, version 3.4.1 (R Development Core Team 2007).

2.1.1. Exponential-Exponential System

Consider $\{X_n, Y_n\}$ a repairable system, where X has a probability density function (pdf) given by $f(x|\mu) = \frac{1}{\mu}e^{-x/\mu}, \quad x > 0$, and Y has a pdf $f(y|\lambda) = \frac{1}{\lambda}e^{-y/\lambda}, \quad y > 0$. In this case $\mu_X = \mu$, $\mu_Y = \lambda$ and $A = \frac{\mu}{\mu+\lambda}$. The simulation study considers 1000 repetitions for each simulation. Now, we present some notations and summary statistics used in the simulations.

- μ and λ are the parameters of the failure time and repair time, respectively.
- n, m are the number of observations from X and Y.
- $\overline{\widehat{A}}$ corresponds to the mean of the limiting availability estimates, i.e., $\sum_{i=1}^{1000} \widehat{A}_i/1000$.
- $\overline{\hat{\sigma}^2}$ is the mean of the estimated variances defined in (7).
- $S^2_{\widehat{A}}$ is the empirical quadratic error of the mean limiting availability, which is defined by, $S^2_{\widehat{A}} = \sum_{i=1}^{1000} \left(\widehat{A}_i - \overline{\widehat{A}}\right)^2/1000$.
- CPT is the probability of containing the true simulate limiting availability based on 95% confidence intervals.
- MA is the mean of amplitudes of the intervals in the 1000 repetitions for the limiting availability estimate.
- $95\%-Interval$ corresponds to the mean of the $95\%-$confidence interval over the 1000 repetitions for the limiting availability.

Table 1: Simulation for a repairable system with $\mu = 1/1.5$, $\lambda = 1/1.5$ and limiting availability $A = 0.5$.

n	m	$\overline{\hat{A}}$	$S^2_{\hat{A}}$	$\overline{\hat{\sigma}^2}$	MA	CPT	$95\% - Interval$
25	25	0.4963	0.0051	0.0057	0.2948	0.9480	(0.3489; 0.6437)
50	40	0.5017	0.0028	0.0022	0.1827	0.9000	(0.4103; 0.5930)
65	70	0.4971	0.0018	0.0022	0.1830	0.9670	(0.4056; 0.5886)
100	100	0.5005	0.0012	0.0013	0.1407	0.9490	(0.4301; 0.5709)

Table 2: Simulation for a repairable system with $\mu = 1/1.2$, $\lambda = 1/5.5$ and limiting availability $A = 0.8209$.

n	m	$\overline{\hat{A}}$	$S^2_{\hat{A}}$	$\overline{\hat{\sigma}^2}$	MA	CPT	$95\% - Interval$
25	25	0.8178	0.0018	0.0021	0.1772	0.9460	(0.7292; 0.9064)
50	40	0.8215	0.0010	0.0008	0.1077	0.9070	(0.7677; 0.8753)
65	70	0.8193	0.0006	0.0008	0.1087	0.9690	(0.7649; 0.8736)
100	100	0.8207	0.0005	0.0005	0.0830	0.9360	(0.7792; 0.8622)

The results of the simulation are presented in Tables 1 and 2.

In Table 1, the quadratic error of the estimator and the mean of variabilities present a similar behavior, depending on sample sizes. Probabilities of containing the true limiting availability tend to 0.95 when the sample sizes are similar. In Table 2 the quadratic error of the estimator and the variance converge to the same value faster than in the first case.

2.1.2. Gamma-Exponential System

Consider a repairable system, where X has a pdf $f(x \mid \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$, $x > 0$ and Y has a pdf $f(y \mid \lambda) = \frac{1}{\lambda} e^{-y/\lambda}, \quad y > 0$. In this case, $\mu_X = \alpha\beta$, $\mu_Y = \lambda$ and the limiting availability is $A = \alpha\beta/(\alpha\beta + \lambda)$. The simulation study considers 1000 repetitions for each simulation. We use the same notations and summary statistics used in the previous simulation. Table 3 and Table 4 present the results of the simulation.

Table 3: Simulation for a repairable system with $\alpha = 2.2$, $\beta = 2.1$, $\lambda = \alpha\beta$ and limiting availability $A = 0.5$.

n	m	$\overline{\hat{A}}$	$S^2_{\hat{A}}$	$\overline{\hat{\sigma}^2}$	MA	CPT	$95\% - Interval$
25	25	0.5035	0.0036	0.0033	0.2225	0.9300	(0.3922; 0.6148)
50	40	0.4996	0.0021	0.0020	0.1738	0.9300	(0.4127; 0.5865)
65	70	0.5015	0.0013	0.0013	0.1396	0.9460	(0.4317; 0.5713)
100	100	0.5001	0.0009	0.0009	0.1162	0.9480	(0.4420; 0.5582)

From the results of the simulation (Tables 3 and 4), approximate variances via delta method and quadratic error of estimators are similar as the sample size increases. In the same way, probabilities of containing the true limiting availability tend to 0.95.

TABLE 4: Simulation for a repairable system with $\alpha = 5$, $\beta = 3$, $\lambda = 2$ and limiting availability $A = 0.8824$.

n	m	$\overline{\hat{A}}$	$S^2_{\hat{A}}$	$\overline{\hat{\sigma}^2}$	MA	CPT	$95\% - Interval$
25	25	0.8826	0.0005	0.0005	0.0850	0.9230	(0.8401; 0.9251)
50	40	0.8825	0.0003	0.0003	0.0674	0.9200	(0.8487; 0.9162)
65	70	0.8821	0.0002	0.0002	0.0527	0.9340	(0.8558; 0.9085)
100	100	0.8823	0.0001	0.0001	0.0440	0.9410	(0.8602; 0.9043)

3. Bayesian Estimation

In this section, we apply the Bayesian approach to the problem of estimation of the limiting availability. Specifically, we analyze the two repairable systems, which were described in the previous section. In this context, we must elicit a prior distribution for the limiting availability. This can be made by defining in a first stage prior distributions for the parameters of the observational model. The inference of the limiting availability will be based on its posterior distribution. We are considering two cases: informative and less informative prior distributions.

3.1. Elicitation of the Prior Distribution

Consider X the failure time with pdf $f(x \mid \mu)$ and Y the repairable time with pdf $f(y \mid \lambda)$, such that $\mu_X = \mu$, $\mu_Y = \lambda$, so $A = \mu/(\mu + \lambda)$. Let $\mu \sim \pi(\mu)$ and $\lambda \sim \pi(\lambda)$ be the prior densities for μ and λ, respectively. Assuming independence, the joint prior for (μ, λ) is given by $\pi(\mu, \lambda) = \pi(\mu) \times \pi(\lambda)$.

We are interested in determining the prior density of the limiting availability A. So, consider the following change of variables $\theta = A = \frac{\mu}{\mu+\lambda}$, $\phi = \mu + \lambda$. Then the joint distribution of the vector (θ, ϕ) is given by

$$\pi(\theta, \phi) = \begin{cases} \pi_\mu(\mu(\theta,\phi))\pi_\lambda(\lambda(\theta,\phi)) \left| J\big((\mu,\lambda),(\theta,\phi)\big)\right|, & \theta \in \Theta,\ \phi \in \Phi, \\ 0, & \theta \notin \Theta,\ \phi \notin \Phi. \end{cases}$$

where $\mu(\theta, \phi) = \theta\phi$ and $\lambda(\theta, \phi) = \phi(1 - \theta)$. Also, Θ is the parameter space of θ and Φ is the parameter space of ϕ.

The Jacobian of the transformation is given by

$$J\big((\mu,\lambda),(\theta,\phi)\big) = \begin{vmatrix} \dfrac{\partial\mu}{\partial\theta} & \dfrac{\partial\mu}{\partial\phi} \\ \dfrac{\partial\lambda}{\partial\theta} & \dfrac{\partial\lambda}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \phi & \theta \\ -\phi & 1-\theta \end{vmatrix} = \phi.$$

Marginalizing, the prior density of $\theta = A$ is

$$\pi(\theta) = \int_\Phi \pi_\mu(\theta\phi)\pi_\lambda(\phi(1-\theta))\phi d\phi. \tag{8}$$

The following result is a direct extension of the characterization of the Beta distribution, through two independent Gamma random variables, with the same scale parameter. Although the proposition is a direct consequence of the work of Libby & Novick (1982), we prefer to include the demonstration in order to have a greater understanding in the reading of the article.

Proposition 1. *If* $\mu \sim Gamma(a_1, b_1)$ *and* $\lambda \sim Gamma(a_2, b_2)$ *then* $\theta = A$ *is Generalized Beta distributed with parameters* $(a_1, a_2, b_1/b_2)$*, i.e.,* $\theta = A \sim \mathcal{GB}3(a_1, a_2, b_1/b_2)$.

Proof . We have

$$\pi(\theta) = \int_0^\infty \pi_\mu(\theta\phi)\pi_\lambda(\phi(1-\theta))\phi d\phi.$$

Then

$$\begin{aligned}
\pi(\theta) &= \int_0^\infty \frac{b_1^{a_1}}{\Gamma(a_1)}(\theta\phi)^{a_1-1}e^{-b_1\theta\phi}\frac{b_2^{(a_2)}}{\Gamma a_2}(\phi(1-\theta))^{a_2-1}e^{-b_2\phi(1-\theta)}\phi d\phi \\
&= \frac{b_1^{a_1}b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\theta^{a_1-1}(1-\theta)^{a_2-1}\int_0^\infty \phi^{a_1+a_2-1}e^{-\phi(b_1\theta+b_2(1-\theta))}d\phi \\
&= \frac{b_1^{a_1}b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\theta^{a_1-1}(1-\theta)^{a_2-1}\frac{\Gamma(a_1+a_2)}{(b_1\theta+b_2(1-\theta))^{a_1+a_2}}
\end{aligned}$$

So,

$$\pi(\theta) \propto \frac{\theta^{a_1-1}(1-\theta)^{a_2-1}}{\left(1-(1-\frac{b_1}{b_2})\theta\right)^{a_1+a_2}}, \; 0<\theta<1, \tag{9}$$

this last expression corresponds to the kernel of a Generalized Beta density with parameters $(a_1, a_2, b_1/b_2)$, see Chen & Novick (1996). □

Note that, if $b_1 = b_2$ we obtain the usual beta distribution of parameters (a_1, a_2).

The calculus of the noninformative prior for the limiting availability is analogous. In this part, we calculate the Jeffreys (1996) prior distribution for the limiting availability A. Considering the same scheme, the likelihood function of (μ, λ) is $L(\mu, \lambda; \boldsymbol{x}, \boldsymbol{y}) = L(\mu; \boldsymbol{x})L(\lambda; \boldsymbol{y})$.

Then the Fisher information matrix of (μ, λ) is given by

$$I(\mu,\lambda) = \begin{pmatrix} E\left(-\frac{\partial^2 \log L(\mu,\lambda;\boldsymbol{x},\boldsymbol{y})}{\partial\mu^2}\right) & E\left(-\frac{\partial^2 \log L(\mu,\lambda;\boldsymbol{x},\boldsymbol{y})}{\partial\mu\partial\lambda}\right) \\ E\left(-\frac{\partial^2 \log L(\mu,\lambda;\boldsymbol{x},\boldsymbol{y})}{\partial\mu\partial\lambda}\right) & E\left(-\frac{\partial^2 \log L(\mu,\lambda;\boldsymbol{x},\boldsymbol{y})}{\partial\lambda^2}\right) \end{pmatrix},$$

so, the Jeffreys prior for (μ, λ) is $\pi(\mu, \lambda) \propto |I(\mu, \lambda)|^{1/2}$.

Analogously,

$$\pi(\theta,\phi) \quad \propto \quad |I(\mu(\theta,\phi),\lambda(\theta,\phi))|^{1/2}\left|J\big((\mu,\lambda),(\theta,\phi)\big)\right|,$$

where $\mu=\mu(\theta,\phi)=\theta\phi$ and $\lambda=\lambda(\theta,\phi)=\phi(1-\theta)$. When it is possible, one must marginalize $\pi(\theta,\phi)$ to obtain the noninformative prior for the limiting availability $\theta=A$.

3.2. Calculus of the Posterior Distribution

In this section, we calculate the posterior density function of the limiting availability A. Recall, the failure time X has a pdf $f(x\mid\mu)$, $\mu_X=\mu$ and the repair time Y has a pdf $f(x\mid\lambda)$, $\mu_Y=\lambda$. The limiting availability is $A=\mu/(\mu+\lambda)$. First, we reparametrize the likelihood function, considering $\theta=\mu/(\mu+\lambda)$ and $\phi=\mu+\lambda$, which was used in the determination of the prior distribution in the previous section.

The likelihood function is $L(\mu,\lambda;\boldsymbol{x},\boldsymbol{y})=L(\mu,\boldsymbol{x})\times L(\lambda,\boldsymbol{y})$. Letting $\mu=\theta\phi$ and $\lambda=\phi(1-\theta)$, the likelihood function becomes

$$L(\theta,\phi;\boldsymbol{x},\boldsymbol{y})=L(\mu(\theta,\phi),\boldsymbol{x})\times L(\lambda(\theta,\phi),\boldsymbol{y}),\ \text{where } \theta\in\Theta,\ \phi\in\Phi.$$

By the Bayes rule, the joint posterior density is

$$\pi(\theta,\phi\mid\boldsymbol{x},\boldsymbol{y})=\frac{L(\theta,\phi;\boldsymbol{x},\boldsymbol{y})\pi(\theta,\phi)}{\int_\Phi\int_\Theta L(\theta,\phi;\boldsymbol{x},\boldsymbol{y})\pi(\theta,\phi)d\theta d\phi},$$

i.e.,

$$\pi(\theta,\phi\mid\boldsymbol{x},\boldsymbol{y})\propto L(\theta,\phi;\boldsymbol{x},\boldsymbol{y})\times\pi(\theta,\phi). \tag{10}$$

Marginalizing this last expression, we can obtain the posterior density of $\theta=A$,

$$\pi(\theta\mid\boldsymbol{x},\boldsymbol{y})=\int_\Phi\pi(\theta,\phi\mid\boldsymbol{x},\boldsymbol{y})d\phi. \tag{11}$$

3.3. Bayesian estimation in an Exponential-Exponential System

Consider a repairable system, where $\boldsymbol{X}=(X_1,\dots,X_n)$ is a random sample from the failure time with pdf $f(x\mid\mu)=\mu e^{-\mu x}, x>0$, and $\boldsymbol{Y}=(Y_1,\dots,Y_m)$ is a random sample from the repair time with pdf $f(y\mid\lambda)=\lambda e^{-\lambda y},\ y>0,\ m\leq n$.

In this case the limiting availability is given by $A=\frac{1/\mu}{(1/\mu)+(1/\lambda)}=\frac{\lambda}{\mu+\lambda}$.

The likelihood function is

$$L(\mu,\lambda;\boldsymbol{x},\boldsymbol{y})=\mu^n\lambda^m\exp\left\{-\mu\sum_{i=1}^n x_i-\lambda\sum_{j=1}^m y_j\right\}. \tag{12}$$

In the following proposition, we show that if one considers the usual prior Gamma for the parameters μ and λ, the Generalized Beta distribution is conjugated. Furthermore, Bayes estimator and credible interval for the limiting availability are computed.

Proposition 2. *Suppose* $\mu \sim Gamma(a_1, b_1)$, $\lambda \sim Gamma(a_2, b_2)$, *i.e.,* $\pi(\mu) \propto \mu^{a_1-1}e^{-b_1\mu}$, $\mu > 0$, $\pi(\lambda) \propto \lambda^{a_2-1}e^{-b_2\lambda}$, $\lambda > 0$ *and,* μ *and* λ *are independent. Then,*

(i) The prior distribution of A *is a Beta Generalized distribution of parameters* $(a_2, a_1, b_2/b_1)$, *i.e.,* $A \sim \mathcal{GB}3(a_2, a_1, b_2/b_1)$.

(ii) $A \mid \boldsymbol{x}, \boldsymbol{y} \sim \mathcal{GB}3(a_2', a_1', b_2'/b_1')$, *where* $a_1' = a_1 + n$, $a_2' = a_2 + m$, $b_1' = b_1 + \sum x_i$ *and* $b_2' = b_2 + \sum y_j$.

(iii) The Bayes estimator of A *and its risk are given by*

$$\widehat{\theta}_B = \frac{b_1'}{b_2'} \sum_{j=0}^{\infty} \left(1 - \frac{b_1'}{b_2'}\right)^j \prod_{r=0}^{j} \frac{a_2' + r}{a_1' + a_2' + r},$$

$$\begin{aligned} \widehat{\sigma}_B^2 \quad = \quad & \left(\frac{b_1'}{b_2'}\right)^2 \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \left(1 - \frac{b_1'}{b_2'}\right)^{i+j} \prod_{k=0}^{i+j+1} \frac{a_2' + k}{a_1' + a_2' + k} \right. \\ & \left. - \left(\sum_{n=0}^{\infty} \left(1 - \frac{b_1'}{b_2'}\right)^n \prod_{r=0}^{n} \frac{a_2' + r}{a_1' + a_2' + r}\right)^2\right] \end{aligned}$$

(iv) A credible interval of level $1 - \alpha$ *for* A *is given by*

$$\left(\frac{a_2' b_1'}{a_1' b_2' F_{1-\alpha/2}(2a_1', 2a_2') + a_2' b_1'}; \frac{a_2' b_1'}{a_1' b_2' F_{\alpha/2}(2a_1', 2a_2') + a_2' b_1'}\right),$$

where $F_\alpha(\nu, \mu)$ *is the* α*-percentile of the* F *distribution with* (ν, μ) *degrees of freedom.*

***Proof*.**
(i) It is a direct consequence of Proposition 1.
(ii) Consider the variable changes of $\theta = A = \lambda/(\lambda + \mu)$, $\phi = \lambda + \mu$ for this case. Then marginalizing the joint posterior density $\pi(\theta, \phi \mid \boldsymbol{x}, \boldsymbol{y})$, we obtain

$$\pi(\theta \mid \boldsymbol{x}, \boldsymbol{y}) \propto \frac{\theta^{a_2'-1}(1-\theta)^{a_1'-1}}{(1 - (1 - \frac{b_2'}{b_1'})\theta)^{a_1'+a_2'}}, \qquad 0 < \theta < 1,$$

which corresponds to the kernel of a $\mathcal{GB}3(a_2', a_1', b_2'/b_1')$ distribution.
(iii) They are consequences of properties of Generalized Beta distribution. See Chen & Novick (1996).
(iv) Consider $\Pr(\theta \leq a \mid \boldsymbol{x}, \boldsymbol{y}) = \Pr(\theta \geq b \mid \boldsymbol{x}, \boldsymbol{y}) = \frac{\alpha}{2}$, and use the fact that $\frac{a_2' b_2'}{a_1' b_1'}(\frac{A}{1-A})$ follows a F distribution with $2a_1', 2a_2'$ degrees of freedom. □

Remark. The non-informative case can be obtained by considering $a_1 \to 0$, $b_1 \to 0, a_2 \to 0$, and $b_2 \to 0$, in the prior densities.

3.4. Bayesian estimation in a Gamma-Exponential System

Consider a repairable system, where $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a random sample from the failure time with pdf $f(x \mid \alpha, \tau) = \frac{1}{\Gamma(\alpha)\tau^{\alpha}} x^{\alpha-1} e^{-x/\tau}, \quad x > 0$, and $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ is a random sample from the repair time with pdf $f(y \mid \lambda) = \frac{1}{\lambda} e^{-y/\lambda}, \quad y > 0,\ m \leq n$, where we suppose that α is known. In this case the limiting availability is given by $A = \frac{\alpha\tau}{\alpha\tau + \lambda}$.

The likelihood function is

$$
\begin{aligned}
&L(\tau, \lambda; \alpha, \boldsymbol{x}, \boldsymbol{y}) \\
&\qquad = \frac{\tau^{-n\alpha}}{\lambda^{m}(\Gamma(\alpha))^{n}} \exp\left\{(\alpha - 1)\sum_{i=1}^{n} \log(x_i) - \frac{1}{\tau}\sum_{i=1}^{n} x_i - \frac{1}{\lambda}\sum_{j=1}^{m} y_j\right\}. \qquad (13)
\end{aligned}
$$

In the following proposition, we show that if one considers the usual inverse-Gamma prior for the parameters τ and λ, the Generalized Beta distribution is conjugated. Furthermore, Bayes estimator and credible interval for the limiting availability are computed.

Proposition 3. *Suppose $\tau \sim inverse - Gamma(c_1, d_1)$, $\lambda \sim inverse - Gamma(c_2, d_2)$, i.e., $\pi(\tau) \propto \mu^{c_1 - 1} e^{-d_1 \mu}$, $\mu > 0$, $\pi(\lambda) \propto \lambda^{c_2 - 1} e^{-d_2 \lambda}$, $\lambda > 0$ and, μ and λ are independents. Then,*

(i) The prior distribution of A is a Generalized Beta distribution of parameters $(c_2, c_1, d_2/d_1)$, i.e., $A \sim \mathcal{GB}3(c_2, c_1, d_2/d_1)$.

(ii) $A \mid \boldsymbol{x}, \boldsymbol{y} \sim \mathcal{GB}3(c_2', c_1', d_2'/d_1')$, where $c_1' = c_1 + n\alpha$, $c_2' = c_2 + m$, $d_1' = \alpha(d_1 + \sum x_i)$ and $d_2' = d_2 + \sum y_j$.

(iii) The Bayes estimator of A and its risk are given by

$$
\widehat{\theta}_B = \frac{d_1'}{d_2'} \sum_{j=0}^{\infty} \left(1 - \frac{d_1'}{d_2'}\right)^{j} \prod_{r=0}^{j} \frac{c_2' + r}{c_1' + c_2' + r},
$$

$$
\begin{aligned}
\widehat{\sigma}_B^2 \quad = \quad & \left(\frac{d_1'}{d_2'}\right)^2 \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(1 - \frac{d_1'}{d_2'}\right)^{i+j} \prod_{k=0}^{i+j+1} \frac{c_2' + k}{c_1' + c_2' + k}\right. \\
& \left. - \left(\sum_{n=0}^{\infty}\left(1 - \frac{d_1'}{d_2'}\right)^{n} \prod_{r=0}^{n} \frac{c_2' + r}{c_1' + c_2' + r}\right)^2\right].
\end{aligned}
$$

(iv) A credible interval of level $1-\alpha$ for A is given by

$$\left(\frac{c_2' d_1'}{c_1' d_2' F_{1-\alpha/2}(2c_1', 2c_2') + c_2' d_1'}; \frac{c_2' d_1'}{c_1' b_2' F_{\alpha/2}(2c_1', 2c_2') + c_2' d_1'}\right),$$

where $F_\alpha(\nu, \mu)$ is the α-percentile of F distribution with (ν, μ) degrees of freedom.

Proof. It is similar to the proof of Proposition 2. □

3.5. Application and Simulation Study

Analogously to the classical case, we consider two systems with the same characteristics, where the times of the first system follow exponential distributions. In the other system, the failure time follows a Gamma distribution and the repair time, an exponential law. The computational implementation considers the following cases: conjugated informative priors, semi-informative priors, flat conjugated priors and non-informative priors.

3.5.1. Exponential-Exponential System

The parameters chosen for the simulations are presented in Table 5:

Table 5: Parameters for Simulations (Exp-Exp).

True parameters				Prior parameters			
Case	μ	λ	A	a_1	b_1	a_2	b_2
1.1	1.5	1.5	0.5000	0.5	1.0	0.5	1.0
2.1	1.2	5.5	0.8209	1.048	0.04	1.22	0.04
1.2	1.5	1.5	0.5000	1.0	1.5	1.0	1.5
2.2	1.2	5.5	0.8209	7.24	5.2	29.6	5.2
1.3	1.5	1.5	0.5000	61	30	61	30
2.3	1.2	5.5	0.8209	25.24	20.2	112.1	20.2

In cases 1.1 and 1.2, the parameters of the prior distributions are interesting to analyze. Cases 1.2 and 2.1 correspond to flat priors. Case 2.2 is a semi-informative prior. Cases 1.3 and 2.3 are informative priors, where the distribution is centred on the true parameters.

The distributions were simulated using R software version 3.4.1 (R Development Core Team 2007), each Bayesian estimator was obtained over 1000 realizations and for different values of n and m. The estimators shown in Table 6 correspond to the average of the Bayesian estimators obtained on the 1000 repetitions. Also, n corresponds to the sampled units from the failure times and m from the repair times.

TABLE 6: Bayesian Estimation of the Limiting Availability (Exp-Exp).

Case	n	m	μ	λ	A	$\overline{\widehat{A}}_{.1}$	$\overline{\widehat{A}}_{.2}$	$\overline{\widehat{A}}_{.3}$	$\overline{\widehat{A}}_{non-inf}$
1	25	25	1.5	1.5	0.5000	0.4973	0.5008	0.4998	0.4975
2	25	25	1.2	5.5	0.8209	0.8154	0.8157	0.8172	0.8153
3	50	40	1.5	1.5	0.5000	0.4988	0.4976	0.5039	0.5038
4	50	40	1.2	5.5	0.8209	0.8179	0.8187	0.8182	0.8164
5	65	70	1.5	1.5	0.5000	0.5000	0.5005	0.4987	0.5010
6	65	70	1.2	5.5	0.8209	0.8191	0.8168	0.8187	0.8194
7	100	100	1.5	1.5	0.5000	0.5007	0.4998	0.4995	0.4979
8	100	100	1.2	5.5	0.8209	0.8197	0.8192	0.8201	0.8188

In Table 6, the subscript of the Bayesian estimator are in relation to the prior distribution used, for example if $\mu = 1.5$ and $\lambda = 1.5$, and the estimate has subscript 2, then the parameters of the prior correspond to case 1.2 of Table 5.

We note that for the limiting availability $A = 0.5$, Bayesian estimators behave quite well. However, when limit availability increases, a fairly light underestimation is observed. The behavior of the estimate improves as the sample size increases.

TABLE 7: Mean Standard Deviation of the Bayesian Estimates (Exp-Exp) of the Limiting Availability.

Case	n	m	μ	λ	A	$\overline{sd}(\widehat{A}_{.1})$	$\overline{sd}(\widehat{A}_{.2})$	$\overline{sd}(\widehat{A}_{.3})$	$\overline{sd}(\widehat{A}_{non-inf})$
1	25	25	1.5	1.5	0.5000	0.0682	0.0675	0.0379	0.0686
2	25	25	1.2	5.5	0.8209	0.0414	0.0332	0.0246	0.0423
3	50	40	1.5	1.5	0.5000	0.0519	0.0516	0.0342	0.0522
4	50	40	1.2	5.5	0.8209	0.0311	0.0264	0.0209	0.0317
5	65	70	1.5	1.5	0.5000	0.0425	0.0423	0.0311	0.0426
6	65	70	1.2	5.5	0.8209	0.0252	0.0231	0.0191	0.0254
7	100	100	1.5	1.5	0.5000	0.0350	0.0349	0.0278	0.0351
8	100	100	1.2	5.5	0.8209	0.0208	0.0193	0.0166	0.0210

Table 7 describes the mean of the standard deviations (sd) of the Bayesian estimates of the limiting availability. We observe that standard deviations decrease as prior distributions are more informative which is an expected behavior. Also, the sample sizes have some influence on the value of the standard deviations, but this influence is slight compared to the informative priors.

The 95% credibility regions are reported in Table 8, using the same notation as in Table 6. The length of the intervals tends to decrease as the sample size increases or when the priors distributions are more informative. When the limit availability is greater ($A = 0.8209$), the interval is more shifted to the left, this is due to the asymmetry of the posterior distribution of the limiting availability.

3.5.2. Gamma-Exponential System

The parameters chosen for the simulations are presented in Table 9.

Table 8: Mean Credibility Regions (CR) for Limiting Availability (Exp-Exp).

Case	A	$\overline{CR}_{.1}$	$\overline{CR}_{.2}$	$\overline{CR}_{.3}$	$\overline{CR}_{non-inf}$
1	0.5000	(0.3643; 0.6307)	(0.3689; 0.6325)	(0.4256; 0.5740)	(0.3637; 0.6316)
2	0.8209	(0.7247; 0.8861)	(0.7452; 0.8749)	(0.7663; 0.8625)	(0.7223; 0.8874)
3	0.5000	(0.3968; 0.5998)	(0.3961; 0.5981)	(0.4368; 0.5707)	(0.4011; 0.6052)
4	0.8209	(0.7509; 0.8726)	(0.7630; 0.8663)	(0.7751; 0.8570)	(0.7481; 0.8721)
5	0.5000	(0.4170; 0.5832)	(0.4178; 0.5834)	(0.4379; 0.5596)	(0.4177; 0.5844)
6	0.8209	(0.7659; 0.8647)	(0.7688; 0.8591)	(0.7796; 0.8543)	(0.7658; 0.8653)
7	0.5000	(0.4322; 0.5693)	(0.4314; 0.5682)	(0.4452; 0.5539)	(0.4292; 0.5666)
8	0.8209	(0.7764; 0.8577)	(0.7792; 0.8548)	(0.7861; 0.8511)	(0.7752; 0.8572)

Table 9: Parameters for Simulations (Gamma-Exp).

True Parameters					Prior Parameters			
Case	α	τ	λ	A	c_1	d_1	c_2	d_2
1.1	2.2	2.1	4.62	0.5000	1.095	4.4	1.095	9.68
2.1	5	3	2	0.8824	3.333	7	21	40
1.2	2.2	2.1	4.62	0.5000	4.810	12.2	4.810	26.84
2.2	5	3	2	0.8824	2	3	8.5	15
1.3	2.2	2.1	4.62	0.5000	18.191	40.3	18.191	88.66
2.3	5	3	2	0.8824	16.667	50	126	250

The parameters in Table 9 were chosen to study the behavior of the Bayesian estimation of limiting availability. The cases 1.1 and 2.1 correspond to a less informative prior, the cases 1.2 and 2.2 correspond to semi-informative priors and the rest to very informative priors. As in the previous case, the simulations are performed over 1000 repetitions for different values of m and n, where the resulting estimator is the mean value of the 1000 Bayes estimators obtained. Table 10 summarizes the results of the estimates obtained.

Table 10: Bayesian Estimation of the Limiting Availability (Gamma-Exp).

Case	n	m	α	τ	λ	A	$\overline{\widehat{A}}_{.1}$	$\overline{\widehat{A}}_{.2}$	$\overline{\widehat{A}}_{.3}$	$\overline{\widehat{A}}_{non-inf}$
1	25	25	2.2	2.1	4.62	0.5000	0.4970	0.4956	0.4981	0.5000
2	25	25	5	3	2	0.8824	0.8813	0.8815	0.8818	0.8786
3	50	40	2.2	2.1	4.62	0.5000	0.4976	0.4981	0.4987	0.4996
4	50	40	5	3	2	0.8824	0.8826	0.8826	0.8818	0.8807
5	65	70	2.2	2.1	4.62	0.5000	0.5000	0.4995	0.5011	0.5000
6	65	70	5	3	2	0.8824	0.8821	0.8817	0.8820	0.8817
7	100	100	2.2	2.1	4.62	0.5000	0.5009	0.5000	0.4997	0.5001
8	100	100	5	3	2	0.8824	0.8825	0.8824	0.8822	0.8817

We note that the estimates closely resemble the true value of the limiting availability. These estimates are more accurate as the sample size increases. The standard deviations are presented in Table 11.

The standard deviations are smaller when considered more informative priors, in the non-informative case the dispersion is greater. In addition, if the sample size increases the estimates are more accurate.

TABLE 11: Standard Deviation (sd) of Bayesian Estimates of Limiting Availability (Gamma-Exp).

Case	n	m	α	τ	λ	A	$\overline{sd}_{.1}$	$\overline{sd}_{.2}$	$\overline{sd}_{.3}$	$\overline{sd}_{non-inf}$
1	25	25	2.2	2.1	4.62	0.5000	0.0582	0.0552	0.0476	0.0592
2	25	25	5	3	2	0.8824	0.0181	0.0206	0.0122	0.0238
3	50	40	2.2	2.1	4.62	0.5000	0.0451	0.0436	0.0392	0.0456
4	50	40	5	3	2	0.8824	0.0149	0.0164	0.0103	0.0181
5	65	70	2.2	2.1	4.62	0.5000	0.0360	0.0352	0.0329	0.0362
6	65	70	5	3	2	0.8824	0.0124	0.0132	0.0093	0.0138
7	100	100	2.2	2.1	4.62	0.5000	0.0299	0.0294	0.0280	0.0300
8	100	100	5	3	2	0.8824	0.0105	0.0110	0.0083	0.0115

Table 12 reports the 95% credibility regions, using the same notation as in Table 11. The length of the intervals tends to decrease as the sample sizes increase or when the priors are more informative.

TABLE 12: Credibility Region (CR) of Limiting Availability (Gamma-Exp).

Case	A	$\overline{CR}_{.1}$	$\overline{CR}_{.2}$	$\overline{CR}_{.3}$	$\overline{CR}_{non-inf}$
1	0.5000	(0.3812; 0.6084)	(0.3859; 0.6017)	(0.4041; 0.5901)	(0.3819; 0.6130)
2	0.8824	(0.8419; 0.9127)	(0.8360; 0.9163)	(0.8564; 0.9041)	(0.8251; 0.9179)
3	0.5000	(0.4076; 0.5841)	(0.4112; 0.5818)	(0.4208; 0.5745)	(0.4085; 0.5870)
4	0.8824	(0.8505; 0.9088)	(0.8469; 0.9111)	(0.8604; 0.9008)	(0.8410; 0.9118)
5	0.5000	(0.4286; 0.5695)	(0.4297; 0.5677)	(0.4360; 0.5650)	(0.4283; 0.5699)
6	0.8824	(0.8559; 0.9043)	(0.8536; 0.9052)	(0.8627; 0.8993)	(0.8521; 0.9062)
7	0.5000	(0.4418; 0.5587)	(0.4417; 0.5569)	(0.4443; 0.5541)	(0.4407; 0.5583)
8	0.8824	(0.8605; 0.9017)	(0.8592; 0.9024)	(0.8651; 0.8976)	(0.8575; 0.9024)

4. Bayes estimation in a Case Study

In the packing process in a certain glass bottles Chilean factory, a machine (handled by a trained worker) called palletizer is used.

The failure times and descriptive statistics are presented in Table 13 and Table 14, respectively. In the same way, the repair times are presented in Table 15 and Table 16. Figure 1 present the histograms for observed failure and repair times, respectively.

Table 13: Failure times (in hours).

i	x_i	i	x_i	i	x_i
1	655.700	8	670.250	15	2274.500
2	660.883	9	922.333	16	366.583
3	62.133	10	818.283	17	980.833
4	306.200	11	1022.250	18	0.300
5	124.417	12	48.950	19	27.000
6	2141.800	13	292.583	20	2943.033
7	1402.083	14	715.550	21	176.250

Table 14: Descriptive statistics for failure times.

Statistic	*Value*
Sample size	21
Minimum	0.3000
Maximum	2943.0333
Median	660.7000
Mean	791.0437
Variance	648486.8
Standard deviation	805.2868

Table 15: Repair times (in hours).

j	y_j	j	y_j	j	y_j
1	0.4167	8	0.0833	15	0.5000
2	0.2000	9	0.3000	16	0.5000
3	0.0500	10	1.3333	17	0.0333
4	0.1667	11	0.6333	18	0.0833
5	0.0333	12	0.5833	19	0.1333
6	1.4167	13	0.1167	20	1.5000
7	0.2500	14	0.1667	21	0.1667

Table 16: Descriptive statistics for repair times.

Statistic	*Value*
Sample size	21
Minimum	0.0333
Maximum	1.5000
Median	0.2000
Mean	0.4127
Variance	0.2099
Standard deviation	0.4582

The Kolmogorov-Smirnov test is applied both for the failure times and for the repair times considering the exponential distribution as the null hypothesis. This was done using the SAS software Version[9.4] (SAS 2017), obtaining a p-value greater than 0.5. So, the density functions of the failure time and the repair time are considered as $f(x \mid \lambda) = \lambda e^{-\lambda x}, x > 0$ and $f(x \mid \mu) = \mu e^{-\mu y}, y > 0$. Since we do not have information about the parameters, it is possible to apply the results of subsection 3.5.1, with non-informative priors. The posterior density of the limiting availability is given by

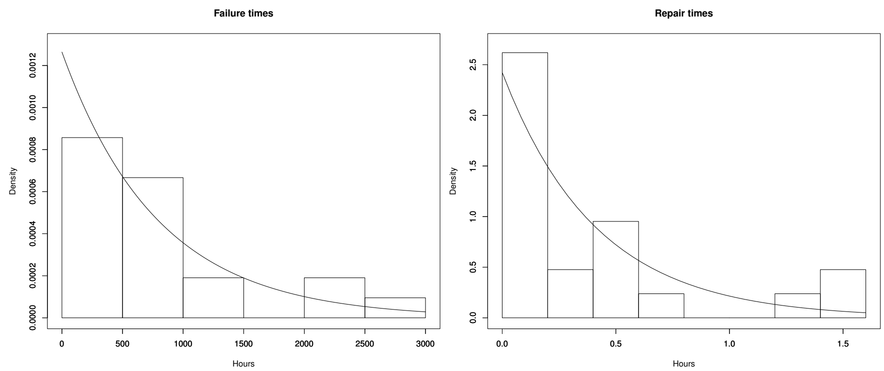

FIGURE 1: Histogram for failures (left) and repair times (right) with exponential densities for the palletizer data.

$$\pi(\theta \mid \boldsymbol{x}, \boldsymbol{y}) \propto \frac{\theta^{21-1}(1-\theta)^{21-1}}{(16611.9 + \theta(8.667 - 16611.9))^{42}}, \; 0 < \theta < 1. \tag{14}$$

Note that, $n = m = 21$, $\sum x_i = 16611.9$ and $\sum y_j = 8.667$. A graph of this density is presented in Figure 2, which accounts for its asymmetry, with high probabilities at the upper end. Consequently, this is reflected in the estimates that are described in Table 17 and in the 95% credibility region which is given by $\mathcal{CR} = (0.99904, 099972)$. The results obtained show that the palletizer is working in optimal conditions. However, periodically its use must be monitored.

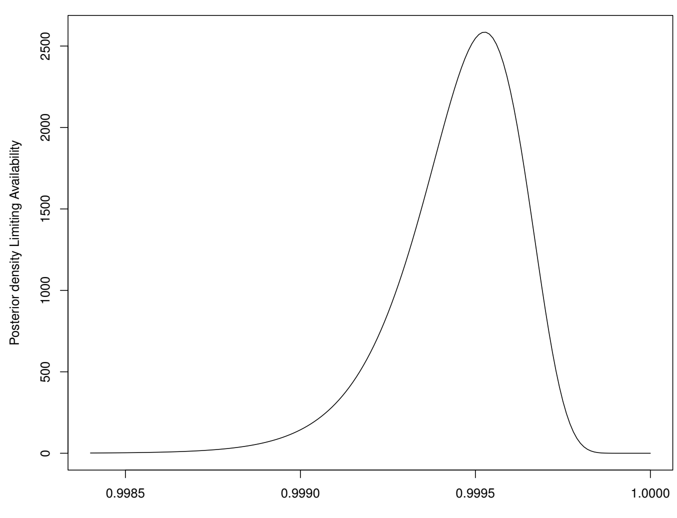

FIGURE 2: Posterior density limiting availability, palletizer data.

Table 17: Bayes estimation of the limiting availability for the palletizer data.

Estimates	*Value*
$\widehat{A}$	0.9994534
$\widehat{Var}(\widehat{A})$	2.697561×10^{-13}
$\widehat{s.d.}(\widehat{A})$	5.193805×10^{-7}

5. Conclusions

In the present work, the problem of estimating the limiting availability in a single-component system is addressed under a Bayesian methodology. Also, the maximum likelihood estimate is revisited.

When implementing the maximum likelihood estimate, the behavior of the estimators is consistent as the sample size is increased. Clearly, the convergence of the estimator is affected by the dispersion of each variable.

In the Bayesian case, the simulations were performed in a rather general way. Exponential and Gamma distributions were considered for the failure time and the repair time. The use of exponential and Gamma distributions for failure and repair times has been motivated, taking into account the referential frame they have in reliability. In addition, in our case, it is possible to perform a conjugate analysis, taking as a priori the generalized Beta distribution. Furthermore, different types of prior distributions for the hyperparameters were considered. In the first instance, priors providing little information and then others more informative. Estimates of limiting availability greater than 0.5 are slightly underestimated in both the classical and Bayesian cases when the failure time and the repair time are exponential. However this does not occur, when the fault time is distributed Gamma and the repair time, exponential.

A relevant point of this work is to have developed a general Bayesian methodology, since this is not limited to the particular distributions considered.

The Bayesian method is applied in the estimation of the limiting availability of a palleitzer of a glass bottles factory, without having prior information. The results reflect the good performance of the machine.

Extensions of this approach include the use of other loss functions, which could help to control underestimation (respectively overestimation). In fact, the Bayesian methodology developed and applied in this paper can be adapted to a coherent system of k independently functioning components. Also, other parametric models used in reliability system can be considered.

One interesting approach is to set up the reliability model treated in this paper in a Bayesian semiparametric framework. This is a topic, for future research.

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