Revista Colombiana de Estadística
January 2018, Volume 41, Issue 1, pp. 1 to 30
DOI: http://dx.doi.org/10.15446/rce.v41n1.59929

Simultaneous Confidence Bands for the Estimation of Expected Discounted Warranty Costs for Coherent Systems under Minimal Repair

Bandas de confianza simultáneas para la estimación del costo medio de garantía descontado para un sistema coherentes bajo reparo mínimo

Carlos M. Lopera-Gómez[a], Nelfi G. González-Álvarez[b]

Escuela de Estadística, Facultad de Ciencias, Universidad Nacional de Colombia, Medellín, Colombia

Abstract

This paper develops simultaneous confidence bands using computer intensive methods based on resampling, for the expected discounted warranty costs in coherent systems under physical minimal repair, that is, when the system is observed at its components level and only the component that causes the fault is minimally repaired. For this purpose, it is shown that, under the framework of the Martingale processes and the central limit resampling theorem (CLRT) over stochastic processes, the discounted warranty cost processes satisfy the conditions set by the central limit resampling theorem (CLRT). Additionally, a simulation study is performed on the most relevant variables, such that the finite sample features of the proposed bands may be assessed, based on their actual coverage probabilities. The results in the considered scenarios show that resampling-based simultaneous confidence bands have coverage probabilities that are close to the nominal coverage. In particular, the agreement is good when there are 100 systems or more where a large number of resamples are used for the approximation.

***Key words*:** Central Limit Theorem; Coverage Probability; Point Process; Resampling; Semimartingales; Weak Convergence.

Resumen

Este trabajo desarrolla bandas de confianza simultáneas usando métodos computacionales intensivos basados en remuestreo, para el costo medio de garantía descontado en sistemas coherentes bajo reparo mínimo físico, esto

[a]PhD. E-mail: cmlopera@unal.edu.co
[b]PhD. E-mail: ngonzale@unal.edu.co

es, cuando el sistema es observado al nivel de sus componentes y sólo la componente que causa la falla es reparada mínimamente. Para ello se prueba que, bajo el marco teórico de los procesos martingala, los procesos de costos de garantía descontados cumplen las condiciones del teorema de límite central de remuestreo (CLRT). Un estudio de simulación Monte Carlo se realiza para evaluar, a través de las probabilidades de cobertura alcanzadas, el desempeño de las bandas propuestas en muestras finitas. Los resultados en los escenarios considerados muestran que las bandas de confianza basadas en remuestreo tienen probabilidades de cobertura con valores cercanos a los esperados, en particular para aquellas basadas en muestras con más de 100 sistemas donde el número de remuestras usado para la aproximación es grande.

***Palabras clave*:** convergencia débil; probabilidad de cobertura; proceso puntual; remuestreo; semimartingalas; teorema de límite central.

1. Introduction

When a manufacturer puts a new product on the market, it is expected that a warranty program will come along with the product, which could become a great cost if it is a low-quality product (Thomas 2005). Bai & Pham (2006) highlight that warranty analysis is very important for cost modeling, especially for discounted costs (Blischke & Murthy 1994, Chien 2005, Ja, Kulkarni, Mitra & Patankar 2002, Murthy & Djamaludina 2002, Nguyen & Murthy 1984). Discounted warranty cost models consider the product age and provide a proper measurement of the costs involved in the warranty program, given that they can be treated as random cash flows. Therefore, it is possible to model their evolution throughout the lifetime of the product under warranty, as well as to estimate the necessary fund reserve levels to meet future warranty claims. Several aspects regarding discounted warranty costs and their corresponding reserves have been studied by Bai & Pham (2004), Duchesne & Marri (2009), Ja et al. (2002), Jain & Maheshwari (2006), Patankar & Mitra (1995), and Thomas (1989).

In practice, many products consist of several components, that is, the product can be seen as a system. If every component has its own warranty, they can be combined to produce one warranty for the system, in which it is necessary to keep in mind both the system structure and warranty service costs at its component level (Thomas 1989). Several previous papers have considered warranty cost analysis for component systems (Ritchken & Fuh 1986, Chukova & Dimitrov 1996, Hussain & Murthy 1998, Bai & Pham 2006, Balachandran, Maschmeyer & Livingstone 1981, Jung, Park & Park 2010). There are many ways to model the impact of repair actions over system failure times. In literature, it is frequently assumed that repairing a system leaves it as good as new (Block, Borges & Savits 1985). Nevertheless, this hypothesis and its implications have been criticized by many authors on the argument that repairing can only, in many practical cases, restoring the system back to the performance conditions right before the failure (Block et al. 1985, Ascher 1968, Ouali, Tadj, Yacout & Ait-Kadi 2011), which is called minimal repair. In complex systems, repairing is frequently assumed as minimal (Blischke &

Murthy 1994). The definition of the state of the system immediately before failure depends on the information level one has about the system (Aven & Jensen 1999), so that minimal repairs are classified into two types: statistical minimal repair, which implies replacing the full system for another one just as old as the other one would be if it had not failed (Nguyen & Murthy 1984, Aven 1983, Sheu, Griffith & Nakagawa 1995, Ja, Kulkarni, Mitra & Patankar 2001), and physical minimal repair, in which the system is supposed to be observed at its component level and, therefore, only the critical component that caused the system to fail, gets minimal repair (Aven & Castro 2008, González & Bueno 2011).

González & Bueno (2011) propose a Martingale estimator for the expected discounted warranty costs for a coherent system under physical minimal repair and include the calculation of specific confidence limits which do not make up a simultaneous confidence band, given that the aforementioned set of limits generally has no correct coverage probability (Fleming & Harrington 1991). The main purpose of constructing simultaneous confidence bands is to assess an estimator's precision, which can be described by the distribution (or a function of it) of that estimator's deviations from its real value (Belyaev 2007). The problem is that the aforementioned distribution is unknown, even if asymptotic convergence results of distribution can be obtained (González & Bueno 2011). In the practice, sample sizes are not always large enough for those approximations to work properly. In general, computer intensive (CI) methods provide a way to find asymptotically precise approximations of the estimator deviation distributions from the real unknown parameters (Belyaev 2000). The bootstrap, introduced by Efron (1979) is a rather universal method, nevertheless, the need to find a proper estimator of the real parameter which can describe data distribution, may be a difficult problem, which is why resampling can be used alternatively (Belyaev 2000).

Resampling is used in this paper to develop simultaneous confidence bands for the mean function of the discounted warranty costs of a system under physical minimal repair. For this, based on the theoretical framework of Martingale process and the Central Limit of Resampling Theorem (CLRT) over stochastic processes proposed by Belyaev (2000) and Belyaev & Seleznjev (2000), proof is presented that the discounted warranty costs processes under the general lifetime model comply with CLRT conditions. In addition, a simulation study was conducted on the most relevant variables to test the finite sample features of the proposed simultaneous confidence bands by means of the actual coverage probability.

Section 2 presents the theoretical framework that is necessary in the development of this paper. The proposal of constructing simultaneous confidence bands is developed in Section 3. In Section 4, the performance of the proposed simultaneous confidence bands is assessed by means of a simulation study. Section 5 presents the paper's most important conclusions and recommendations.

2. Theoretical Framework

In this paper it is assumed that there is a coherent system under physical minimal repair, that is, under the physical approach of the general failure model

(Aven & Jensen 1999). Next section summarizes some theoretical results which are necessary for the development of the remaining sections in the paper.

2.1. Physical Minimal Repair Model for a Coherent System and Discounted Warranty Costs

Let us suppose a system with m components, where T is the system lifetime, S_i is the lifetime of component i, $i = 1, \ldots, m$ and $\tilde{N}_t$ is the number of minimal repairs of the system in the interval $[0, t]$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$,

$$\mathcal{F}_t = \sigma\left(\widetilde{N}_s, I\left(S_i > s\right), 0 \leq s \leq t, i = 1, \ldots, m\right), \tag{1}$$

where $I(A)$ is the indicator of the event A. Therefore, the system repair/failure process is observed at the level of its m components. Suppose the following conditions are hold:

a) $0 < S_i < \infty$ P-a.s., $i = 1, \ldots, m$, where P-a.s. denotes that an event E happens almost surely with respect to the probability P.

b) For every $i \neq j, P\left(S_i = S_j\right) = 0$, that is, there are not two components failing simultaneously.

c) All lifetimes S_i are totally inaccessible $\mathcal{F}_t$-stopping times, and consequently all the compensators A^i of the respective simple counting processes $N_t^i = I\left(S_i \leq t\right)$, are continuous P-a.s., with the Doob-Meyer decomposition $N_t^i = A_t^i + M_t^i, \quad M_t^i \in \mathcal{M}_0^2, \quad i = 1, \ldots, m$.

Under these assumptions and according to Aven & Jensen (1999) can be shown that

$$T = \min_{i: Y_i < \infty} S_i, \tag{2}$$

where Y_i is the critical level of component i, that is, the first instant after which failure of component i causes the system failure. González & Bueno (2011) show that the system failure indicator process $N_t = I\left(T \leq t\right)$, given by

$$N_t = A_t + M_t, \quad M_t \in \mathcal{M}_0^2, \tag{3}$$

has the compensator A_t as follows

$$A_t = \int_0^t I\left(T > s\right) \sum_{i=1}^m I\left(Y_i < s\right) \lambda^i\left(s\right) ds, \tag{4}$$

where $\lambda^i(t)$ is the failure rate function for component i.

From (4), it is clear that the $\mathcal{F}_t$-intensity of the system on $\{T > t\}$ is given by

$$\lambda_t = \sum_{i=1}^m I\left(Y_i < t\right) \lambda^i\left(t\right). \tag{5}$$

Let $\widetilde{N}_t^i$ be the number of minimal repairs on the component i in the interval $[0,t]$ and let $H_i\left(t\right)$ be the minimal repair discounted cost of component i at time t, a deterministic, continuous, decreasing, bounded, and integrable function on the interval $[0,t]$, such that $\int_0^t H_i\left(s\right)\lambda^i\left(s\right)ds < \infty, \forall\ 0 \leq t < \infty$, and let $\widehat{B}_t^i = \int_0^t H_i\left(s\right)d\widetilde{N}_s^i = \sum_{j=1}^{\widetilde{N}_t^i} H_i\left(S_{ij}\right)$ be the accumulated cost process by minimal repairs on component i on the interval $[0,t]$, where S_{ij} is the time of the j-th minimal repair of component i and $S_{i1} = S_i$. González & Bueno (2011) show that $\widehat{B}_t^i$ has $\mathcal{F}_t$-compensator B_t^i given by

$$B_t^i = \int_0^t H_i\left(s\right)\lambda^i\left(s\right)ds < \infty, \forall\ 0 \leq t < \infty, \tag{6}$$

and that when only the critical component causing the system failure is minimally repaired, that is, considering the set $\mathcal{C}^i = \left\{\omega \in \Omega : S_i\left(\omega\right) > Y_i\left(\omega\right)\right\}$ the $\mathcal{F}_t$-compensator of $\widehat{B}_t^i$ is the process

$$\begin{aligned} B_t^{i*} &= \int_0^t I\left(Y_i < s\right)H_i\left(s\right)\lambda^i\left(s\right)ds \\ &= \int_{Y_i}^t H_i\left(s\right)\lambda^i\left(s\right)ds < \infty, \quad \forall\ 0 \leq t < \infty,\ P\text{-a.s.} \end{aligned} \tag{7}$$

Definition 1. (González & Bueno 2011) For each $\omega \in \Omega$, the set of components which survive its critical level is defined as

$$\mathcal{C}^{\Phi}\left(\omega\right) = \left\{i \in \{1,\ldots,m\} : S_i\left(\omega\right) > Y_i\left(\omega\right)\right\}. \tag{8}$$

For each $i = 1,\ldots,m$ define $\quad C^i\left(\omega\right) = \begin{cases} 1 & \text{if } i \in \mathcal{C}^{\Phi}\left(\omega\right) \\ 0 & \text{otherwise.} \end{cases}$

Then, the minimal repair counting process of the coherent system in $[0,t]$ is $\widetilde{N}_t\left(\omega\right) = \sum\limits_{i=1}^{m} C^i\left(\omega\right)\widetilde{N}_t^i\left(\omega\right)$ and the warranty cost process is given by $\widehat{B}_t\left(\omega\right) = \sum\limits_{i=1}^{m} C^i\left(\omega\right)\widehat{B}_t^i\left(\omega\right)$, whose compensator process is $B_t\left(\omega\right) = \sum\limits_{i=1}^{m} C^i\left(\omega\right)B_t^i\left(\omega\right)$.

Then, from Definition 1 the expected cost of minimal repairs carried out over the system in interval $[0,t]$ is

$$B^*\left(t\right) = E\left[\widehat{B}_t\right] = \sum_{i=1}^{m} B^{i*}\left(t\right), \tag{9}$$

with

$$B^{i*}\left(t\right) = P\left(S_i > Y_i\right)E\left[\int_{Y_i}^t H_i\left(s\right)\lambda^i\left(s\right)ds \middle| S_i > Y_i\right]. \tag{10}$$

Based on the results above, next section gives an estimation method for $B^*\left(t\right)$ based on a random sample of n identical systems (or n independent copies of the process).

2.2. Estimation Based on a Sample of n Identical Systems

Consider n independent copies of the process $\left(\widehat{B}_t^i, C^i, i = 1, \dots, m\right)_{t \geq 0}$ which are denoted by $\left(\widehat{B}_t^{i(j)}, C^{\,i(j)},\ i = 1, \dots, m,\ j = 1, \dots, n\right)$. For each j, let $\mathcal{C}^{\Phi(j)}$ be the set of critical components for the j-th system, which is defined as in (8), then the minimal repair costs for the j-th system is

$$\widehat{B}_t^{(j)} = \sum_{i=1}^{m} C^{\,i(j)} \int_0^t H_i(s)\, d\widetilde{N}_s^{\,i(j)}, \tag{11}$$

whose compensator process is

$$B_t^{(j)} = \sum_{i=1}^{m} C^{\,i(j)} \int_{Y_i^{(j)}}^t H_i(s)\, \lambda^i(s)\, ds. \tag{12}$$

From the n independent copies, the following mean processes are obtained,

$$\overline{\widehat{B}}_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \widehat{B}_t^{(j)} \quad \text{y} \quad \overline{B}_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} B_t^{(j)}.$$

The following are results for the mean processes defined above.

Theorem 1. *(González & Bueno 2011)*

i. $\overline{\widehat{B}}_t^{(n)}$ *is a consistent and unbiased estimator for* $B^*(t)$.

ii. A consistent and unbiased estimator for $Var\left[\overline{\widehat{B}}_t^{(n)}\right]$ *is*

$$\widehat{Var}\left[\overline{\widehat{B}}_t^{(n)}\right] = \sum_{i=1}^{m} \varrho_t^{2i(n)}/n.$$

iii. An approximate $100(1-\gamma)\,\%$ *confidence interval for* $B^*(t)$, *is*

$$\overline{\widehat{B}}_t^{(n)} \pm Z_{\gamma/2} \sqrt{\frac{\sum_{i=1}^{m} \varrho_t^{2i(n)}}{n}}, \tag{13}$$

where $Z_{\gamma/2}$ *is the* $(1-\gamma/2)$ *quantile of the standard normal distribution and*

$$\varrho_t^{2\,i(n)} = \left(\frac{n}{n-1}\right) \frac{1}{n} \sum_{j=1}^{n} C^{i(j)} \left[\left(\widehat{B}_t^{i(j)} - B^{i*}(t)\right)^2 - \left(\overline{\widehat{B}}_t^{i(n)} - B^{i*}(t)\right)^2\right],$$

with $\widehat{B}_t^{i(j)} = \int_0^t H_i(s)\, d\widetilde{N}_s^{\,i(j)}$.

Despite having a set of approximately $100(1-\gamma)\,\%$ level pointwise confidence limits in $[0, t]$, given by (13), they do not form $100(1-\gamma)\,\%$ level simultaneous confidence bands. The following section introduces a general definition of simultaneous confidence bands for random functions.

2.3. Simultaneous Confidence Bands for Random Functions

Suppose the goal is to estimate and bound a function $f(s)$ in the interval $[0, t]$. That is, given a coverage probability of $(1-\gamma)$, we want to find two random functions $b_1(s)$ and $b_2(s)$ with the property

$$P\left[b_1(s) \leq f(s) \leq b_2(s),\ \forall\ s \in [0, t]\right] \approx 1-\gamma, \tag{14}$$

Except for functions having a very simple structure, there are no simultaneous confidence bands with an exact $(1-\gamma)$ coverage probability (Knowles 1988).

Let $\left(\widehat{f}_t^{(n)}\right)_{t \geq 0}$ be an estimator of the function $f(t)$, based on a random sample of size n, then the weak convergence of processes with the form $\sqrt{n}\left(\widehat{f}_t^{(n)} - f(t)\right)$ provides a general method for calculating simultaneous confidence bands for the function $f(t)$ (Fleming & Harrington 1991). When $\sqrt{n}\left(\widehat{f}_t^{(n)} - f(t)\right)$ converges in distribution ($\xrightarrow{w}$) on interval $[0, t]$ to a limit process Q, the Continuous Mapping Theorem implies that

$$\sup_{0 \leq s \leq t} \sqrt{n}\left|\widehat{f}_s^{(n)} - f(s)\right| \xrightarrow{w} \sup_{0 \leq s \leq t} |Q(s)|. \tag{15}$$

If $q_\gamma(t)$ satisfies

$$P\left(\sup_{0 \leq s \leq t} |Q(s)| \leq q_\gamma(t)\right) \approx 1-\gamma, \tag{16}$$

where $q_\gamma(t)$ is the $(1-\gamma)$ quantile in the distribution of $\sup\limits_{0 \leq s \leq t} |Q(s)|$, then, asymptotically,

$$P\left(\sup_{0 \leq s \leq t} \sqrt{n}\left|\widehat{f}_s^{(n)} - f(s)\right| \leq q_\gamma(t)\right) \approx 1-\gamma. \tag{17}$$

Then, the construction of simultaneous confidence bands is based on finding $q_\gamma(t)$ which satisfies the desirable coverage probability on the interval $[0, t]$.

Simultaneous confidence bands based on $\sup_{0 \leq s \leq t} \sqrt{n}\left|\widehat{f}_s^{(n)} - f(s)\right|$, will be useful only when sufficient conditions for the convergence of $\sqrt{n}\left(\widehat{f}_t^{(n)} - f(t)\right)$ on reasonable intervals $[0, t]$ are not too restrictive, when $q_\gamma(t)$ is easy to calculate, and when the resulting bands have appealing properties (Fleming & Harrington 1991). Even when the general conditions for weak convergence could be fulfilled, calculating $q_\gamma(t)$ requires determining the limit process $\sup_{0 \leq s \leq t} |Q(s)|$ to which the process $\sup_{0 \leq s \leq t} \sqrt{n}\left|\widehat{f}_s - f(s)\right|$ converges, and this is not easy when only a sample of n systems is available.

The following section presents the weak approach of processes introduced by Belyaev (2000) and Belyaev & Seleznjev (2000) as an extension of weak convergence of processes, which justifies the use of resampling in the approximation of asymptotic distributions.

2.4. Weakly Approaching Distributions

Let $\{\mathcal{L}(U_n)\}_{n\geq 1}$ and $\{\mathcal{L}(V_n)\}_{n\geq 1}$ be two sequences of distributions of random variables U_n and V_n which have values on $\mathcal{H}_n$, a metric space with metric d_n, and let $\mathscr{C}_b(\mathcal{H}_n, d_n)$ be the set of all bounded real-valued continuous functions on $\mathcal{H}_n$.

Definition 2. (Belyaev 2000) The distributions $\{\mathcal{L}(U_n)\}_{n\geq 1}$ weakly approach $\{\mathcal{L}(V_n)\}_{n\geq 1}$, denoted by $\mathcal{L}(U_n) \overset{wa}{\longleftrightarrow} \mathcal{L}(V_n)$, $n \to \infty$, if for all $h = h(\cdot) \in \mathscr{C}_b(\mathcal{H}_n, d_n)$ it holds that

$$E[h(U_n)] - E[h(V_n)] \to 0, \ n \to \infty. \tag{18}$$

Let W_n be a random variable (in a metric space $\mathcal{W}_n$) defined on the same probability space of U_n previously defined. Suppose that the regular conditional distribution $\mathcal{L}(U_n | W_n)$ exists.

Definition 3. (Belyaev 2000) The Random distributions $\{\mathcal{L}(U_n | W_n)\}_{n\geq 1}$ weakly approach $\{\mathcal{L}(V_n)\}_{n\geq 1}$ in probability, denoted by $\mathcal{L}(U_n | W_n) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(V_n)$, $n \to \infty$, if for all $h \in \mathscr{C}_b(\mathcal{H}_n, d_n)$ the condition $E[h(U_n)| W_n] - E[h(V_n)] \overset{P}{\longrightarrow} 0$, $n \to \infty$ is satisfied. Here $\overset{P}{\longrightarrow}$ denotes convergence in probability.

Lemma 1. *(Belyaev 2000) Let U_n, W_n and V_n be as defined before. Suppose that $\mathcal{L}(U_n | W_n) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(V_n)$ and let Z_n be an $\mathcal{H}_n$-valued random variable defined on the same probability space of U_n, such that $Z_n \underset{w}{\overset{P}{\longrightarrow}} 0, n \to \infty$. Then,*

$$\mathcal{L}(U_n + Z_n | W_n) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}(V_n), \ n \to \infty. \tag{19}$$

The notion of weakly approaching establishes a variant of Lyapunov's Central Limit Theorem (CLT) for $\mathbb{R}^k$, as follows. Let $\mathbb{U}_n = \{\mathbf{U}_{1n}, \ldots, \mathbf{U}_{nn}\}$, $n = 1, 2, \ldots$, be a triangular scheme of independent vector-valued random variables, where for each n, $\mathbf{U}_{in} = (U_{1in}, \ldots, U_{kin})^T$. Let $\mathbb{C}_{\,in}$, $i = 1, \ldots, n$ be the covariance matrix between the k variables of $\mathbf{U}_{in}$ and define $\mathbf{U}_{\cdot n} = \sum_{i=1}^n \mathbf{U}_{in}$, $\boldsymbol{\mu}_{\cdot n} = \sum_{i=1}^n E(\mathbf{U}_{in})$ and $\overline{\mathbb{C}}_{\cdot n} = (1/n) \sum_{i=1}^n \mathbb{C}_{\,in}$.

Theorem 2. *(Belyaev 2000, CLT in $\mathbb{R}^k$) Suppose that for some constants $\delta > 0$ and $c = c(2+\delta) < \infty, E|\sqrt{n}\, U_{hin}|^{2+\delta} \leq c$ for all $(i, n) \in \mathcal{T}$. Then it holds that*

$$\mathcal{L}\left(\sqrt{n}\,(\boldsymbol{U}_{\cdot n} - \boldsymbol{\mu}_{\cdot n})\right) \overset{wa}{\leftrightarrow} N_k\left(\boldsymbol{0}_k, \overline{\mathbb{C}}_{\cdot n}\right), n \to \infty, \tag{20}$$

where $N_k\left(\boldsymbol{0}_k, \overline{\mathbb{C}}_{\cdot n}\right)$ is the k-dimensional normal distribution with mean $\boldsymbol{0}_k$ and the covariance matrix $\overline{\mathbb{C}}_{\cdot n}$.

Now consider a triangular scheme $\mathbb{U}_n = \{\mathbf{U}_{1n}, \ldots, \mathbf{U}_{nn}\}$ of independent vector-valued random variables with $\mathbf{U}_{in} \in \mathbb{R}^k$, $(i, n) \in \mathcal{T}$, where $\mathcal{T} = \{(i', n')\, i' = 1, \ldots, n,$ $n' = 1, 2, \ldots\}$. Let $J_{1n}^\star, \ldots, J_{nn}^\star$ be the indexes of a resample from $\mathbb{U}_n$, indicating which of the observations $\mathbf{U}_{in}$ is chosen as the i-th element in the resample. Let

$N_{hn}^{\star}=\sum_{i=1}^{n} I\left(J_{in}^{\star}=h\right)$ be the number of times that the observation h is drawn in the resample and define the vector-valued random variable

$$\mathbf{U}_{\cdot n}^{\star 0}:=\sum_{i=1}^{n}\left(N_{in}^{\star}-1\right)\mathbf{U}_{in},$$

which can be interpreted as a centered sum of values, n times randomly sampled with replacement from the components in the statistical data $\mathbb{U}_n$.

Theorem 3. *(Belyaev 2000, CLRT in $\mathbb{R}^k$) Suppose the assumptions of Theorem 2 are fulfilled and that $E\left[U_{hin}\right]=\mu_{hn}$, that is, the expectation does not dependent of i. Then, it holds that*

$$\mathcal{L}\left(\boldsymbol{U}_{\cdot n}^{\star 0}\middle|\,\mathbb{U}_n\right)\overset{wa(P)}{\longleftrightarrow}\mathcal{L}\left(\boldsymbol{U}_{\cdot n}-\boldsymbol{\mu}_{\cdot n}\right),\ n\to\infty. \tag{21}$$

In general, the resampling process consists of simulating G copies $\{J_{1n}^{\star r},\ldots,J_{nn}^{\star r}\}$, $r=1,\ldots,G$, (which are used to approximate $\mathcal{L}\left(\mathbf{U}_{\cdot n}^{\star 0}\middle|\,\mathbb{U}_n\right)$). Then, $\mathbf{U}_{\cdot n}^{\star 0r}=\sum_{i=1}^{n}\left(N_{in}^{\star r}-1\right)\mathbf{u}_{in}$ is obtained for $r=1,\ldots,G$, where $\mathbf{u}_{in}$ is the observed value of $\mathbf{U}_{in}$. So, for any Borel set $A\subset\mathbb{R}^k$ it has that

$$\frac{1}{G}\sum_{r=1}^{G} I\left(\mathbf{U}_{\cdot n}^{\star 0r}\in A\right)\overset{P}{\longrightarrow}P\left(\mathbf{U}_{\cdot n}^{\star 0}\in A\middle|\,\mathbb{U}_n\right),\ G\to\infty. \tag{22}$$

Belyaev (2000) shows that to assess the accuracy of some nonparametric estimators, it is necessary to consider more general spaces than $\mathbb{R}^k$. For instance, for many non-parametric statistical models it is necessary to consider the Skorokhod space $D\left[0,t\right], t>0$, of the so-called *cadlag* functions. That is, the set of all real functions $v\left(\cdot\right)$ defined in $[0,t]$ such that for all $s\in(0,t)$ there are limit values $v\left(s\pm 0\right)=\lim_{h\downarrow 0} v\left(s\pm h\right)$ and $v\left(s\right)=v\left(s+0\right)$.

Theorem 4. *(Belyaev 2000, CLRT in Skorokhod space). For the $\mathbb{U}_n=\{U_{1n}\left(t\right),\ldots,U_{nn}\left(t\right)\}$ triangular scheme of $D[0,t]$-valued random variables which are independent for each $n=1,2,\ldots$. Let $U_{\cdot n}\left(t\right)=\sum_{i=1}^{n}U_{in}\left(t\right)$. Suppose that $E\left[U_{in}\left(s\right)\right]=\mu_n\left(s\right), i=1,\ldots,n, s\in[0,t]$ and that there are positive constants c_1,c_2,c_3 and $\delta>0$ such that for every $0\le t_1\le s\le t_2\le t$.*

C-1. $nE[(U_{jn}\left(t_2\right)-U_{jn}\left(t_1\right))^2]\le c_1\left|t_2-t_1\right|^{(1+\delta)/2}$,

C-2. $n^2E[(U_{jn}\left(s\right)-U_{jn}\left(t_1\right))^2\left(U_{jn}\left(t_2\right)-U_{jn}\left(s\right)\right)^2]\le c_2\left|t_2-t_1\right|^{1+\delta}$, *and*

C-3. $E\left[\left|\sqrt{n}U_{jn}\left(s\right)\right|^{2+\delta}\right]\le c_3$.

Then, it holds that

$$\mathcal{L}\left(\sum_{j=1}^{n}\left(N_{jn}^{\star}-1\right)U_{jn}\left(\cdot\right)\middle|\,\mathbb{U}_n\right)\overset{wa(P)}{\longleftrightarrow}\mathcal{L}\left(U_{\cdot n}\left(\cdot\right)-\mu_n\left(\cdot\right)\right),\ n\to\infty. \tag{23}$$

3. Simultaneous Confidence Bands for the Expected Discounted Warranty Cost

In constructing simultaneous confidence bands for $B^*(t)$ it is important to assess the distribution of the unobservable processes $\sqrt{n}\left(\overline{\widehat{B}}_t^{(n)} - B^*(t)\right)$. Theorem 4 and the use of resampling allow approximating relevant distribution when information about an initial sample of n systems is available. For this purpose, define

$$U_{\cdot n}(t) = \sqrt{n}\left(\overline{\widehat{B}}_t^{(n)} - B^*(t)\right) = \sum_{j=1}^{n} \frac{1}{\sqrt{n}}\left(\widehat{B}_t^{(j)} - B^*(t)\right) = \sum_{j=1}^{n} U_{jn}(t) \tag{24}$$

where,

$$U_{jn}(t) = \frac{1}{\sqrt{n}}\left(\widehat{B}_t^{(j)} - B^*(t)\right), \tag{25}$$

are stochastic processes in $D[0,t]$, which can be arranged in a triangular scheme $\mathbb{U}_n$.

To use the CLRT in $D[0,t]$, it must be verified that the processes $U_{jn}(t)$ satisfy the conditions *C-1*, *C-2* and *C-3* established in Theorem 4, which is summarized in the following theorem (see proof in the Appendix B)

Theorem 5. *Let $U_{jn}(t)$ be as defined in (25). Then for every $0 \le t_1 \le s \le t_2 \le t < \infty$, there exist positive constants c_1, c_2, c_3 and $\delta > 0$, such that*

C-1. $nE[(U_{jn}(t_2) - U_{jn}(t_1))^2] \le c_1 |t_2 - t_1|^{(1+\delta)/2}$,

C-2. $n^2 E[(U_{jn}(s) - U_{jn}(t_1))^2 (U_{jn}(t_2) - U_{jn}(s))^2] \le c_2 |t_2 - t_1|^{1+\delta}$, *and*

C-3. $E\left[|\sqrt{n}\, U_{jn}(s)|^{2+\delta}\right] \le c_3$.

The following corollary formalizes the application of the CLRT to warranty cost processes.

Corollary 1. *Consider a sample of n independent copies of*

$$\left(\widehat{B}_t^i, C^i, i = 1, \dots, m\right)_{t \ge 0},$$

which are denoted by $\left(\widehat{B}_t^{i(j)}, C^{\,i(j)}, i = 1, \dots, m, j = 1, \dots, n\right)_{t \ge 0}$ and the triangular scheme $\mathbb{U}_n(t) = \{U_{n1}(t), \dots, U_{nn}(t)\}$ with $U_{jn}(t)$ given in (25). Then, for the process

$$U_{\cdot n}^{\star 0}(t) = \sum_{j=1}^{n} \left(N_{jn}^{\star} - 1\right) U_{jn}(t), \tag{26}$$

one can show that

$$\mathcal{L}\left(U_{\cdot n}^{\star 0}(t) \middle|\, \mathbb{U}_n\right) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}\left(U_{\cdot n}(t)\right), \; n \to \infty. \tag{27}$$

***Proof*.** By applying Theorem 5 over the process $U_{jn}(t)$, the necessary conditions for Theorem 4 are obtained, from whose application the result is obtained. $\square$

In practice the processes $U_{jn}(t)$ are unknown and they need to be estimated. The following section uses estimations of the processes $U_{jn}(t)$ to construct simultaneous confidence bands based on resampling.

3.1. A Proposal for Simultaneous Confidence Bands for the Expected Warranty Cost

The unobservable process $U_{jn}(t)$ can be rewritten as follows:

$$U_{jn}(t) = \widehat{U}_{jn}(t) + \widetilde{U}_{jn}(t), \tag{28}$$

where

$$\widehat{U}_{jn}(t) = \frac{1}{\sqrt{n}}\left(\widehat{B}_t^{(j)} - \overline{\widehat{B}}_t^{(n)}\right) \quad \text{y} \quad \widetilde{U}_{jn}(t) = \frac{1}{\sqrt{n}}\left(\overline{\widehat{B}}_t^{(n)} - B^*(t)\right).$$

The following result establishes that the weakly approaching of the process $U_{\cdot n}^{\star 0}(t)$, given in (26), is kept in an estimated version of itself.

Corollary 2. *The process* $\widehat{U}_{\cdot n}^{\star 0}(t) = \sum_{j=1}^{n}\left(N_{jn}^{\star} - 1\right)\widehat{U}_{jn}(t)$ *has the property that*

$$\mathcal{L}\left(\widehat{U}_{\cdot n}^{\star 0}(t)\Big|\,\mathbb{U}_n\right) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}\left(U_{\cdot n}(t)\right), \ n \to \infty. \tag{29}$$

***Proof*.** By using (28) in (26), the identity

$$U_{\cdot n}^{\star 0}(t) = \widehat{U}_{\cdot n}^{\star 0}(t) + \widetilde{U}_{\cdot n}^{\star 0}(t), \tag{30}$$

is obtained, where $\widehat{U}_{\cdot n}^{\star 0}(t) = \sum_{j=1}^{n}\left(N_{jn}^{\star} - 1\right)\widehat{U}_{jn}(t)$ and $\widetilde{U}_{\cdot n}^{\star 0}(t) = \sum_{j=1}^{n}\left(N_{jn}^{\star} - 1\right)$ $\widetilde{U}_{jn}(t)$.

Notice that $\widetilde{U}_{\cdot n}^{\star 0}(t) \equiv 0$ (since $\widetilde{U}_{\cdot n}^{\star 0}(t)$ does not dependent of j by definition and $\sum_{j=1}^{n} N_{jn}^{\star} = n$). Then, $\widetilde{U}_{\cdot n}^{\star 0}(t) \xrightarrow{P} 0$, therefore by using Lemma 1 the result is obtained. $\square$

The latter allows proposing simultaneous confidence bands for the expected discounted warranty cost in coherent systems under physical minimal repair.

Theorem 6. *An approximate* $100(1-\gamma)\,\%$ *simultaneous confidence band for* $B^*(t)$*, the expected discounted warranty cost process for a coherent system under physical minimal repair on the interval* $[0,t]$*, is*

$$\overline{\widehat{B}}_t^{(n)} \pm \frac{q_{\gamma}^{\star}(t)}{\sqrt{n}}, \ 0 \leq s \leq t, \tag{31}$$

where, $q_{\gamma}^{\star}(t)$ *is the* $(1-\gamma)$ *quantile of the empirical distribution of* $\sup\limits_{0\leq s\leq t}\left|\widehat{U}_{\cdot n}^{\star 0}(s)\right|$.

Proof. González & Bueno (2011) showed that the process $U_{\cdot n}(t)$ weakly converges in the space $D[0,t]$ to a Gaussian stochastic process denoted by $Q(t)$ which implies

$$\sup_{0\leq s\leq t}|U_{\cdot n}(s)| \xrightarrow{w} \sup_{0\leq s\leq t}|Q(s)|. \tag{32}$$

By following the definition of simultaneous confidence bands introduced in Section 2.3 and from (32) the idea is finding $q_\gamma(t)$ such that:

$$P\left(\sup_{0\leq s\leq t}|Q(s)| \leq q_\gamma(t)\right) \approx 1-\gamma, \tag{33}$$

where the process $\sup_{0\leq s\leq t}|Q(s)|$, to which the process $\sup_{0\leq s\leq t}|U_{\cdot n}(s)|$ weakly converges is unknown. Using Corollary 1

$$\mathcal{L}\left(U^{\star 0}_{\cdot n}(t)\middle|\,\mathbb{U}_n\right) \overset{wa(P)}{\longleftrightarrow} \mathcal{L}\left(U_{\cdot n}(t)\right),\ n\to\infty.$$

Thus, instead of finding $q_\gamma(t)$ satisfying (33), the purpose is obtaining $q^\star_\gamma(t)$ such that:

$$P\left(\sup_{0\leq s\leq t}\left|U^{\star 0}_{\cdot n}(s)\right| \leq q^\star_\gamma(t)\right) \approx 1-\gamma. \tag{34}$$

This is equivalent, using Corollary 2, to find the value of $q^\star_\gamma(t)$ such that

$$P\left(\sup_{0\leq s\leq t}\left|\widehat{U}^{\star 0}_{\cdot n}(s)\right| \leq q^\star_\gamma(t)\right) \approx 1-\gamma. \tag{35}$$

Thus, $q^\star_\gamma(t)$ can be chosen as the $(1-\gamma)$ quantile of the empirical distribution based on resampling from $\sup_{0\leq s\leq t}\left|\widehat{U}^{\star 0}_{\cdot n}(s)\right|$. Thus, considering that

$$\sup_{0\leq s\leq t}\left|\widehat{U}^{\star 0}_{\cdot n}(s)\right| = \lim_{M\to\infty}\ \max_{t_j\in\mathcal{A}}\left|\widehat{U}^{\star 0}_{\cdot n}(t_j)\right|, \tag{36}$$

where $\mathcal{A}=\{t_0=0\leq t_1\leq\cdots\leq t_{M-1}\leq t_M=t\}$ is a partition of interval $[0,t]$, then, with a value of M large enough, it holds that

$$P\left(\overline{\widehat{B}}{}^{(n)}_t - \frac{q^\star_\gamma(t)}{\sqrt{n}} \leq B^*(s) \leq \overline{\widehat{B}}{}^{(n)}_t + \frac{q^\star_\gamma(t)}{\sqrt{n}}:\ 0\leq s\leq t\right) \approx 1-\gamma, \tag{37}$$

such that $\overline{\widehat{B}}{}^{(n)}_t \pm q^\star_\gamma(t)/\sqrt{n}$, $0\leq s\leq t$, is an approximate $(1-\gamma)$ level simultaneous confidence band for $B^*(t)$ within interval $[0,t]$. □

4. Simulation Study

This section uses simulation to assesses the properties of the proposed simultaneous confidence bands. The simulation study considers different scenarios which depend on factors that may affect the performance of the bands established in Theorem 6.

4.1. Simulation Factors and Parameters

The following are the factors considered in the simulation study:

- **System type.** A three-component parallel system and a 2-out-of-4 components system are considered. To avoid confusion, both systems are treated as k-out-of-m systems, which are denoted by: a) $\Phi_t^{1:3}$ for the three-components parallel system or 1-out-of-3 components system, and b) $\Phi_t^{2:4}$ for the 2-out-of-4 components system.
- **Number of systems under warranty.** This is denoted by n and corresponds to the number of independent and identical copies of the repair/cost process used for constructing simultaneous confidence bands. The levels considered are $n = 10, 30, 50, 100, 500$ and 1000.
- **Discount function.** Denoted by $H_i(t)$ describes the consumer share of physical minimal repair costs for the system during the W warranty term. $H_i(t) = c_i e^{-t},\ c_i \left(1 - tW^{-1}\right) e^{-t},\ i = 1, \ldots, m$ were used.
- **Number of resamples.** This is denoted by G. The levels of G are 500, 1000, 5000 and 10000.
- **Partition size.** This is denoted by M and it determines how thin is the partition of the warranty period, for the approximation of the supreme of the limit process given in (32). The levels of M are 100, 500 and 1000.

Table 1 summarizes the levels considered in the simulation factors.

Table 1: Simulation Factors and Their Levels.

Factor	Levels
$\Phi_t^{k:m}$	$\Phi_t^{1:3}$, $\Phi_t^{2:4}$
n	$10, 30, 50, 100, 500, 1000$
$H_i(t)$	$c_i e^{-t},\ c_i \left(1 - tW^{-1}\right) e^{-t}$
G	$500, 1000, 5000, 10000$
M	$100, 500, 1000$

The following are fixed values or simulation parameters:

- **Component failure distributions.** For the systems considered ($\Phi_t^{1:3}$ and $\Phi_t^{2:4}$) in each component i, whose associated lifetime is denoted S_i, the respective cumulative distribution function F_i is considered, thus: $F_i(t) =$ Weibull$(\eta = 1.0,\ \beta = 1.5)$, $i = 1, 2$; $F_i(t) =$ Weibull$(\eta = 2.0,\ \beta = 2.0)$, $i = 3, 4$.
- **Warranty term.** W denotes the time period within which the system is under warranty. The simulation uses $W = 5$. This can be interpreted as representing five years or five thousand use cycles.
- **Minimal repair cost by component.** It corresponds to the minimal repair value in the i-th component and is denoted c_i. For this study, $c_1 = c_2 = 3$ and $c_3 = c_4 = 5$ were considered.

- **Nominal coverage probability.** Denoted by $(1-\gamma)$. It specifies the expected probability that the true mean cost function is bounded. A value of $(1-\gamma)=0.95$ was considered.

- **Number of simulations.** This is denoted by N. A total of $N=10000$ simulations were used.

The Weibull distribution for component lifetime was chosen for its frequent use in industrial reliability. Besides, the values of Weibull distribution parameters set for each component ensure the distributions have increasing failure rates, and a record of failures with at least one event during the established warranty term. The constrains $F_1 \equiv F_2$ and $F_3 \equiv F_4$ were used to simplify the simulation scenarios.

Table 2 summarizes the fixed values for the parameters considered in the simulation study.

TABLE 2: Fixed Values of Simulation Parameters.

Parameter	**Fixed value**
$F_i(t),\ i=1,2$	Weibull $(\eta=1.0, \beta=1.5)$
$F_i(t),\ i=3,4$	Weibull $(\eta=2.0, \beta=2.0)$
W	5
c_i	$c_1=c_2=3,\ c_3=c_4=5$
$(1-\gamma)$	0.95
N	10000

The purpose, at this point, is to assess the performance of the simultaneous confidence band proposed in each scenario, by estimating the coverage probabilities.

4.2. Coverage Probabilities

Let $\text{SCB}_t^{(n)}$ be a simultaneous confidence band for the function $B^*(t)$ in $[0,t]$, based on a sample of n systems, then the coverage probability (CP) for $\text{SCB}_t^{(n)}$ is defined as:

$$\text{CP} = P\left(B^*(t) \in \text{SCB}_t^{(n)}\right). \tag{38}$$

If simultaneous confidence band $\text{SCB}_t^{(n)}$ has a $(1-\gamma)$ level, then $\text{CP} \approx 1-\gamma$.

The following is the procedure followed during the simulations:

i. For each scenario, generate N simulations of n systems under warranty.

ii. For each simulation:

   a) Observe the component failure processes in $[0,W]$, record $C^{i(j)}$, $\widehat{B}_t^{i(j)}$, $\widehat{B}_t^{(j)}$, $i=1,\dots,m$, $j=1,\dots,n$ and calculate $\widehat{\overline{B}}_t^{(n)}$.

   b) Obtain G resamples of size n say $\{J_{1n}^{\star r},\dots,J_{nn}^{\star r}\}$, $r=1,\dots,G$. Calculate $\max\limits_{t_j\in\mathcal{A}}\left|\widehat{U}_{\cdot n}^{\star 0}(t_j)\right|$ for each r, where $\mathcal{A}$ is a partition of size M of $[0,W]$.

c) Obtain $q_{\gamma}^{\star}(t)$, the $(1-\gamma)$ quantile of the estimated approximate distribution for $\sup_{0\leq s\leq t}\left|\widehat{U}_{\cdot n}^{\star 0}(s)\right|$.

d) Using (31) and the information on n systems, obtain a $(1-\gamma)$ level simultaneous confidence band for $B^{*}(t)$ ($\mathrm{SCB}_t^{l(n)}$, $l=1,\ldots,N$).

iii. For each scenario, calculate the actual coverage probability for the simultaneous confidence band,

$$\widehat{\mathrm{CP}}=\frac{1}{N}\sum_{l=1}^{N} I\left(B^{*}(t)\in \mathrm{SCB}_t^{l(n)}\right), \tag{39}$$

where the indicator variables determine if the function $B^{*}(t)$ is totally contained within the resulting bands in each simulation.

Since the function of the expected cost $B^{*}(t)$ is unknown, it is approximated with $\widehat{\overline{B}}_t^{(n)}$ for a sample of 100000 systems. Figure 1 shows the functions $B^{*}(t)$ for coherent systems and discount functions under study.

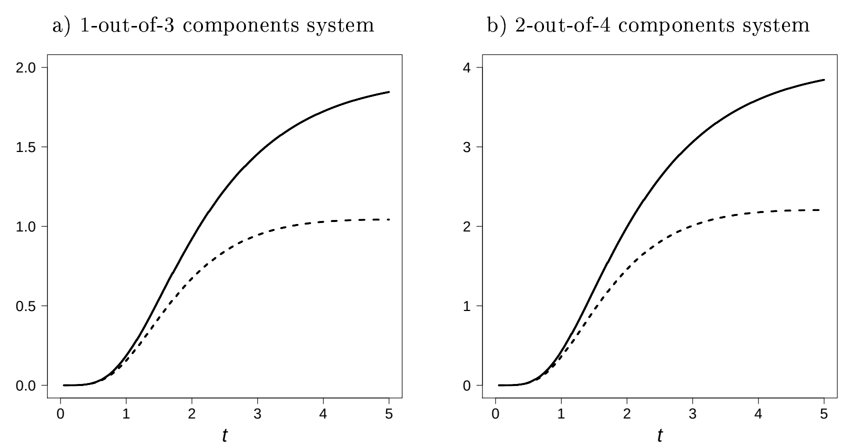

FIGURE 1: Approximate $B^{*}(t)$ for systems under study. The solid curve is calculated for $H_i(t)=c_i\exp(-t)$ and the dashed cure is calculated for $H_i(t)=c_i\left(1-tW^{-1}\right)\exp(-t)$.

Then, the actual coverage probability for the proposed simultaneous confidence bands is obtained from (39).

The following are the results of the simulation study for each of the coherent systems considered.

4.3. Actual Coverage Probabilities for the 1-out-of-3 Components System

For each value considered of the partition size M of the warranty period, Figure 2 shows the results for analyzing the effects of the size of resamples G and the size of sample n, over the actual coverage probabilities for both discount functions.

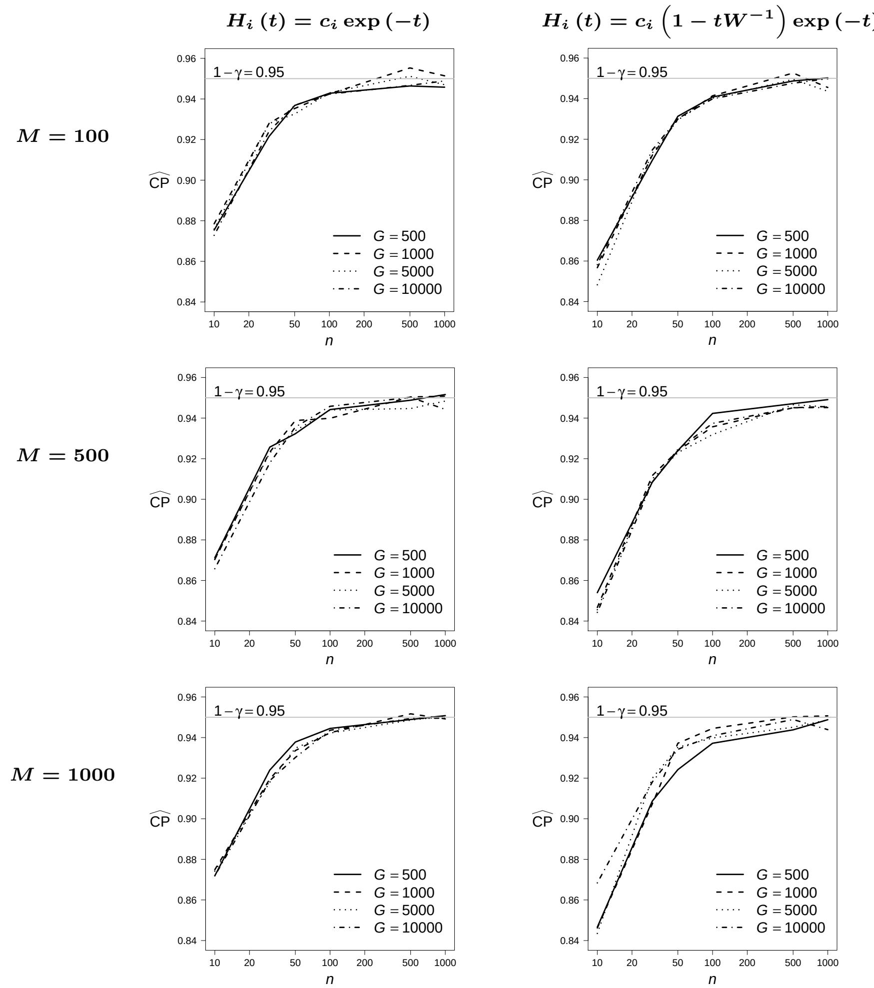

FIGURE 2: $\widehat{\text{CP}}$ for $\text{SCB}_t^{(n)}$, for the 1-out-of-3 components system, varying G for both discount functions.

Note that for the values of $M = 100$ and 500 considered for the partition size, there are just small differences between the $\widehat{\text{CP}}$ curves for the resample sizes G studied in each discounted cost model considered. This suggests that for this study the resample size G does not affect the behavior of the actual coverage probabilities of the proposed bands. Nevertheless when dealing with fine partitions $M = 1000$, the actual coverage probabilities increase because the larger number of resamples. Figure 2 shows that the differences in actual coverage probabilities achieved by the two discount functions decrease when the number of systems increases, reaching values close to the nominal level of $(1-\gamma)$ used in the simulation.

4.4. Actual Coverage Probabilities for the 2-out-of-4 Components System

Figure 3 shows the effect of the resample size G over the actual coverage probabilities under different sample sizes, for both discount functions.

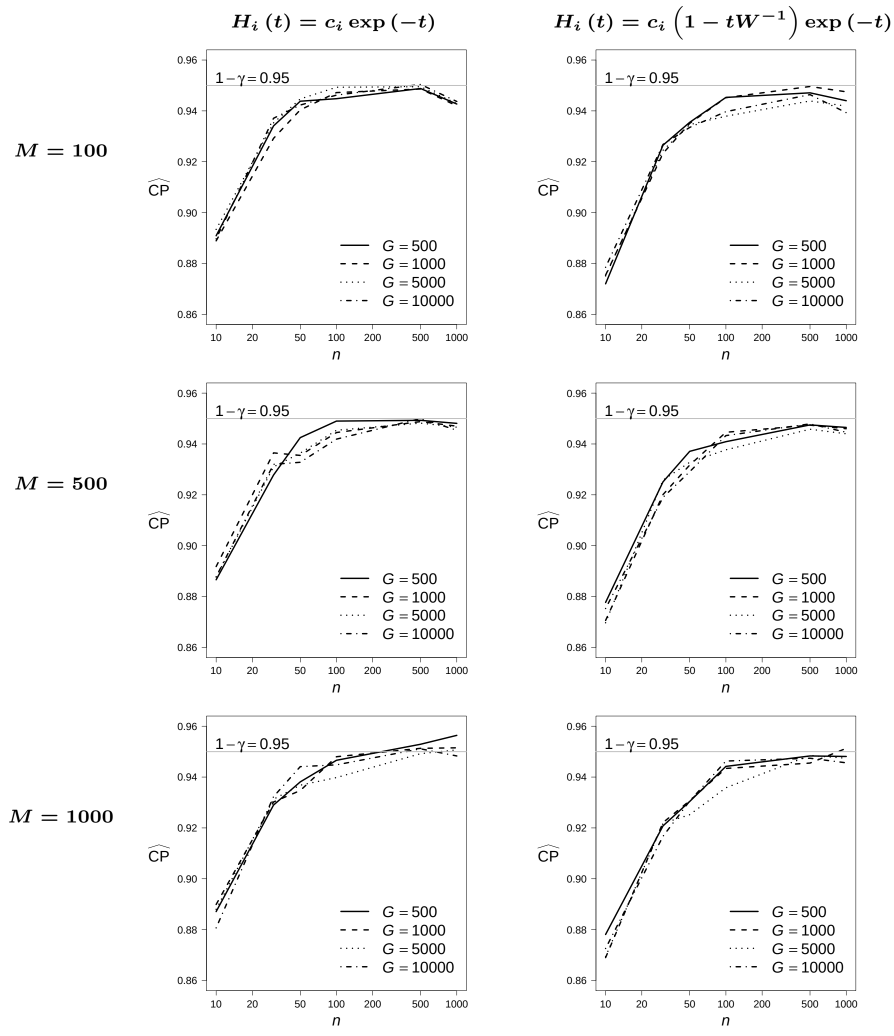

FIGURE 3: $\widehat{\text{CP}}$ for $\text{SCB}_t^{(n)}$, for the 1-out-of-3 components system, varying G for both discount functions.

Figure 3 shows that, similar to, the 1-out-of-3 system, when the number of systems under warranty increases, the actual coverage probabilities of the simultaneous confidence bands increase toward the confidence level $(1-\gamma)$. At each level of the partition size M, the smaller coverage probabilities suggest an improvement when the resample size G increases. It is worth noting that there are large differences between the coverage probabilities for both discount functions when the number of systems is smaller or equal than 100. But similar values to the nominal $(1-\gamma)$ level are achieved in both discount functions when the number of systems is greater than 100.

4.5. Examples of Simultaneous Confidence Bands

Simultaneous confidence bands are illustrated with simulated data for both the 1-out-of-3 components system and the 4-out-of-2 components system for fixed values: $W = 5$; $\alpha = 0.05$; $c_1 = c_2 = 3$; $c_3 = c_4 = 5$; $F_i(t) = \text{Weibull}(\eta = 1.0,\ \beta = 1.5)$, $i = 1, 2$; $F_i(t) = \text{Weibull}(\eta = 2.0,\ \beta = 2.0)$, $i = 3, 4$; $M = 1000$ and $R = 10000$.

Figures 4 and 5 show the precision changes within the simultaneous confidence bands when the number of systems n under warranty varies, for the 1-out-of-3 and 2-out-of-4 components systems, respectively.

Notice that for both systems in the study, the number of systems under warranty is a key factor in the precision of the proposed simultaneous confidence bands. Observe that the bands are narrow when $n \geq 500$. Few systems under warranty ($n \leq 100$) generate high variability of the estimation of the expected discounted warranty cost as reflected in wider simultaneous confidence bands.

5. Conclusions and Future Research

For some statistical models, assessment of the precision of the statistical inferences may be carried out by means of intensive computer methods (Efron 1979, Davison & Hinkley 1997, Belyaev 2007). Resampling was efficiently used in this work to obtain simultaneous confidence bands, for the expected discounted warranty cost under physical minimal repair. This is a useful tool to assess the precision of the estimator, avoiding the complications of the asymptotic analysis of the related stochastic processes. For instance, it avoids finding the specific distributions for those processes. Besides, it is worth noting that the technique used to estimate the warranty costs, and the subsequent computation of the simultaneous confidence bands is non-parametric, and therefore no distributional assumption about the failure/repair processes is required by the proposed method.

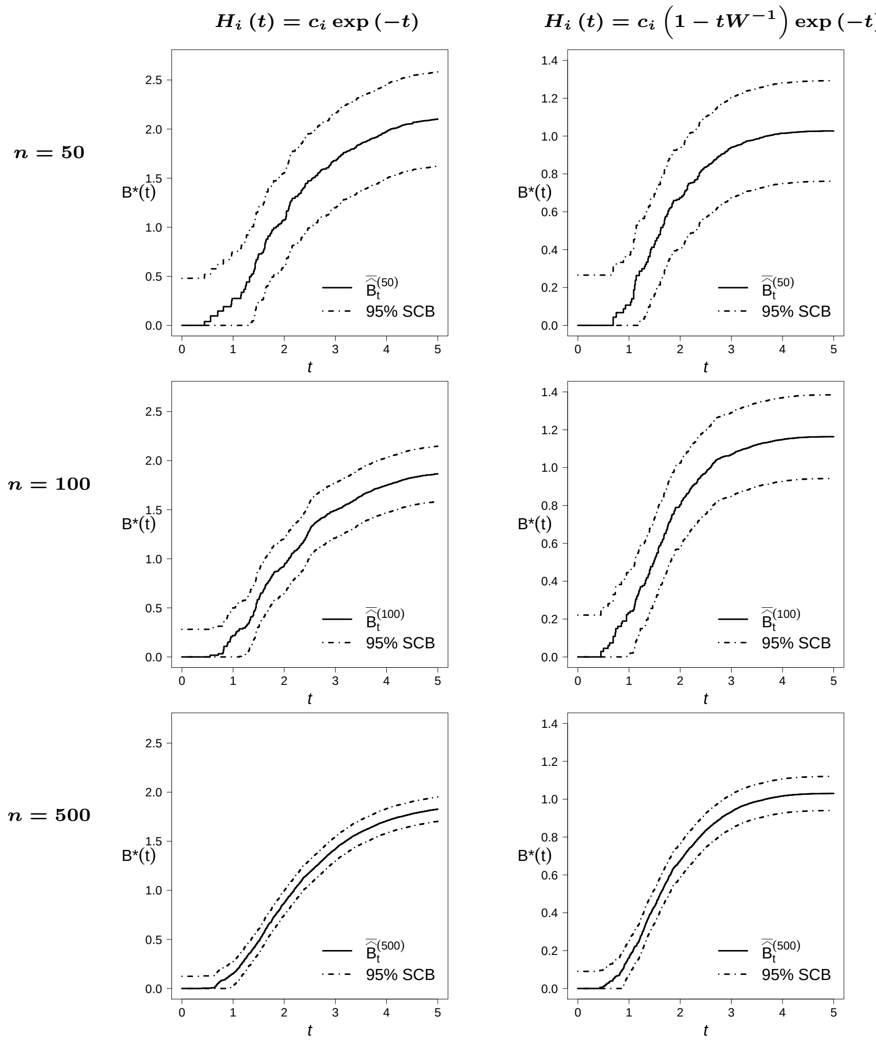

FIGURE 4: Simultaneous confidence bands of $(1-\gamma) = 0.95$ level for the 1-out-of-3 components system.

The proposed computation of the simultaneous confidence bands is valid in a wide range of models that satisfy with the general conditions identified in Section 2.

The proposed computation of the simultaneous confidence bands for the expected discounted warranty cost of coherent systems under minimal repair is easy to implement in current statistical package, since it only involves random sampling with replacement. Also, in the simulation scenarios studied in Section 4, reasonable actual coverage probabilities were obtained, particularly when it there was a number of systems under warranty greater than 100, a fine partition of the warranty term and a large number of resamples.

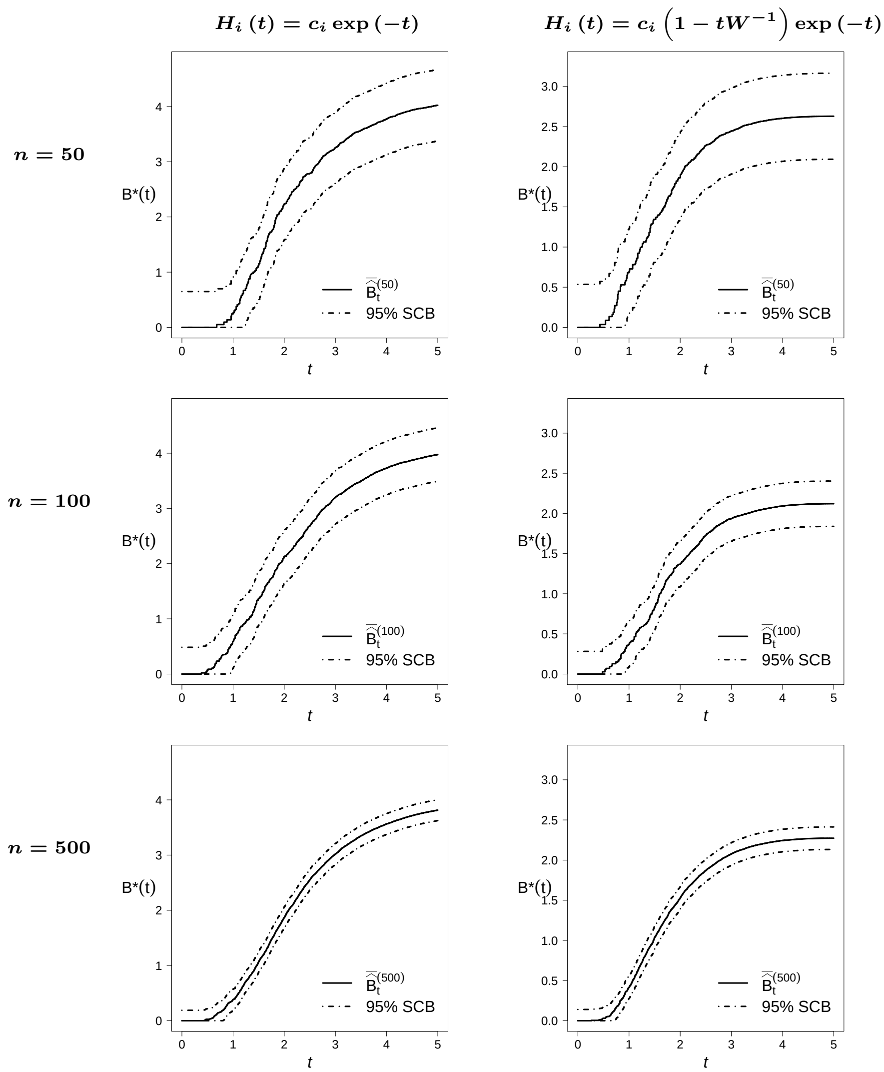

Figure 5: Simultaneous confidence bands of $(1-\gamma) = 0.95$ level for the 2-out-of-4 components system.

This paper has focused on the minimal repair model and on the warranty policy in which repair cost is shared by both the customer and the manufacturer (PRW policy). In the literature there are references to other types of repair besides minimal repair, such as: perfect repair, in which the product is as good as new after repair, and imperfect repair, in which the product is better than used, but not as good as new. Both types of repair make use of well-known renewal processes (Park & Pham 2010, Su & Shen 2012). In addition, it is possible to consider Free Replacement Policies (FRW) or combination of FRW and PRW policies

(Chien 2008, Chien 2010). Generalizing the results described here into previously described situations is considered relevant.

[Received: September 2016 — Accepted: November 2017]

References

Ascher, H. (1968), 'Evaluation of repairable system reliability using the "bad-as-old" concept', *IEEE Transactions on Reliability* **R-17**(2), 103–110.

Aven, T. (1983), 'Optimal replacement under a minimal repair strategy: A general failure model', *Advances in Applied Probability* **15**(1), 198–211.

Aven, T. & Castro, I. (2008), 'A minimal repair replacement model with two types of failure and a safety constraint', *European Journal of Operational Research* **188**(2), 506–515.

Aven, T. & Jensen, U. (1999), *Stochastic models in reliability*, 1 edn, Springer-Verlag Inc, New York.

Bai, J. & Pham, H. (2004), 'Discounted warranty cost of minimally repaired series systems', *IEEE Transactions on Reliability* **53**(1), 37–42.

Bai, J. & Pham, H. (2006), 'Cost analysis on renewable full-service warranties for multi-component systems', *European Journal of Operation Research* **168**(2), 492–508.

Balachandran, K., Maschmeyer, R. & Livingstone, J. (1981), 'Product warranty period: A markovian approach to estimation and analysis of repair and replacement costs', *The Accounting Review* **56**(1), 115–124.

Belyaev, Y. (2000), Computer intensive methods based on resampling in analysis of reliability and survival data, *in* 'In Recent advances in reliability theory: methodology, practice and inference', Birkhäuser, Boston.

Belyaev, Y. (2007), Resampling for lifetime distributions, *in* 'In Encyclopedia of Statistics: Quality and Reliability', Wiley, Chichester.

Belyaev, Y. & Seleznjev, O. (2000), 'Approaching in distribution with applications to resampling of stochastic processes', *Scandinavian Journal of Statistics* **27**(2), 371–384.

Blischke, W. & Murthy, D. (1994), *Warranty cost analysis*, 1 edn, Marcel Dekker Inc, New York.

Block, H., Borges, W. & Savits, T. (1985), 'Age-dependent minimal repair', *Journal of Applied Probability* **22**(2), 370–385.

Chien, Y. (2005), 'Determining optimal warranty periods form the seller's perspective and optimal out-of-warranty replacement age from the buyer's perspective', *International Journal of Systems Science* **36**(10), 631–637.

Chien, Y. (2008), 'A general age-replacement model with minimal repair under renewing free-replacement warranty', *European Journal of Operation Research* **186**(3), 1046–1058.

Chien, Y. (2010), 'Optimal age for preventive replacement under a combined fully renewable free replacement with a pro-rata warranty', *International Journal of Production Economics* **124**(1), 198–205.

Chukova, S. & Dimitrov, B. (1996), *Warranty analysis for complex systems, In Product warranty handbook*, 1 edn, Marcel Dekker Inc, New York.

Davison, A. & Hinkley, D. (1997), *Bootstrap methods and Their Applications*, 1 edn, Cambridge University Press, Cambridge.

Duchesne, T. & Marri, F. (2009), 'General distributional properties of discounted warranty costs with risk adjustment under minimal repair', *IEEE Transactions on Reliability* **58**(1), 143–151.

Efron, B. (1979), 'Bootstrap methods: another look at the jacknife', *The Annals of Statistics* **7**(1), 1–26.

Fleming, T. & Harrington, D. (1991), *Counting processes and survival analysis*, 1 edn, John Wiley and Sons, Inc, New York.

González, N. & Bueno, V. (2011), 'Estimating the discounted warranty cost of a minimally repaired coherent system', *Revista Colombiana de Estadística* **34**(3), 513–543.

Herman, J., Kučera, R. & Šimša, J. (2000), *Equations and Inequalities*, 1 edn, Springer-Verlag, Inc., New York.

Hussain, A. & Murthy, D. (1998), 'Warranty and redundancy design with uncertain quality', *IIE Transactions* **30**(12), 1191–1199.

Ja, S., Kulkarni, V., Mitra, A. & Patankar, J. (2001), 'A nonrenewable minimal-repair warranty policy with time-dependent costs', *IEEE Transactions on Reliability* **50**(4), 346–352.

Ja, S., Kulkarni, V., Mitra, A. & Patankar, J. (2002), 'Warranty reserves for nonstationary sales processes', *Naval Research Logistics* **49**(5), 499–513.

Jain, M. & Maheshwari, S. (2006), 'Discounted costs for repairable units under hybrid warranty', *Applied Mathematics and Computation* **173**(2), 887–901.

Jung, K., Park, M. & Park, D. (2010), 'System Maintenance Cost Dependent on Life Cycle under Renewing Warranty Policy', *Reliability Engineering and System Safety* **95**(7), 816–821.

Knowles, M. (1988), Simultaneous confidence bands for random functions, Technical Report 113, SIAM Institute for Mathematics and Society.

Lopera, C. (2014), Bandas de confianza simultáneas para la estimación del costo medio de garantía descontado para un sistema coherentes bajo reparo mínimo, Tesis de doctorado en ciencias, Universidad Nacional de Colombia, Escuela de Estadística, Medellín, Colombia.

Loève, M. (1977), *Probability theory, volumes I and II*, 4 edn, Springer-Verlag Inc, New York.

Murthy, D. & Djamaludina, I. (2002), 'New product warranty: a literature review', *International Journal of Production Economics* **79**(3), 231–260.

Nguyen, D. & Murthy, D. (1984), 'Cost analysis of warranty policies', *Naval Research Logistics* **31**(4), 525–541.

Ouali, M., Tadj, L., Yacout, S. & Ait-Kadi, D. (2011), A survey of replacement models with minimal repair, *in* 'Replacement Models with Minimal Repair', Springer-Verlag Inc, London.

Park, M. & Pham, H. (2010), 'Altered quasi-renewal concepts for modeling renewable warranty costs with imperfect repairs', *Mathematical and Computer Modelling* **52**(9-10), 1435–1450.

Patankar, J. & Mitra, A. (1995), 'Effect of warranty execution on warranty reserve costs', *Management Science* **41**(3), 395–400.

Riordan, J. (1937), 'Moment Recurrence Relations for Binomial, Poisson and Hypergeometric Frequency Distributions', *The Annals of Mathematical Statistics* **8**(2), 103–111.

Ritchken, P. & Fuh, D. (1986), 'Optimal replacement policies for irreparable warranty item', *IEEE Transactions on Reliability* **35**(5), 621–624.

Sheu, S., Griffith, W. & Nakagawa, T. (1995), 'Extended optimal replacement model with random minimal repair costs', *European Journal of Operational Research* **85**(3), 636–649.

Su, C. & Shen, J. (2012), 'Analysis of extended warranty policies with different repair options', *Engineering Failure Analysis* **25**, 49–62.

Thomas, M. (1989), 'A prediction model for manufacturer warranty reserves', *Management Science* **35**(12), 1515–1519.

Thomas, M. (2005), 'Engineering economic decisions and warranties', *The Engineering Economist* **50**(4), 307–326.

Appendix A. Properties of Discounted Warranty Cost Processes

Consider the mean cost function for minimal repair associated to a realization (j) of the process $\widehat{B}_t^{(j)}$ given in (11) and to the mean cost function for minimal repair $B^*(t)$ given in (9). Define the process of deviations between the mean cost and the estimated cost for minimal repairs of the system as

$$\widehat{B}_t^{(j)} - B^*(t). \tag{A1}$$

Note that $B^*(t)$ does not correspond to the compensator of the Martingale estimator $\widehat{B}_t^{(j)}$, observed on the j-th copy of the process, since that role is played by the process $B_t^{(j)}$ given in (12). Then the deviation process in (A1) can be decomposed as

$$\widehat{B}_t^{(j)} - B^*(t) \quad = \quad R_t^{(j)} + D_t^{(j)}, \tag{A2}$$

where,

$$R_t^{(j)} = \widehat{B}_t^{(j)} - B_t^{(j)} \tag{A3}$$

is a martingale process observed on the j-th copy of the cost process with predictable variation process

$$\langle R \rangle_t = \langle \widehat{B} - B \rangle_t = \sum_{i=1}^{m} C^i(\omega) \int_0^t H_i^2(s)\, I(Y_i < s)\, \lambda^i(s)\, ds.$$

and

$$D_t^{(j)} = B_t^{(j)} - B^*(t) \tag{A4}$$

is a continuous P-a.s. process obtained as the difference between (12) and (9).

The following are some features of the processes in (A3)-(A4) (see Lopera, 2014 for details).

Proposition A.1. *Let $H(t)$ be a non-negative, bounded and continuous function, in $[a,b]$ with $0 \leq a \leq b \leq t < \infty$. Let $\lambda(t)$ be a non-negative function representing a failure rate that is likely to be associated to an increasing failure rate (IFR) distribution, or to a decreasing failure rate (DFR) distribution, or with one mixing both patterns, for instance a bathtub-shape failure rate. Assume that $\int_a^b H(u)\lambda(u)\,du < \infty$, that is $f(t) = H(t)\lambda(t)$ is Riemann integrable in $[a,b]$. Then, there is $\kappa \geq 0$ such that*

$$\int_a^b H(u)\,\lambda(u)\,du \leq \kappa\,|b-a|. \tag{A5}$$

Proposition A.2. *Let $R_t^{(j)}$ be defined as in (A3) and let t_1, t_2 be such that $0 \leq t_1 \leq t_2 \leq t < \infty$. Then,*

a) It holds that

$$E\left[\left(R_{t_2}^{(j)} - R_{t_1}^{(j)}\right)^2 \middle| \mathcal{F}_{t_1}\right] = E\left[\langle R^{(j)}\rangle_{t_2} - \langle R^{(j)}\rangle_{t_1} \middle| \mathcal{F}_{t_1}\right], \tag{A6}$$

and therefore,

$$E\left[\left(R_{t_2}^{(j)} - R_{t_1}^{(j)}\right)^2\right] = E\left[\langle R^{(j)}\rangle_{t_2} - \langle R^{(j)}\rangle_{t_1}\right] \tag{A7}$$

b) There is $L_1 > 0$, such that

$$E\left[\left(R_{t_2}^{(j)} - R_{t_1}^{(j)}\right)^2\right] \leq L_1 \left|t_2 - t_1\right|. \tag{A8}$$

Proposition A.3. *Let $D_t^{(j)}$ be defined as in (A4), the j-th realization of process D_t within interval $[0, t]$. Let t_1, t_2 be two times such that $0 \leq t_1 \leq t_2 \leq t < \infty$. Then, there is $L_2 > 0$, such that*

$$\left(D_{t_2}^{(j)} - D_{t_1}^{(j)}\right)^2 \leq L_2 \left|t_2 - t_1\right|^2 \ \ P\text{-a.s.} \tag{A9}$$

Consequently

$$E\left(D_{t_2}^{(j)} - D_{t_1}^{(j)}\right)^2 \leq L_2 \left|t_2 - t_1\right|^2. \tag{A10}$$

Proposition A.4. *Let $R_t^{(j)}$ be defined as in (A3). Let s, t_1, t_2 be times such that $0 \leq t_1 \leq s \leq t_2 \leq t < \infty$. Then, there is $L_3 > 0$, such that*

$$E\left[\left(R_{t_2}^{(j)} - R_{s}^{(j)}\right)^2 \left(R_{s}^{(j)} - R_{t_1}^{(j)}\right)^2\right] \leq L_3 \left|t_2 - t_1\right|^2. \tag{A11}$$

Appendix B. Proof of Theorem 5

Using (33), $U_{jn}(t)$ can be written as

$$U_{jn}(t) \quad = \quad \frac{1}{\sqrt{n}}\left(R_t^{(j)} + D_t^{(j)}\right), \tag{B1}$$

where $R_t^{(j)}$ and $D_t^{(j)}$ are defined in (A3) and (A4), respectively.

Proof of condition C-1

Consider $\delta = 1$. We want to show that

$$n\, E\left[U_{jn}(t_2) - U_{jn}(t_1)\right]^2 \leq c_1 \left|t_2 - t_1\right|. \tag{B2}$$

Using (B1)

$$\begin{aligned} n\left[U_{jn}(t_2) - U_{jn}(t_1)\right]^2 &= \left[\left(R_{t_2}^{(j)} - R_{t_1}^{(j)}\right) + \left(D_{t_2}^{(j)} - D_{t_1}^{(j)}\right)\right]^2 \\ &\leq 2\left(R_{t_2}^{(j)} - R_{t_1}^{(j)}\right)^2 + 2\left(D_{t_2}^{(j)} - D_{t_1}^{(j)}\right)^2 \quad P\text{-a.s.}, \end{aligned} \tag{B3}$$

Then, using expectations on both sides of (B3)

$$n\, E\left[U_{jn}(t_2) - U_{jn}(t_1)\right]^2 \leq 2E\left[\left(R_{t_2}^{(j)} - R_{t_1}^{(j)}\right)^2\right] + 2E\left[\left(D_{t_2}^{(j)} - D_{t_1}^{(j)}\right)^2\right]. \tag{B4}$$

Applying Propositions A.2 and A.3 is found that

$$\begin{aligned} n\, E\left[\left(U_{jn}\left(t_2\right)-U_{jn}\left(t_1\right)\right)^2\right] &\leq 2L_1\left|t_2-t_1\right|+2L_2\left|t_2-t_1\right|^2 \\ &= 2\left(L_1+L_2\left|t_2-t_1\right|\right)\left|t_2-t_1\right| \\ &\leq 2\left(L_1+L_2\kappa_3\right)\left|t_2-t_1\right| \\ &= c_1\left|t_2-t_1\right| \text{ with } c_1=2\left(L_1+L_2\kappa_3\right), \end{aligned}$$

for some $\kappa_3>0$ such that $|t_2-t_1|\leq\kappa_3$.

Proof of condition C-2

Consider $\delta=1$. We want to show that

$$n^2\, E\left[\left(U_{jn}\left(t_2\right)-U_{jn}\left(s\right)\right)^2\left(U_{jn}\left(s\right)-U_{jn}\left(t_1\right)\right)^2\right]\leq c_2\left|t_2-t_1\right|^2. \tag{B5}$$

Using (B3)

$$n\left[U_{jn}\left(t_2\right)-U_{jn}\left(s\right)\right]^2\leq 2\left(R_{t_2}^{(j)}-R_s^{(j)}\right)^2+2\left(D_{t_2}^{(j)}-D_s^{(j)}\right)^2 \quad P\text{-a.s.}$$

and

$$n\left[U_{jn}\left(s\right)-U_{jn}\left(t_1\right)\right]^2\leq 2\left(R_s^{(j)}-R_{t_1}^{(j)}\right)^2+2\left(D_s^{(j)}-D_{t_1}^{(j)}\right)^2 \quad P\text{-a.s.}$$

Consequently,

$$\begin{aligned} &n^2\left[U_{jn}\left(t_2\right)-U_{jn}\left(s\right)\right]^2\left[U_{jn}\left(s\right)-U_{jn}\left(t_1\right)\right]^2 \\ &\leq 4\left[\left(R_{t_2}^{(j)}-R_s^{(j)}\right)^2+\left(D_{t_2}^{(j)}-D_s^{(j)}\right)^2\right]\left[\left(R_s^{(j)}-R_{t_1}^{(j)}\right)^2+\left(D_s^{(j)}-D_{t_1}^{(j)}\right)^2\right] \\ &= 4\left[\left(R_{t_2}^{(j)}-R_s^{(j)}\right)^2\left(R_s^{(j)}-R_{t_1}^{(j)}\right)^2+\left(R_{t_2}^{(j)}-R_s^{(j)}\right)^2\left(D_s^{(j)}-D_{t_1}^{(j)}\right)^2\right. \\ &\quad\left.+\left(D_{t_2}^{(j)}-D_s^{(j)}\right)^2\left(R_s^{(j)}-R_{t_1}^{(j)}\right)^2+\left(D_{t_2}^{(j)}-D_s^{(j)}\right)^2\left(D_s^{(j)}-D_{t_1}^{(j)}\right)^2\right] \quad P\text{-a.s.} \end{aligned} \tag{B6}$$

Using result (A9) in (B6)

$$\begin{aligned} &n^2\left[U_{jn}\left(t_2\right)-U_{jn}\left(s\right)\right]^2\left[U_{jn}\left(s\right)-U_{jn}\left(t_1\right)\right]^2 \\ &\leq 4\left[\left(R_{t_2}^{(j)}-R_s^{(j)}\right)^2\left(R_s^{(j)}-R_{t_1}^{(j)}\right)^2+\left(R_{t_2}^{(j)}-R_s^{(j)}\right)^2 4m^2\kappa_1'^2\left|s-t_1\right|^2\right. \\ &\quad\left.+4m^2\kappa_1''^2\left|t_2-s\right|^2\left(R_s^{(j)}-R_{t_1}^{(j)}\right)^2+16m^4\kappa_1'^2\kappa_1''^2\left|t_2-s\right|^2\left|s-t_1\right|^2\right] \quad P\text{-a.s.} \end{aligned}$$

Thus

$$\begin{aligned} &n^2\, E\left[U_{jn}\left(t_2\right)-U_{jn}\left(s\right)\right]^2\left[U_{jn}\left(s\right)-U_{jn}\left(t_1\right)\right]^2 \\ &\leq 4\left\{E\left[\left(R_{t_2}^{(j)}-R_s^{(j)}\right)^2\left(R_s^{(j)}-R_{t_1}^{(j)}\right)^2\right]+4m^2\kappa_1'^2\left|s-t_1\right|^2 E\left[\left(R_{t_2}^{(j)}-R_s^{(j)}\right)^2\right]\right. \\ &\quad\left.+4m^2\kappa_1''^2\left|t_2-s\right|^2 E\left[\left(R_s^{(j)}-R_{t_1}^{(j)}\right)^2\right]+16m^4\kappa_1'^2\kappa_1''^2\left|t_2-s\right|^2\left|s-t_1\right|^2\right\}. \end{aligned} \tag{B7}$$

Applying Proposition A.4 to the first expectation on the right hand side of inequality (B7) and Proposition A.2 to the other two expectations, it follows that

$$
\begin{aligned}
& n^2 E\left[\left(U_{jn}(t_2)-U_{jn}(s)\right)^2\left(U_{jn}(s)-U_{jn}(t_1)\right)^2\right] \\
& \leq 4\left[L_3\left|t_2-t_1\right|^2+4m^2\kappa_1'^2\left|s-t_1\right|^2 L_1''\left|t_2-s\right|\right. \\
& \quad \left.+4m^2\kappa_1''^2\left|t_2-s\right|^2 L_1'\left|s-t_1\right|+16m^4\kappa_1'^2\kappa_1''^2\left|s-t_1\right|^2\left|t_2-s\right|^2\right] \\
& \leq 4\left[L_3\left|t_2-t_1\right|^2+4m^2\kappa_1'^2 L_1''\left|t_2-t_1\right|^3+4m^2\kappa_1''^2 L_1'\left|t_2-t_1\right|^3+16m^4\kappa_1'^2\kappa_1''^2\left|t_2-t_1\right|^4\right] \\
& =4\left[L_3+4m^2\kappa_1'^2 L_1''\left|t_2-t_1\right|+4m^2\kappa_1''^2 L_1'\left|t_2-t_1\right|+16m^4\kappa_1'^2\kappa_1''^2\left|t_2-t_1\right|^2\right]\left|t_2-t_1\right|^2 \\
& \leq 4\left[L_3+4m^2\kappa_1'^2 L_1''\kappa_3+4m^2\kappa_1''^2 L_1'\kappa_3+16m^4\kappa_1'^2\kappa_1''^2\kappa_3^2\right]\left|t_2-t_1\right|^2 \\
& =c_2\left|t_2-t_1\right|^2, \text{ with } c_2=4\left[L_3+4m^2\kappa_1'^2 L_1''\kappa_3+4m^2\kappa_1''^2 L_1'\kappa_3+16m^4\kappa_1'^2\kappa_1''^2\kappa_3^2\right],
\end{aligned}
$$

where the second inequality holds because for every $0 \leq t_1 \leq s \leq t_2 \leq t < \infty$, $|t_2 - s| \leq |t_2 - t_1|$, $|s - t_1| \leq |t_2 - t_1|$, and the fourth inequality is given because there is a $\kappa_3 > 0$ such that $|t_2 - t_1| \leq \kappa_3$.

Proof of condition C-3

Consider $\delta = 1$. We want to show that

$$E\left[\left|\sqrt{n}\, U_{jn}(s)\right|^3\right] \leq c_3. \tag{B8}$$

Using (25), the triangular inequality, and that $g(x) = x^3$ is increasing when $x \geq 0$, it can shown that

$$\left|\sqrt{n}\, U_{jn}(s)\right|^3 \leq \left(\widehat{B}_s^{(j)} + B^*(s)\right)^3 \;\; P\text{-a.s.} \tag{B9}$$

Using inequality C_r (Loève 1977), it follows that $\left|\sqrt{n}\, U_{jn}(s)\right|^3 \leq 4\left(\widehat{B}_s^{(j)}\right)^3 + 4\left(B^*(s)\right)^3$ P-a.s. Thus

$$E\left[\left|\sqrt{n}\, U_{jn}(s)\right|^3\right] \leq 4E\left[\left(\widehat{B}_s^{(j)}\right)^3\right] + 4E\left[\left(B^*(s)\right)^3\right]. \tag{B10}$$

Now, from (9) and (10) it follows that

$$B^*(s) = \sum_{i=1}^{m} B^{i*}(s) = \sum_{i=1}^{m} P\left(S_i > Y_i\right) E\left[\int_0^s I\left(Y_i < u\right) H_i(u)\, \lambda^i(u)\, du \,\middle|\, S_i > Y_i\right],$$

Note that for every $i = 1, \ldots, m$,

$$\left[\int_0^s I\left(Y_i < u\right) H_i(u)\, \lambda^i(u)\, du \,\middle|\, S_i > Y_i\right] \leq \int_0^s H_i(u)\, \lambda^i(u)\, du \;\; P\text{-a.s.}$$

Thus $E\left[\int_0^s I\left(Y_i < u\right) H_i\left(u\right) \lambda^i\left(u\right) du \middle| S_i > Y_i\right] \leq \int_0^s H_i\left(u\right) \lambda^i\left(u\right) du$, which implies

$$\begin{aligned} B^{i*}\left(s\right) &= P\left(S_i > Y_i\right) E\left[\int_0^s I\left(Y_i < u\right) H_i\left(u\right) \lambda^i\left(u\right) du \middle| S_i > Y_i\right] \\ &\leq \int_0^s H_i\left(u\right) \lambda^i\left(u\right) du. \end{aligned}$$

Consequently, $B^*\left(s\right) = \sum_{i=1}^m B^{i*}\left(s\right) \leq \sum_{i=1}^m \int_0^s H_i\left(u\right) \lambda^i\left(u\right) du$.

Using Proposition A.1 on each integral on the right hand side of the previous inequality, it is observed that there is $\kappa_1 > 0$ such that for every $i = 1, \ldots, m$ it holds that $\int_0^s H_i\left(s\right) \lambda^i\left(s\right) ds \leq \kappa_1 s \leq \kappa_1 \zeta_1$ P-a.s., for some $\zeta_1 > 0$ such that for all $0 \leq s \leq t < \infty$, and $s \leq \zeta_1$. Consequently

$$B^*\left(s\right) = \sum_{i=1}^m \kappa_1 \zeta_1 = m \kappa_1 \zeta_1. \tag{B11}$$

On the other hand,

$$\widehat{B}_s^{(j)} = \sum_{i=1}^m C^{\,i(j)} \widehat{B}_s^{\,i(j)} = \sum_{i=1}^m C^{\,i(j)} \sum_{l=1}^{\widetilde{N}_s^{\,i(j)}} H_i\left(S_{il}^{(j)}\right). \tag{B12}$$

Because $H_i\left(s\right)$ with $s \in [0, t]$ are non-negative, bounded and continuous functions in $[0, t]$, there is $0 < \upsilon < \infty$ such that $H_i\left(s\right) \leq \upsilon$ for every $i = 1, \ldots, m$ and since $g\left(x\right) = x^3$ with $x \geq 0$, is an increasing function, from (B12), it follows that

$$\left(\widehat{B}_s^{(j)}\right)^3 \leq \upsilon^3 \left(\sum_{i=1}^m C^{\,i(j)} \widetilde{N}_s^{\,i(j)}\right)^3 \;\; P\text{-a.s.} \tag{B13}$$

In addition using inequality $\left(\sum_{k=1}^m d_k\right)^3 \leq m^2 \left(\sum_{k=1}^m d_k^3\right)$ (Herman, Kučera & Šimša 2000),

$$\left(\widehat{B}_s^{(j)}\right)^3 \leq \upsilon^3 m^2 \sum_{i=1}^m C^{\,i(j)} \left(\widetilde{N}_s^{\,i(j)}\right)^3 \;\; P\text{-a.s.}$$

Applying the expected value it gives

$$E\left[\left(\widehat{B}_s^{(j)}\right)^3\right] \leq= \upsilon^3 m^2 \sum_{i=1}^m P\left(S_i^{(j)} > Y_i^{(j)}\right) E\left[\left(\widetilde{N}_s^{\,i(j)}\right)^3 \middle| S_i^{(j)} > Y_i^{(j)}\right]. \tag{B14}$$

Now, recall that for every $i \in \mathcal{C}^\Phi$, $\widetilde{N}_s^{\,i(j)} \geq 0$, while for $i \notin \mathcal{C}^\Phi$, $\widetilde{N}_s^{\,i(j)} = 0$ P-a.s.. On the other hand, $\widetilde{N}_s^i\left(\omega\right)$ restricted to $\mathcal{C}^i = \{\omega \in \Omega : S_i\left(\omega\right) > Y_i\left(\omega\right)\}$ is a Non-Homogeneous Poisson Process (NHPP). Therefore, for each realization (j), it can be shown that $\left[\widetilde{N}_s^{\,i(j)} \middle| S_i^{(j)} > Y_i^{(j)}\right]$ satisfies

$$E\left[\left(\widetilde{N}_s^{\,i(j)}\right)^3 \middle| S_i^{(j)} > Y_i^{(j)}\right] = \sum_{x=0}^3 \mathcal{S}_{x,3} \left(\int_0^s \lambda^i\left(u\right) du\right)^x, \tag{B15}$$

where $\mathcal{S}_{x,3}$ are the Stirling numbers (Riordan 1937) and $\int_0^s \lambda^i(u)\,du < \infty$ for each $i = 1, \ldots, m$. Then, by applying Proposition A.1 it follows that there is $\kappa_4 > 0$ such that $\int_0^s \lambda^i(u)\,du \leq \kappa_4 s \leq \kappa_4\zeta_2$ for some $\zeta_2 > 0$ with $s \leq \zeta_2$. Then (B15) gives

$$E\left[\left(\widetilde{N}_s^{i(j)}\right)^3 \middle| S_i^{(j)} > Y_i^{(j)}\right] \leq \sum_{x=0}^{3} \mathcal{S}_{x,3}\left(\kappa_4\zeta_2\right)^x. \tag{B16}$$

Note that, in (B16) the sum on the right side of the inequality is a non-negative finite number and therefore there is $\kappa_5 > 0$ such that for each $i = 1, \ldots, m$, it holds that

$$E\left[\left(\widetilde{N}_s^{i(j)}\right)^3 \middle| S_i^{(j)} > Y_i^{(j)}\right] \leq \kappa_5,$$

consequently, for each $i = 1, \ldots, m$

$$P\left(S_i^{(j)} > Y_i^{\,,(j)}\right) E\left[\left(\widetilde{N}_s^{i(j)}\right)^3 \middle| S_i^{(j)} > Y_i^{(j)}\right] \leq \kappa_5. \tag{B17}$$

From (B17) in (B14), it follows that

$$E\left[\left(\widehat{B}_s^{(j)}\right)^3\right] \leq \upsilon^3 m^2 \sum_{i=1}^{m} \kappa_5 = \upsilon^3 m^3 \kappa_5. \tag{B18}$$

Finally, applying (B11) and (B18) in (B10), it gives $E\left[\left|\sqrt{n}\,U_{jn}(s)\right|^3\right] \leq c_3$, with $c_3 = 4\upsilon^3 m^3 \kappa_5 + 4m\kappa_1\zeta_1$.

Appendix C. Results of Simulation Study

Some results from the simulation study described in Section 4 are showed now.

Table C1: Actual Coverage Probabilities for the 1-out-of-3 Components System with $H_i(t) = c_i \exp(-t)$.

M	G	$n = 10$	$n = 30$	$n = 50$	$n = 100$	$n = 500$	$n = 1000$
100	500	0.8754	0.9216	0.9368	0.9428	0.9464	0.9458
	1000	0.8785	0.9280	0.9353	0.9425	0.9553	0.9514
	5000	0.8759	0.9279	0.9326	0.9430	0.9512	0.9470
	10000	0.8728	0.9241	0.9369	0.9424	0.9467	0.9489
500	500	0.8712	0.9257	0.9323	0.9442	0.9488	0.9516
	1000	0.8704	0.9233	0.9388	0.9399	0.9504	0.9508
	5000	0.8703	0.9229	0.9338	0.9440	0.9447	0.9484
	10000	0.8658	0.9179	0.9356	0.9458	0.9502	0.9443
1000	500	0.8719	0.9240	0.9378	0.9445	0.9489	0.9508
	1000	0.8749	0.9196	0.9336	0.9424	0.9517	0.9492
	5000	0.8742	0.9180	0.9348	0.9422	0.9486	0.9504
	10000	0.8719	0.9188	0.9302	0.9436	0.9494	0.9496

Table C2: Actual Coverage Probabilities for the 1-out-of-3 Components System with $H_i\left(t\right) = c_i\left(1 - tW^{-1}\right)\exp\left(-t\right)$.

M	G	$n = 10$	$n = 30$	$n = 50$	$n = 100$	$n = 500$	$n = 1000$
100	500	0.8604	0.9096	0.9313	0.9408	0.9487	0.9502
	1000	0.8567	0.9121	0.9299	0.9414	0.9526	0.9455
	5000	0.8483	0.9132	0.9295	0.9402	0.9495	0.9436
	10000	0.8582	0.9147	0.9302	0.9399	0.9476	0.9498
500	500	0.8540	0.9084	0.9240	0.9423	0.9471	0.9491
	1000	0.8467	0.9117	0.9239	0.9357	0.9451	0.9457
	5000	0.8456	0.9100	0.9230	0.9319	0.9466	0.9455
	10000	0.8443	0.9085	0.9246	0.9374	0.9451	0.9452
1000	500	0.8466	0.9087	0.9242	0.9372	0.9438	0.9488
	1000	0.8462	0.9073	0.9372	0.9445	0.9502	0.9507
	5000	0.8435	0.9200	0.9352	0.9398	0.9451	0.9487
	10000	0.8684	0.9181	0.9342	0.9409	0.9489	0.9439

Table C3: Actual Coverage Probabilities for the 2-out-of-4 Components System with $H_i\left(t\right) = c_i\exp\left(-t\right)$.

M	G	$n = 10$	$n = 30$	$n = 50$	$n = 100$	$n = 500$	$n = 1000$
100	500	0.8909	0.9341	0.9438	0.9448	0.9488	0.9427
	1000	0.8889	0.9292	0.9404	0.9472	0.9486	0.9420
	5000	0.8934	0.9353	0.9447	0.9493	0.9495	0.9429
	10000	0.8897	0.9371	0.9423	0.9462	0.9504	0.9437
500	500	0.8866	0.9278	0.9425	0.9490	0.9493	0.9481
	1000	0.8918	0.9365	0.9355	0.9445	0.9488	0.9470
	5000	0.8877	0.9310	0.9364	0.9454	0.9482	0.9466
	10000	0.8878	0.9320	0.9328	0.9419	0.9500	0.9454
1000	500	0.8871	0.9290	0.9381	0.9466	0.9529	0.9564
	1000	0.8899	0.9302	0.9347	0.9480	0.9513	0.9515
	5000	0.8880	0.9313	0.9368	0.9398	0.9492	0.9507
	10000	0.8807	0.9323	0.9441	0.9448	0.9511	0.9483

Table C4: Actual Coverage Probabilities for the 2-out-of-4 Components System with $H_i\left(t\right) = c_i\left(1 - tW^{-1}\right)\exp\left(-t\right)$.

M	G	$n = 10$	$n = 30$	$n = 50$	$n = 100$	$n = 500$	$n = 1000$
100	500	0.8720	0.9265	0.9355	0.9453	0.9471	0.9440
	1000	0.8754	0.9233	0.9351	0.9452	0.9496	0.9475
	5000	0.8751	0.9245	0.9346	0.9379	0.9439	0.9419
	10000	0.8785	0.9268	0.9335	0.9397	0.9464	0.9393
500	500	0.8777	0.9251	0.9371	0.9409	0.9475	0.9465
	1000	0.8706	0.9201	0.9317	0.9446	0.9474	0.9460
	5000	0.8697	0.9258	0.9328	0.9377	0.9458	0.9440
	10000	0.8753	0.9190	0.9291	0.9433	0.9479	0.9446
1000	500	0.8781	0.9208	0.9304	0.9442	0.9483	0.9481
	1000	0.8690	0.9221	0.9307	0.9434	0.9455	0.9513
	5000	0.8694	0.9218	0.9252	0.9358	0.9485	0.9474
	10000	0.8726	0.9167	0.9306	0.9463	0.9474	0.9456