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A Bivariate Model based on Compound Negative Binomial Distribution

Un modelo basado en bivariadas compuesto distribución binomial negativa

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Abstract

A new bivariate model is introduced by compounding negative binomial and geometric distributions. Distributional properties, including joint, marginal and conditional distributions are discussed. Expressions for the product moments, covariance and correlation coefficient are obtained. Some properties such as ordering, unimodality, monotonicity and self-decomposability are studied. Parameter estimators using the method of moments and maximum likelihood are derived. Applications to traffic accidents data are illustrated.

***Key words*:** Bivariate distribution; Compound distribution; Correlation coefficient; Divisibility; Geometric distribution; Moments; Negative binomial distribution;Total positivity.

Resumen

Un nuevo modelo de dos variables se introduce mediante la composición distribuciones binomiales negativos y geométricos. propiedades distributivas, incluyendo distribuciones conjuntas, marginales y condicionales se discuten. se obtienen las expresiones para los momentos de productos, la covarianza y el coeficiente de correlación. Se estudian algunas propiedades tales como pedidos, unimodalidad, monotonía y la auto-decomposability. estimadores de parámetros utilizando el método de los momentos y de máxima verosimilitud se derivan. Aplicaciones a los datos de accidentes de tráfico se ilustran.

***Palabras clave*:** coeficiente de correlación; distribución binomial negativa; distribución bivariada; distribución compuesto; distribución geométrica; divisibilidad; momentos; positividad total.

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1. Introduction

Let Y_1 be a negative binomial random variable with parameters $0 < p_1 < 1$, $r > 0$, and probability mass function (pmf)

$$f_{Y_1}(y_1) = \binom{y_1 + r - 1}{y_1} p_1^r (1 - p_1)^{y_1}, y_1 = 0, 1, \dots, \tag{1}$$

and let $W_i, i = 1, 2, \dots$ be independent identically distributed (i.i.d.) non-negative, integer-valued random variables distributed as Q-distribution, independent of Y_1. The random sum $Y_2 = \sum_{i=0}^{Y_1} W_i$ has a compound negative binomial distribution (CQNB) with compounding distribution Q, where $W_0 = 0$ with probability 1.

The univariate compound negative binomial models arise naturally in insurance and actuarial sciences and were studied by several authors (see Drekic & Willmot (2005)). Panjer & Willmot (1981) studied compound negative binomial with exponential distribution. Subrahmaniam (1966) derived the Pascal-Poisson distribution (compound negative binomial with Poisson distributions) as a limiting case of a more general contagious distribution (see Johnson, Kemp & Kotz (2005)). Subrahmaniam (1978) investigated the parameters estimates for the Pascal-Poisson distribution by method of moments and maximum likelihood procedures. Jewell & Milidiu (1986) suggested three methods to approximate the evaluation of the compound Pascal distribution where the compounding distribution is defined on both negative and positive integers. Ramsay (2009) derived expression for the cumulative distribution function of compound negative binomial where the compounding distribution is Pareto distribution. Wang (2011) presented recursion on the pdf of compound beta negative binomial distribution. Willmot & Lin (1997) constructed upper bound for the tail of the compound negative binomial distribution. Cai & Garrido (2000) derived two sided-bounds for tails of compound negative binomial distributions. Vellaisamy & Upadhye (2009b) studied convolutions of compound negative binomial distributions. Gerber (1984), dhaene (1991), Vellaisamy & Upadhye (2009a) and Upadhye & Vellaisamy (2014) considered the problem of approximating a compound negative binomial distribution by a compound Poisson distribution. Hanagal & Dabade (2013) introduced compound negative binomial frailty model with three baseline distributions.

Joint modeling of the bivariate random vector (Y_1, Y_2) has been studied by several authors. A variety of bivariate models such as Poisson-Bernoulli, Poisson-Poisson and Poisson-Geometric are discussed by Leiter & Hamdan (1973), Cacoullos & Papageorgiou (1980), Papageorgiou (1985) and Papageorgiou (1995). Cacoullos & Papageorgiou (1982) introduced and studied a three parameter bivariate discrete distribution, which they called the negative binomial-Poisson, to analyze traffic accidents. Papageorgiou & Loukas (1988) derived maximum likelihood estimators for the parameters of the bivariate negative binomial-Poisson distribution. Recently, Alzaid, Almuhayfith & Omair (2017) obtained some general forms for density, cumulative distribution, moments, cumulants and correlation coefficient of (Y_1, Y_2), when Y_1 has a Poisson distribution, and different assumptions for the compounding distribution, namely Poisson, binomial and negative binomial distributions, denoted by BPPM, BBPM and BNBPM, respectively. Özel (2011*b*) proposed a bivariate compound Poisson distribution and introduced bivariate versions of the Neyman Type A, Neyman type B, geometric-Poisson and Thomas distributions. Earthquake data was used to illustrate the application of these distributions. Özel (2011*a*) defined a bivariate compound Poisson distribution to model the occurences of forshock and aftershock sequences in Turkey.

In this paper, we study the random vector (Y_1, Y_2) where $Y_1 \sim NB(r, p_1)$ and $W_i \sim geo(p_2)$. We refer to this distribution as BGNBD, which stands for bivariate geometric-negative binomial distribution. The BGNBD distribution can be used as appropriate model for many problems of social, income and physical nature. For instance, the number of purchased order and the number of total soled items per day, the total number of insurance claimed and the number of claimants per unit time, the total number of injury accidents and number of fatalities and the number of visits and number of drugs prescribed.

Our paper is organized as follows. In Section 2, the bivariate geometric-negative binomial distribution is derived and distributional properties are discussed. Parameter estimators of BGNBD are derived using the methods of moment and maximum likelihood in Section 3. Applications on real data sets are presented in Section 4 to illustrate the BGNBD. Finally, some conclusions are drawn in Section 5.

2. Bivariate Geometric Negative Binomial Distribution

Definition 1. A random vector (Y_1, Y_2) with the stochastic representation $(Y_1, Y_2) =^d (Y_1, \sum_{i=0}^{Y_1} W_i)$ where Y_1 is a negative binomial variable given in (1) and the W_i's are i.i.d. geometric variables (p_2), independent of the Y_1, is said to have a bivariate geometric-negative binomial distribution with parameters r,p_1 and p_2. This distribution is denoted by BGNBD(r, p_1, p_2).

The random variable Y_2 is distributed according to the compound geometric-negative binomial distribution (CGNB) with parameters r, p_1 and p_2, denoted by $CGNB(r, p_1, p_2)$.

2.1. General Properties of Compound Geometric-Negative Binomial Distribution

- *The probability mass function*

  By using conditional argument on Y_1, it is easy to show that the pmf of $Y_2 \sim CGNB(r, p_1, p_2)$ is given by

$$f_{Y_2}(y_2) = rp_2q_1p_1^r q_2^{y_2}\,{}_2F_1(y_2 + 1, r + 1; 2; p_2q_1), y_2 = 0, 1, \ldots, \qquad (2)$$

  where ${}_2F_1$ is the Gaussian hypergeometric function (see Abramowitz & Stegun (1972), chapter 15). Recurrence for the pmf in (2) can be derived using the recurrence relation of the Gaussian hypergeometric function,

$$(c-a)_2F_1(a-1,b;c;z)+(2a-c-(a-b)z)_2F_1(a,b;c;z)+ \\ a(z-1)_2F_1(a+1,b;c;z)=0.$$

Thus, we have:

$$f_{Y_2}(0)=[\frac{p_1}{1-p_2q_1}]^r,$$

$$f_{Y_2}(1)=\frac{rp_1^rp_2q_2q_1}{(1-p_2q_1)^{r+1}}$$

and, $\forall y_2 \geq 2$

$$f_{Y_2}(y_2+1)= \\ \frac{q_2}{(y_2+1)(p_2q_1-1)}\{(y_2(p_2q_1-2)-rp_2q_1)f_{Y_2}(y_2)+q_2(y_2-1)f_{Y_2}(y_2-1)\}. \quad (3)$$

- *Moments properties*

  Using properties of compound distribution, the moment generating function (mgf) can be derived as

$$M_{Y_2}(t)=[\frac{p_1}{1-q_1M_W(t)}]^r=[\frac{p_1(1-q_2e^t)}{1-q_2e^t-p_2q_1}]^r, \quad (4)$$

  The mean and variance of CGNBD are obtained as follows:

$$E(Y_2)=r\frac{q_1}{p_1}E(W)=r\frac{q_1q_2}{p_1p_2}, \quad (5)$$

$$Var(Y_2)=r\frac{q_1}{p_1}Var(W)+r\frac{q_1}{p_1^2}E^2(W)=r\frac{q_1q_2(p_1+q_2)}{p_1^2p_2^2}. \quad (6)$$

  The Skewness of $Y_2 \sim CGNB(r,p_1,p_2)$ is given by

$$Skew(Y_2)=E(\frac{Y_2-r\frac{q_1}{p_1}E(W)}{\sigma_{Y_2}})^3=\frac{1}{\sigma_{Y_2^{\frac{3}{2}}}}[3r\frac{q_1^2}{p_1^2}\mu_1'\mu_2'+2r\frac{q_1^3}{p_1^3}\mu_1'^3+r\frac{q_1}{p_1}\mu_3'], \quad (7)$$

  where $\mu_i', i=1,2,3$ are the first three moments about zero of W. As r and p_1 and the moments of W are positive, it follows that the compound geometric negative binomial distribution is positively skewed.

Proposition 1. *If $r=1$ then CGNBD random variable has the representation $Y_2=IU$ where $I \sim Bernoulli(p_1)$ independent of $U \sim Geo(\frac{p_1p_2}{1-p_2q_1})$ i.e. Y_2 has zero inflated geometric distribution.*

Proof. From equation (4), the pgf of Y_2 when $r = 1$ is

$$G_{Y_2}(t) = \frac{p_1(1-q_2t)}{1-q_2t-p_2q_1} = \frac{p_1(1-q_2t-p_2q_1)+p_1p_2q_1}{1-q_2t-p_2q_1} = p_1 + q_1\frac{\frac{p_1p_2}{1-p_2q_1}}{1-\frac{q_2}{1-p_2q_1}t}.$$

which is a mixture of degenerate distribution at 1 with probability p_1 and $Geo(\frac{p_1p_2}{1-p_2q_1})$ with probability q_1. Hence, the proof is complete. □

Since the negative binomial distribution can be represented as a compound Poisson distribution with logarithmic compounding distribution. Then, the compound negative binomial distribution is a compound Poisson distribution with a compound logarithmic distribution as the compounding distribution. This is stated in the following proposition.

Proposition 2. *The compound negative binomial distribution with parameters r and p_1, and compounding distribution with pgf G_W, can be regarded as compound Poisson distribution with mean $\lambda = -r\log p_1$ and compounding distribution with pgf of the form*

$$G_{W^*}(t) = \frac{\log(1-q_1G_W(t))}{\log p_1}.$$

- *Monotonicity Properties.*

Proposition 3. *The pmf of $Y_2 \sim CGNB(r, p_1, p_2)$ is log-concave for $r > 1$ and log-convex for $r < 1$.*

Proof. The relation

$$f^2_{Y_2}(y_2+1) \geq f_{Y_2}(y_2)f_{Y_2}(y_2+2)$$

is equivalent to

$${}_2F_1(y_2+1, r+1; 2; p_2q_1)^2 \geq {}_2F_1(y_2, r+1; 2; p_2q_1)\,{}_2F_1(y_2+2, r+1; 2; p_2q_1),$$

Using the fact that ${}_2F_1(a, b; c; x)$ is log-concave in a for $0 < x < 1$, $b > c > 0$ and log-convex in a for $-\infty < x < 1$, $c > b > 0$ (Theorem 6 and 7 of Karp & Sitnik 2010), we get the result. □

Note that the log-concavity is equivalent to strongly unimodal, and it implies that the distribution is unimodal and has increasing hazard (failure) rate.

- *Divisibility and Self-decomposability*
Useful theorems from Steutel & van Harn (2004) regarding the representation of infinitely divisible and self-decomposable for distributions on the set of nonnegative integers are quoted here. The results of these theorems enable us to prove the self-decomposability of the compound negative binomial distribution.

Theorem 1 (Theorem 3.2 of Steutel & van Harn (2004), Chapter II, Section 3)**.** *A pgf G is infinitely divisible iff it is compound Poisson, i.e., if it has the form*

$$G(t) = e^{-\lambda(1-Q(t))}$$

with $\lambda > 0$ and Q a pgf with $Q(0) = 0$.

Theorem 2 (4.13 of Steutel & van Harn (2004), Chapter V, Section 4)**.** *Let $(p_n)_0^\infty$ and $(r_n)_0^\infty$ be sequences of real numbers with $p_n \geq 0$, $p_0 > 0$, and let p_n and r_n be related by*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} r_{n-k}p_k, n = 0, 1, 2, \ldots,$$

where the r's satisfy $r_n \geq 0$, and necessarily $\sum_{n=0}^{\infty} \frac{r_n}{n+1} < \infty$. Then $(p_n)_0^\infty$ is self-decomposable iff it is infinitely divisible and has a canonical sequence r_n that is non-increasing.

- *Remark.*

Note that the probability generating function of the compound negative binomial distribution is given by

$$\begin{aligned}
G(t) &= (\frac{p_1}{1-q_1G_W(t)})^r \\
&= e^{-r\log\frac{p_1}{1-q_1G_W(t)}} \\
&= e^{-r\log p_1(1-\frac{\log(1-q_1G_W(t))}{\log p_1})} \\
&= e^{-r\log p_1(1-G_{W^*}(t))} \\
&= e^{-\lambda(1-Q(t))}.
\end{aligned}$$

Therefore by Theorem 1 and Proposition 2, the compound negative binomial distribution is infinitely divisible.

Proposition 4. *The compound negative binomial distribution has canonical sequence representation of the form*

$$r_k = r(k+1)\sum_{i=1}^{\infty} \frac{q_1^i}{i} f_W^{*i}(k+1). \tag{8}$$

***Proof*.** From Proposition 2, the compound negative binomial distribution can be regarded as compound Poisson distribution with $\lambda = -r\log p_1$ and compounding distribution with pgf of the form

$$G_{W^*}(t) = \frac{\log(1-q_1G_W(t))}{\log p_1} = -\frac{1}{\log p_1}\sum_{i=1}^{\infty}\frac{q_1^i}{i}G_W^i(t).$$

Since f_W^{*i} is the probability mass function of G_W^i, we get

$$f_{W^*}(k+1) = -\frac{1}{\log p_1} \sum_{i=1}^{\infty} \frac{q_1^i}{i} f_W^{*i}(k+1). \tag{9}$$

It is easily seen that the canonical representation of the compound Poisson distribution is given by

$$r_k = \lambda(k+1) f_W(k+1). \tag{10}$$

Substituting $\lambda = -r \log p_1$ and (9) in (10), we get the relation (8). □

Corollary 1. *The compound negative binomial distribution is self-decomposable iff the canonical sequence in (8) is non-increasing in k.*

***Proof*.** Follows directly from Theorem 2. □

Example 1. In case of compound geometric-negative binomial distribution, we have

$$\begin{aligned} r_k &= r(k+1) \sum_{i=1}^{\infty} \frac{q_1^i}{i} f_W^{*i}(k+1) \\ &= r(k+1) \sum_{i=1}^{\infty} \frac{q_1^i}{i} \binom{k+i}{k+1} p_2^i q_2^{k+1} \\ &= r\left(\frac{q_2}{1-q_1 p_2}\right)^{k+1}, \end{aligned}$$

which is non-increasing function. Hence, the compound geometric-negative binomial distribution is self-decomposable.

Definition 2. If X_1 and X_2 are two rv's with pmf's $f_1(x)$ and $f_2(x)$, respectively. Then X_1 is less than X_2 in likelihood ratio order (denoted by $X_1 \leq_{lr} X_2$) if $\frac{f_2(x)}{f_1(x)}$ is increasing in x.

Proposition 5. *Let $\{W_i : i = 1, 2, \ldots\}$ be sequence of independent geo(p_2) random variables, and let $Y_1 \sim NB(r, p_1)$ and $Y_1^* \sim NB(r^*, p_1^*)$ be two random variables which are independent of the W_i's. Then*

$$\sum_{i=1}^{Y_1} W_i \leq_{lr} \sum_{i=1}^{Y_1^*} W_i$$

if and only if $r \geq r^*$ and $r(1-p_1) \leq r^*(1-p_1^*)$.

***Proof*.** The result follows from application of Theorem 1.C.11 of Shaked & Shanthikumar (2007), and likelihood ordering of negative binomial distribution and log-concavity of geometric distribution. □

2.2. Basic Properties of $(Y_1, Y_2) \sim BGNBD(r, p_1, p_2)$

Using conditional argument on Y_1 we can obtain the followings;

- The joint pmf of (Y_1, Y_2) is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \binom{y_1 + r - 1}{y_1}\binom{y_1 + y_2 - 1}{y_2} p_1^r (1-p_1)^{y_1} p_2^{y_1} (1-p_2)^{y_2}, \quad (11)$$

$y_1 \geq 1, y_2 = 0, 1, \ldots$, $f_{Y_1,Y_2}(0, y_2) = 0$ for $y_2 = 1, 2, \ldots$, and $f_{Y_1,Y_2}(0, 0) = p_1^r$.

- The Moment generating function of (Y_1, Y_2) is

$$M_{Y_1,Y_2}(u, v) = [\frac{p_1}{1 - q_1 e^u M_W(v)}]^r; M_W(v) = \frac{p_2}{1 - q_2 e^v}. \quad (12)$$

- Covariance structure of $(Y_1, Y_2) \sim BGNBD(r, p_1, p_2)$.

  The covariance matrix of (Y_1, Y_2) takes the form

$$\begin{bmatrix} r\frac{(1-p_1)}{p_1^2} & r\frac{(1-p_1)(1-p_2)}{p_1^2 p_2^2} \\ r\frac{(1-p_1)(1-p_2)}{p_1^2 p_2^2} & r\frac{(1-p_1)(1-p_2)(p_1+q_2)}{p_1^2 p_2^2} \end{bmatrix} \quad (13)$$

  and the correlation coefficient of Y_1 and Y_2 is

$$\begin{aligned} Corr(Y_1, Y_2) = E(W)\sqrt{\frac{Var(Y_1)}{Var(Y_2)}} &= \sqrt{\frac{1}{1 + \frac{C.V^2(W)}{E(Y_1)C.V^2(Y_1)}}} \\ &= \sqrt{\frac{1 - p_2}{1 - p_2 + p_1}}. \end{aligned} \quad (14)$$

  where $C.V(W)$ denotes the coefficient of variation of W. It is interesting to note that the correlation does not depend on r. This gives more flexibilty in modeling as one can let the mean and the variance varies without affecting the correlation. Also, One can see that the correlation coefficient is a decreasing function of p_1 and p_2 and assumes only positive values. Obviously, the correlation is bounded by 0 and 1, where the lower bound is attained if $p_2 = 0$ and the upper bound is attained when $p_2 = 1$ which correspond to the trivial cases $Y_2 = 0$ and $Y_1 = Y_2$, respectively.

- Product moments and joint cumulants.

  The (r, s)-th product moment of $(Y_1, Y_2) \sim BGNBD(r, p_1, p_2)$ are given by

$$\begin{aligned}
\mu'_{1,1} &= r\frac{1-p_1}{p_1^2}(1+r(1-p_1))E(W),\\
\mu'_{2,1} &= r\frac{1-p_1}{p_1^3}(1+q_1(1+3r+r^2q_1^2))E(W),\\
\mu'_{1,2} &= r\frac{1-p_1}{p_1^3}[(1-q_1)(1+rq_1)E(W^2)+q_1(r+1)(2+rq_1)E^2(W)],\\
\mu'_{2,2} &= r\frac{1-p_1}{p_1^4}[p_1(1+q_1(1+3r+r^2q_1))E(W^2)\\
&\quad + q_1(r+3+r^2q_1(2+q_1+rq_1)+(3r+1)(1+2q_1+rq_1))E^2(W)].
\end{aligned}$$

and the three first cumulants of $(Y_1, Y_2) \sim BGNBD(r, p_1, p_2)$ are as follows

$$\begin{aligned}
k_{1,1} &= r\frac{1-p_1}{p_1^2}E(W),\\
k_{1,2} &= r\frac{1-p_1}{p_1^3}[p_1E(W^2)+2q_1E^2(W)],\\
k_{2,1} &= r\frac{1-p_1}{p_1^3}(1+q_1)E(W),\\
k_{2,2} &= r\frac{1-p_1}{p_1^4}[(1-q_1^2)E(W^2)+2q_1(2+q_1)E^2(W)],\\
k_{1,3} &= r\frac{1-p_1}{p_1^4}[p_1^2E(W^3)+6p_1q_1E^2(W)E(W^2)+6q_1^2E^3(W)],\\
k_{3,1} &= r\frac{1-p_1}{p_1^4}(1+2q_1)^2E(W).
\end{aligned}$$

- Conditional distribution and regression functions

  1. It is obvious that the conditional distribution of Y_2 given Y_1 is a negative binomial random variable with parameters y_1 and p_1. Thus

     $$E(Y_2|Y_1=y_1) = y_1\frac{1-p_2}{p_2}, \tag{15}$$

     which is a linear in y_1 with regression coefficient $\frac{1-p_2}{p_2}$. As the coefficient is non-negative we have the conditional mean of Y_2 increases with the increase in y_1. Also the conditional variance is

     $$Var(Y_2|Y_1=y_1) = y_1\frac{1-p_2}{p_2^2}, \tag{16}$$

     which has similar properties as the conditional mean.
  2. The conditional pmf of Y_1 given $Y_2 = y_2$ has the form

     $$f_{Y_1|Y_2}(Y_1|Y_2=y_2) = \frac{\theta^{y_1}}{r\theta}\frac{\binom{y_1+r-1}{y_1}\binom{y_1+y_2-1}{y_2}}{{}_2F_1(y_2+1,r+1;2;\theta)}, \tag{17}$$

$\theta = p_2 q_1, y_1 = 0, 1, \ldots$

As a direct consequence of (17), we have the pgf of the conditional distribution as

$$G_{Y_1|Y_2=y_2}(t|y_2) = t\frac{{}_2F_1(y_2+1, r+1; 2; \theta t)}{{}_2F_1(y_2+1, r+1; 2; \theta)}, \tag{18}$$

i.e. the conditional distribution of Y_1 given $Y_2 = y_2$ is shifted (by 1) generalized hypergeometric probability distribution (GHPD) (see Kemp 1968). The first and second derivative of the pgf in (18) yield the following results:

$$E(Y_1|Y_2 = y_2) =$$
$$1 + \frac{(r+1)(y_2+1)}{2} p_2 q_1 \frac{{}_2F_1(y_2+2, r+2; 3; \theta)}{{}_2F_1(y_2+1, r+1; 2; \theta)},$$

$$\begin{aligned} E(Y_1^2|Y_2 = y_2) &= 1 + 3\frac{(r+1)(y_2+1)}{2} p_2 q_1 \frac{{}_2F_1(y_2+2, r+2; 3; \theta)}{{}_2F_1(y_2+1, r+1; 2; \theta)} \\ &+ \frac{(r+1)(r+2)(y_2+1)(y_2+2)}{6}(p_2 q_1)^2 \frac{{}_2F_1(y_2+3, r+3; 4; \theta)}{{}_2F_1(y_2+1, r+1; 2; \theta)}. \end{aligned}$$

The following proposition gives the distribution of the random sum $S = Y_1 + Y_2$.

Proposition 6. *The random variable $S = Y_1 + Y_2$ has compound negative binomial distribution with shifted geometric distribution.*

Proof.

$$\begin{aligned} G_S(t) &= E(e^{tS}) = E(t^{Y_1+Y_2}) = E(E(t^{Y_1+\sum_{i=1}^{Y_1} W_i})|Y_1) \\ &= E(t^{Y_1} E(t^{\sum_{i=1}^{Y_1} W_i})|Y_1) = E((tG_W(t))^{Y_1}) \\ &= G_{Y_1}(tG_W(t)) = [\frac{p_1}{1 - q_1 t G_W(t)}]^r = [\frac{p_1}{1 - q_1 \frac{tp_2}{1-q_2 t}}]^r. \end{aligned}$$

and, the proof is complete. □

- Convolutions of BGNBD

Proposition 7. *Let $(Y_{1i}, Y_{2i}) =^d (Y_{1i}, \sum_{i=0}^{Y_{1i}} W_i)$ be mutually independent BGNBD for $i = 1, 2 \cdots, n$, Y_{1i} is a negative binomial random variable with parameters r_i,p_1, and W_i's are iid random variables distributed as geometric with parameter p_2, and independent of the Y_{1i}'s, then the distribution of $\sum_{i=1}^{n}(Y_{1i}, Y_{2i})$ is BGNBD with parameters $r = \sum_{i=1}^{n} r_i$,p_1 and p_2.*

Proof. The mgf of the random vector (Y_{1i}, Y_{2i}) is given by

$$M_{Y_{1i},Y_{2i}}(u, v) = [\frac{p_1}{1 - q_1 e^u M_W(v)}]^{r_i}.$$

Then, the mgf of the sum of the n random vectors (Y_{1i}, Y_{2i}) is

$$\begin{aligned} E(e^{t\sum_{i=1}^{n}(Y_{1i},Y_{2i})}) &= \prod_{i=1}^{n} E(e^{t(Y_{1i},Y_{2i})}) \\ &= \prod_{i=1}^{n} [\frac{p_1}{1-q_1e^uM_W(v)}]^{r_i} \\ &= [\frac{p_1}{1-q_1e^uM_W(v)}]^{\sum_{i=1}^{n} r_i} \end{aligned}$$

which is the mgf of BGNBD with parameters $r = \sum_{i=1}^{n} r_i, p_1$ and p_2. □

Example 2. Let $(Y_{11}, Y_{21}) \sim BGNBD(r_1, p_1, p_2)$ independent of $(Y_{12}, Y_{22}) \sim BGNBD(r_2, p_1, p_2)$. Then according to Proposition 7, we have $(Y_{11} + Y_{12}, Y_{21} + Y_{22}) \sim BGNBD(r_1 + r_2, p_1, p_2)$. Hence, the conditional distribution is given by

$$\begin{aligned} &Pr(Y_{11} = y_1, Y_{21} = y_2 | Y_{11} + Y_{12} = z_1, Y_{21} + Y_{22} = z_2) \\ &= \frac{f_{Y_1,Y_2}(y_1, y_2; r_1, p_1, p_2) f_{Z_1-Y_1,Z_2-Y_2}(z_1 - y_1, z_2 - y_2; r_2, p_1, p_2)}{f_{Z_1,Z_2}(z_1, z_2; r_1 + r_2, p_1, p_2)} \\ &= \frac{\binom{y_1+r_1-1}{y_1}\binom{z_1-y_1+r_2-1}{z_1-y_1}}{\binom{z_1+r_1+r_2-1}{z_1}} \frac{\binom{y_1+y_2-1}{y_2}\binom{z_1+z_2-y_1-y_2-1}{z_2-y_2}}{\binom{z_1+z_2-1}{z_2}}. \end{aligned}$$

i.e. the conditional distribution is the product of two negative hypergeometric distribution, $NHG(r_1, z_1, r_1 + r_2)$ and $NHG(y_1, z_2, z_1)$.

- Limiting Distribution.

Since the negative binomial distribution with parameters r and p_1 converges to the Poisson distribution with parameter $\lambda = r(p - 1)$ where $r \to \infty$ and $p_1 \to 1$. Thus, we have the following proposition.

Proposition 8. *Under the limiting conditions $r \to \infty$ and $p_1 \to 1$ such that $r(1 - p_1) = \lambda$, the following relation is true*

$$\lim_{r\to\infty} \lim_{p_1\to 1} M_{Y_1,Y_2}(u, v) = e^{\lambda(e^u M_W(v)-1)},$$

where $e^{\lambda(e^u M_W(v)-1)}$ is the mgf of bivariate Poisson-geometric distribution. Hence, the bivariate geometric-negative binomial distribution converges to that of the bivariate geometric-Poisson distribution

- Monotonicity

Definition 3. A function $p(x, y)$ defined for $x \in X$ and $y \in Y$ is totally positive of order 2 (TP_2) if and only if $p(x, y) \geq 0$ for all $x \in X$, $y \in Y$ and

$$\begin{vmatrix} p(x_1, y_1) & p(x_1, y_2) \\ p(x_2, y_1) & p(x_2, y_2) \end{vmatrix} \geq 0$$

whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

Proposition 9. *If* $(Y_1, Y_2) \sim BGNBD(r, p_1, p_2)$*, then the function* $f_{Y_1,Y_2}(y_1, y_2)$ *defined in (11) is* TP_2*.*

Proof. For $z_1 < z_2$, we have

$$\frac{f_{Y_1,Y_2}(y_1, z_1)}{f_{Y_1,Y_2}(y_1, z_2)} = (1-p_2)^{z_1-z_2} \frac{z_2!(y_1+z_1-1)!}{z_1!(y_1+z_2-1)!}$$
$$= (1-p_2)^{z_1-z_2} \frac{z_2!}{z_1!(y_1+z_2-1)\dots(y_1+z_2-z_1)\dots(y_1+z_1)}.$$

which is decreasing function in y_1, hence $f_{Y_1,Y_2}(y_1, y_2)$ defined in (11) is TP_2. □

The TP_2 is very strong positive dependence between random variables in particular it implies association and positive quadrant dependence and hence a nonnegative covariance (see for example Barlow & Proschan (1975)).

Proposition 10.

i For $r < 1$ *and* $y_2 = 0$*, the joint pmf of BGNBD given in (11) is log-convex in* y_1*, otherwise it is log-concave.*

ii The joint pmf of BGNBD given in (11) is log-concave in y_2*.*

Proof.

i In order to prove that the joint pmf of BGNBD given in (11) is log-concave in y_1, we need to show that $\frac{f_{Y_1,Y_2}(y_1+1,y_2)}{f_{Y_1,Y_2}(y_1,y_2)}$ is decreasing in y_1 for every y_2.
But
$$\frac{f_{Y_1,Y_2}(y_1+1, y_2)}{f_{Y_1,Y_2}(y_1, y_2)} = (1-p_1)p_2(1 + \frac{y_2+r-1}{y_1+1} + \frac{ry_2}{y_1(y_1+1)})$$
Thus, the ratio is decreasing in y_1 for $r \geq 1$. For $r < 1$, we have two cases, the first is that $y_2 = 0$ then the ratio increasing and the second case where $y_2 > 0$ which is clearly decreasing in y_1.

ii The log-concavity of BGNBD in y_2 follows from the fact that the $y_1 - th$ convolution of geometric distribution with parameter p_2 is negative binomial distribution with parameters y_1 and p_2 which is a log-concave.

□

- Stochastic Order

Proposition 11. *Let* $\{W_i : i = 1, 2, \dots\}$ *and* $\{W_i^* : i = 1, 2, \dots\}$ *be two sequences of independent* $geo(p_2)$ *and* $geo(p_2^*)$ *random variables, respectively, such that* $p_2 \geq p_2^*$*. Let* $Y_1 \sim NB(r, p_1)$ *and* $Y_1^* \sim NB(r^*, p_1^*)$ *be two random variables which are independent of* W_i*ś and* W_i^**ś, respectively, where* $r \geq r^*$ *and* $r(1-p_1) \leq r^*(1-p_1^*)$*, then* $(Y_1, \sum_{i=1}^{Y_1} W_i) \leq_{st} (Y_1^*, \sum_{i=1}^{Y_1^*} W_i^*)$*.*

Proof. The result follows from an application of Theorem 6.B.3 of Shaked & Shanthikumar (2007) and Proposition 5. □

3. Estimation

Assume that we have n pairs of observations (y_{1i}, y_{2i}); $i = 1, 2, \ldots, n$ from BGNBD with parameters r,p_1 and p_2.

- Method of moments
  The moment estimates $\hat{p}_{1MM}$,$\hat{p}_{2MM}$ and $\hat{r}_{MM}$ of p_1,p_2 and r are obtained from solving the moments equations. Using the moments

$$E(Y_1) = r\frac{q_1}{p_1},$$
$$Var(Y_1) = r\frac{q_1}{p_1^2},$$
$$E(Y_2) = r\frac{q_1 q_2}{p_1 p_2},$$

we get

$$\hat{p}_{1MM} = \frac{\hat{r}_{MM}}{\hat{r}_{MM} + \bar{y}_1},$$
$$\hat{p}_{2MM} = \frac{\bar{y}_1}{\bar{y}_1 + \bar{y}_2},$$
$$\hat{r}_{MM} = \frac{\bar{y}_1^2}{s_1^2 - \bar{y}_1}.$$

As the value of r is non-negative, then the estimate $\hat{r}_{MM}$ has meaning only when $s_1^2 > \bar{y}_1$.

- Maximum Likelihood
  Maximum likelihood estimates (MLE) for the parameters p_1, p_2 and r can be derived by considering the likelihood function given by

$$L = \prod_{i=1}^{n} \binom{y_{1i} + r - 1}{y_{1i}} \binom{y_{1i} + y_{2i} - 1}{y_{2i}} p_1^r (1 - p_1)^{y_{1i}} p_2^{y_{1i}} (1 - p_2)^{y_{2i}}.$$

Then it can be seen that the MLE satisfy

$$\hat{p}_{1MLE} = \frac{\hat{r}_{MLE}}{\hat{r}_{MLE} + \bar{y}_1},$$
$$\hat{p}_{2MLE} = \frac{\bar{y}_1}{\bar{y}_1 + \bar{y}_2}$$

and

$$\frac{\partial LogL}{\partial r} = \sum_{i=1}^{n} [log(p_1) + \psi(y_{1i} + r) - \psi(r)] = 0, \psi(x) = \frac{d \log \Gamma(x)}{dx}.$$

Note that MLE and MM estimate of p_2 are identical. Under mild regularity condition the maximum likelihood estimator $\hat{\Theta} = (\hat{r}, \hat{p_1}, \hat{p_2})$ for large sample has approximately a multivariate normal distribution $N_3(\Theta, I^{-1}(\Theta))$ where $I(\Theta) = -E(\frac{\partial^2 \log L}{\partial\Theta\partial\grave{\Theta}})$.

In order to obtain the asymptotic variance-covariance matrix of p_1, p_2 and r, we need the second partial derivatives of the log likelihood function. These are given by

$$\frac{\partial^2 LogL}{\partial p_1^2} = -\frac{nr}{p_1^2} - \frac{\sum_{i=1}^n y_{1i}}{(1-p_1)^2},$$
$$\frac{\partial^2 LogL}{\partial p_2^2} = -\frac{\sum_{i=1}^n y_{1i}}{p_2^2} - \frac{\sum_{i=1}^n y_{2i}}{(1-p_2)^2},$$
$$\frac{\partial^2 LogL}{\partial r^2} = \sum_{i=1}^n \psi'(y_{1i}+r) - n\psi'(r),$$
$$\frac{\partial^2 LogL}{\partial p_1 \partial r} = \frac{n}{p_1}.$$

and

$$\frac{\partial^2 LogL}{\partial p_1 \partial p_2} = \frac{\partial^2 LogL}{\partial p_2 \partial r} = 0.$$

Hence,

$$Cov(\hat{p}_2, \hat{r}) = Cov(\hat{p}_1, \hat{p}_2) = 0.$$

4. Numerical Example

For comparison purposes, the BGNBD was fitted to the same sets of accident data used by Leiter & Hamdan (1973) and Cacoullos & Papageorgiou (1980), i.e., the total number of injury accidents recorded during 639 days (in 1969 and 1970) in a 50-mile stretch of highway in eastern Virginia (Y_1), and the corresponding number of fatalities (Y_2) for individual years. We look at the data as three sets of data. The first data is the entire study, the second and third set of data representing the total number of injury accidents in 1969 and 1970, respectively. Descriptive statistics of the considered data are presented in Table 1.

As the estimation criterion holds ($s_1^2 > \bar{y}_1$), hence we considered estimating the parameters using both methods the moments and the maximum likelihood. The results are reported in Table 2. Comparing the MM and MLE of the parameters show that they are quite similar. The estimated variance-covariance matrix of the maximum likelihood estimators are computed for each data set.

$$\Sigma_{entire} = \begin{pmatrix} 7.397 & 0.138 & 0 \\ 0.138 & 0.003 & 0 \\ 0 & 0 & 0.0001 \end{pmatrix},$$

Table 1: Descriptive Statistics for accident data.

Data	Variable	Size	Mean	Variance	Min	Max	Skew	Corr (p-value)
entire study	Y_1	639	0.862	0.984	0	5	1.21	0.205 (0)
	Y_2	639	0.058	0.061	0	2	4.41	
Year 1969	Y_1	349	0.880	1.014	0	5	1.21	0.206 (0)
	Y_2	349	0.066	0.067	0	2	4.00	
Year 1970	Y_1	290	0.841	0.951	0	4	1.20	0.204 (0)
	Y_2	290	0.048	0.053	0	2	5.06	

Table 2: Parameter estimates for BGNBD.

Data	Method	$\hat{p_1}$	$\hat{p_2}$	$\hat{r}$	$\hat{\rho}$	Log-lik	AIC
entire study	MLE	0.873	0.937	5.925	0.259	−919.154	1842.307
	MM	0.876	0.937	6.102	0.259		
year 1969	MLE	0.865	0.930	5.649	0.273	−513.839	1031.677
	MM	0.867	0.930	5.751	0.273		
Year 1970	MLE	0.884	0.946	6.383	0.241	−404.892	813.7831
	MM	0.885	0.946	6.486	0.240		

$$\Sigma_{1969} = \begin{pmatrix} 19.11 & 0.396 & 0 \\ 0.396 & 0.008 & 0 \\ 0 & 0 & 0.0002 \end{pmatrix},$$

$$\Sigma_{1970} = \begin{pmatrix} 11.978 & 0.192 & 0 \\ 0.192 & 0.003 & 0 \\ 0 & 0 & 0.0002 \end{pmatrix}.$$

In order to investigate the performance of the BGNBD, we compared the fitting of this model with the results of fitting the bivariate Poisson-Poisson (BPPD), bivariate binomial-Poisson (BBPD), bivariate geometric-Poisson (BGPD), and bivariate negative binomial-Poisson (BNBPD) distributions to the data (For more information about these distributions, see Alzaid et al. (2017)). The BBPD is fitted assuming different values of the parameter m, the BNBPD assuming different values of the parameter r for the first two data sets, in this case the moments estimates coincide with the maximum likelihood estimates. The fit of each model was measured using the Akaike information criterion AIC, SSE values and chi-square goodness-of-fit criterion, where the SSE is defined by $SSE = \sum_{all y_1, y_2}(observed - expected)^2$. The observed and expected values for the bivariate models along with the log-likelihood, AIC, χ^2 values, degrees of freedom (d.f.), corresponding p-values and SSE are given in Tables 3-5. Figure 1 demonstrates the fitted distributions. The values of χ^2, were computed after the grouping of bolded cells in the table. The results show that the log-likelihood and AIC values of all the bivariate models are essentially the same. Note that the fit of the models BPPD, BBPD, BGPD and BNBPD is much better for the individual years, than it is for the entire 639 days. It is obvious from the χ^2 and SSE values in Table 3 that the models BPPD, BBPD, BGPD and BNBPD could not give a satisfactory fit for the data. The fit by BGNBD yields a smaller χ^2 and SSE values as compared with the other models, which implies that this model fits the data well, this is also reflected by the p-value. Same conclusion is reached from Table 4. The p-values of the models in Table 5, suggest acceptable with the superiority of BGNBD as judged by larger p-value and smaller SSE.

Table 3: Bivariate models fitted to accident data entire study (639 days).

Cell no.	y_1	y_2	Observed	Expected BPPD	Expected BBPD $(m=5)$	Expected BGPD	Expected BNBPD $(r=20)$	Expected BGNBD MM	Expected BGNBD MLE
1	0	0	286	269.78	269.78	269.78	269.78	285.25	285.68
2	1	0	198	217.52	217.42	217.99	217.55	201.95	201.51
3	2	0	82	87.69	87.61	88.07	87.71	83.2	83.06
4	3	0	24	23.57	23.54	23.72	23.58	26.07	26.12
5	4	0	13	4.75	4.74	4.79	4.75	6.88	6.94
6	5	0	1	0.77	0.76	0.77	0.77	1.61	1.64
7	1	1	17	14.61	14.8	13.72	14.56	12.71	12.68
8	2	1	10	11.78	11.93	11.08	11.74	10.47	10.45
9	3	1	5	4.75	4.81	4.48	4.73	4.92	4.93
10	4	1	1	1.28	1.29	1.21	1.27	1.73	1.75
11	5	1	0	0.26	0.26	0.24	0.26	0.51	0.52
12	1	2	1	0.49	0.4	0.86	0.51	0.8	0.8
13	2	2	0	0.79	0.73	1.05	0.81	0.99	0.99
14	3	2	1	0.48	0.46	0.56	0.48	0.62	0.62
15	4	2	0	0.17	0.17	0.19	0.17	0.27	0.27
16	5	2	0	0.04	0.04	0.05	0.04	0.1	0.1
			Log-like	-921.753	-921.795	-921.987	-921.749	-	-919.154
			AIC	1847.505	1847.59	1847.974	1847.498	-	1842.307
			χ^2-value	16.896	16.863	17.284	16.907	5.984	6.088
			p-value	0.0097	0.0098	0.0083	0.0096	0.4249	0.4134
			d.f.	6	6	6	6	6	6
			SSE	755.05	750.16	780.69	756.36	80.38	76.15

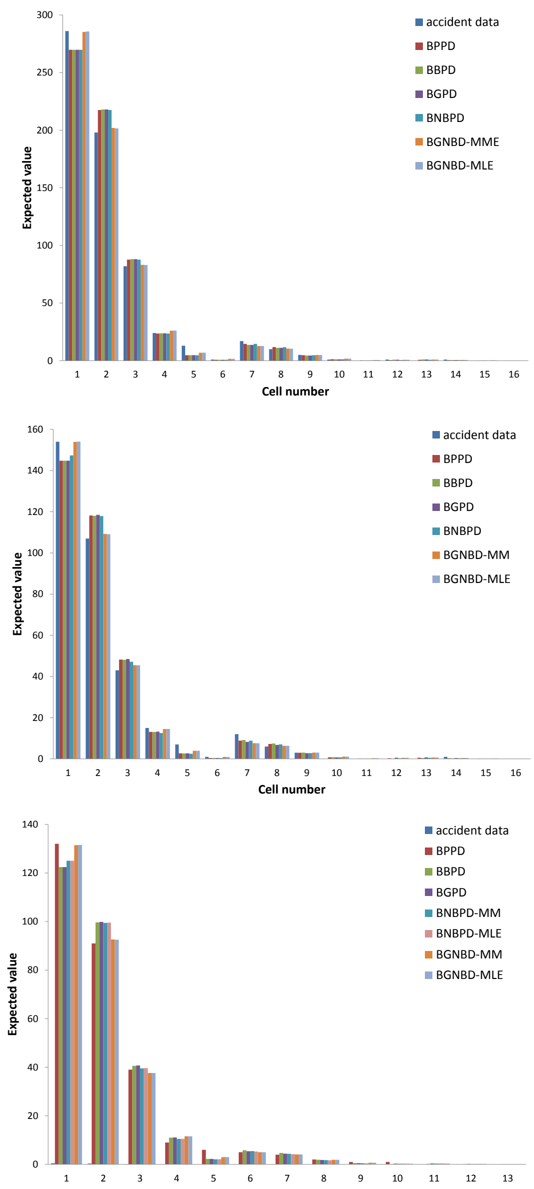
Table 4: Bivariate models fitted to accident data for year 1969 (349 days).

Cell no.	y_1	y_2	Observed	Expected BPPD	Expected BBPD ($m = 2$)	Expected BGPD	Expected BNBPD ($r = 50$)	Expected BGNBD MM	Expected BGNBD MLE
1	0	0	154	144.81	144.81	144.81	147.35	153.94	154.09
2	1	0	107	118.19	118.02	118.5	117.89	109.26	109.11
3	2	0	43	48.23	48.09	48.49	47.16	45.52	45.47
4	3	0	15	13.12	13.06	13.23	12.58	14.51	14.53
5	4	0	7	2.68	2.66	2.71	2.52	3.92	3.94
6	5	0	1	0.44	0.43	0.44	0.4	0.94	0.95
7	1	1	12	8.85	9.19	8.26	8.82	7.62	7.6
8	2	1	6	7.23	7.49	6.76	7.06	6.34	6.34
9	3	1	3	2.95	3.05	2.77	2.82	3.03	3.04
10	4	1	0	0.8	0.83	0.75	0.75	1.09	1.1
11	5	1	0	0.16	0.17	0.15	0.15	0.33	0.33
12	1	2	0	0.33	0.18	0.58	0.34	0.53	0.53
13	2	2	0	0.54	0.44	0.71	0.53	0.66	0.66
14	3	2	1	0.33	0.3	0.39	0.32	0.42	0.42
15	4	2	0	0.12	0.11	0.13	0.11	0.19	0.19
16	5	2	0	0.03	0.03	0.03	0.03	0.07	0.07
			Log-like	-515.1093	-514.7947	-515.6018	-515.0592	-	-513.839
			AIC	1034.219	1033.589	1035.204	1034.118		1031.677
			χ^2-value	11.775	11.603	12.224	12.514	5.172	5.617
			df	5	5	5	5	5	5
			p-value	0.038	0.040659	0.031849	0.02838	0.395275	0.345327
			SSE	272.4608	266.2474	285.2767	219.2574	42.9263	41.978

Table 5: Bivariate models fitted to accident data for year 1970 (290 days).

Cell no.	y_1	y_2	Observed	Expected BPPD	Expected BBPD (m=5)	Expected BGPD	Expected BNBPD	Expected BGNBD MM	Expected BGNBD MLE
1	0	0	132	125.02	122.44	122.44	125.02	131.47	131.57
2	1	0	91	99.33	99.66	99.85	99.56	92.60	92.50
3	2	0	39	39.46	40.56	40.71	39.64	37.64	37.61
4	3	0	9	10.45	11.00	11.07	10.52	11.56	11.58
5	4	0	6	2.08	2.24	2.26	2.09	2.98	2.99
6	1	1	5	5.70	5.78	5.42	5.26	5.02	5.02
7	2	1	4	4.53	4.71	4.42	4.19	4.08	4.08
8	3	1	2	1.80	1.92	1.80	1.67	1.88	1.88
9	4	1	1	0.48	0.52	0.49	0.44	0.65	0.65
10	1	2	1	0.16	0.13	0.29	0.35	0.27	0.27
11	2	2	0	0.26	0.25	0.36	0.39	0.33	0.33
12	3	2	0	0.15	0.16	0.20	0.20	0.20	0.20
13	4	2	0	0.05	0.06	0.07	0.06	0.09	0.09
			log-like	-406.2536	-406.3961	-405.9546	-405.9235	-	-404.892
			AIC	816.5071	816.7921	815.9091	817.8469	-	813.7831
			χ^2-value	2.1398	2.3157	2.0818	1.8648	0.19169	0.1862
			df	4	4	4	3	3	3
			P-value	0.710063	0.677909	0.72071	0.60093	0.97892	0.979785
			SSE	137.5873	189.1432	192.2528	141.008	21.228	20.8874

Figure 1: Accident data (entire study) and fitted distributions (Top). Accident data (year 1969) and fitted distributions (Middle). Accident data (year 1970) and fitted distributions (Bottom).



5. Conclusions

In this paper, the moments, cumulants, skewness of the univariate CGNBD are derived. Some monotonicity and distributional properties of the univariate CGNBD are provided. Then, BGNBD is defined and some important probabilistic characteristics such as moments, cumulants, covariance, and the coefficient of correlation are obtained. Some applications to accident data have been presented to illustrate the usage of the BGNBD. The results showed the superiority of BGNBD among other competitive models in the presented applications.

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