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Bayesian Analysis of the 3-Component Mixture of Exponential Distribution Assuming the Non-Informative Priors

Análisis bayesiano de una mezcla de tres componentes de distribuciones exponenciales asumiendolas a priori no informativas

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Abstract

Bayesian analysis of the 3-component mixture of an Exponential distribution under type-I right censoring scheme is considered in this paper. The Bayes estimators and posterior risks for the unknown parameters are derived under squared error loss function, precautionary loss function and DeGroot loss function assuming the non-informative (uniform and Jeffreys') priors. The Bayes estimators and posterior risks are viewed as a function of the test termination time. A simulation study is given to highlight and compare the properties of the Bayes estimates.

***Key words*:** Mixture Model, Bayes Estimators, Exponential Distribution, Loss Function, Posterior Risks.

Resumen

El análisis bayesiano de una mezcla de tres componentes de una distribución exponencial bajo el esquema de censura a la derecha tipo I se considera en este artículo. Los estimadores de Bayes y los riesgos posteriores de los parámetros desconocidos son derivados bajo una función de perdida de error cuadrático, función de perdida precautelary función de perdida de DeGroot asumiendo a prioris no informativas (uniforme y Jeffreys). Los estimadores de Bayes y los riesgos posteriores seven como una función del tiempo de terminación del test. Un estudio de simulación muestra y compara las propiedades de los estimadores de Bayes.

***Palabras clave*:** modelo de mezcla, estimadores de Bayes, distribución exponencial, función de pérdida, riesgos posteriores.

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1. Introduction

When a population is supposed to comprise a number of subpopulations mixed in an unknown proportion then common available distributions are irrelevant, e.g. a population of lifetime of certain electrical elements or medicines may be divided into a number of subpopulations depending upon the possible cause of failure. To model such populations in the engineering, physical sciences, chemical, biological sciences, and other fields, mixture models are used. As examples, Harris (1983) fitted mixture distributions to modeling the crime and justice data, Kanji (1985) described wind shear data using mixture distributions. Jones & McLachlan (1990) applied the mixture of normal distribution and Laplace distribution to wind shear data.

An Exponential distribution, because of its memory-less property, has many real life applications in testing the objects that do not depend upon their age (Rao 2012). There are many electronic devices whose failure rate does not depend on their age with time, therefore, an Exponential distribution is a suitable choice to model lifetimes of such devices. Suppose, a population of such devices is composed of three subpopulations (taken as Exponential) mixed by two unknown proportions. Some applications of mixture of Exponential distributions are in order. McCullagh (1994) derived some conditions under which quadratic and polynomial Exponential models can be generated as mixtures of Exponential models. Hebert & Scariano (2005) compared the location estimators for Exponential mixtures under Pitman's measure of closeness. Raqab & Ahsanullah (2001) discussed the location and scale parameters of generalized Exponential distribution based on order statistics. Ali, Woo & Nadarajah (2005) studied the Bayes estimators of an Exponential distribution and Abu-Taleb, Smadi & Alawneh (2007) presented the Bayesian estimation of the lifetime parameters for an Exponential distribution when survival time and censoring time are both exponentially distributed.

The mixture of the probability density functions from the same family is known as type-I mixture model and type-II mixture model is defined as a mixture of density functions from several families. Motivated by the above mentioned studies, the current study, a 3-component mixture of Exponential distributions is considered to model such a population provided the random observations are not given for each component but rather for the overall mixture distribution only.

This study provides the Bayesian analysis of the 3-component mixture of an Exponential distribution assuming the uniform prior (UP) and the Jeffreys' prior (JP). Estimations through the squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF) are also discussed. The interaction of using the 3-component mixture model is that, in the current situation, the researchers are able to define estimates and make predictions about the complicated system by using strong and complicated computational methods. The Bayesian technique to analyze the 3-component mixture model has developed an interest between researchers. Most of the researchers worked on the Bayesian analysis of the 2-component mixture models. Sinha (1998) used the Bayesian counterpart of the maximum likelihood estimates of the 2-component mixture model considered by Mendenhall & Hader (1958). Saleem & Aslam (2009) discussed the use of the informative and the non-informative priors for Bayesian analysis of the 2-component mixture of the Rayleigh distribution. Also, Saleem, Aslam & Economou (2010) presented the Bayesian analysis of the 2-component mixture of the Power distribution using the complete and censored sample. Kazmi, Aslam & Ali (2012) described the Bayesian analysis for the 2-component mixture of the Maxwell distribution.

Censoring is an important and valuable aspect of the lifetime data. In real life most of the time, it is not suitable that the testing procedure continues until is reached the last value of the data set. Censoring is a form of primary quality and missing life time data problems. A valuable account of censoring is given in Romeu (2004), Gijbels (2010) and Kalbfleisch & Prentice (2011). There are three main types of censoring i.e., (i) right censoring, (ii) left censoring and (iii) interval censoring. In this paper an ordinary type-I right censoring is used with fixes life-test termination time for all objects.

The rest of the paper is arranged as follows. The 3-component mixture of an Exponential distribution is given in Section 2. The posterior distributions assuming the UP and the JP are presented in Section 3. The Bayes estimators and their posterior risks using the Up and the JP under SELF, PLF and DLF are defined in Sections 4, 5 and 6 respectively. The simulation study is made in Section 7. Finally, the conclusion of this study is given in Section 8.

2. 3-Component Mixture of an Exponential Distribution

The probability density function (p.d.f.) of an Exponential distribution for random variable $X$is:

$$f_m(x;\theta_m) = \theta_m \exp(-\theta_m x), \quad x \geq 0, \theta_m > 0, m = 1, 2, 3 \tag{1}$$

where θ_m is a parameter of an Exponential distribution. The cumulative distribution function for random variable X is:

$$F_m(x) = 1 - \exp(-\theta_m x), \quad m = 1, 2, 3 \tag{2}$$

A finite 3-component mixture model with the unknown mixing proportions p_1 and p_2 is:

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + (1 - p_1 - p_2) f_3(x), \quad p_1, p_2 \geq 0, p_1 + p_2 \leq 1 \tag{3}$$

$$\begin{aligned} f(x;\theta_1,\theta_2,\theta_3,p_1,p_2) = {} & p_1\theta_1 \exp(-\theta_1 x) + p_2\theta_2 \exp(-\theta_2 x) \\ & + (1 - p_1 - p_2)\theta_3 \exp(-\theta_3 x) \end{aligned} \tag{4}$$

The cumulative distribution function of 3-component mixture model is:

$$F(x) = p_1 F_1(x) + p_2 F_2(x) + (1 - p_1 - p_2) F_3(x) \tag{5}$$

$$F(x) = 1 - p_1 \exp(-\theta_1 x) - p_2 \exp(-\theta_2 x) - (1 - p_1 - p_2) \exp(-\theta_3 x) \tag{6}$$

3. The Posterior Distributions Assuming the Non-informative Priors

The posterior distribution of parameters given data $\mathbf{x}$ is derived assuming the non-informative (uniform and Jeffreys') priors.

3.1. The Likelihood Function

Suppose n units are used in a life testing experiment from the 3-component mixture model. Let r units out of n units failed until fixed test termination time t and the remaining $n-r$ units are still working. According to Mendenhall & Hader (1958), there are many practical situations in which the failed objects can be pointed out easily as subsets of either subpopulation-I or subpopulation-II or subpopulation-III. Out of r units, suppose r_1, r_2 and r_3 units belong to subpopulation-I, subpopulation-II and subpopulation-III respectively and such that $r = r_1 + r_2 + r_3$. Now we define x_{lk}, $0 < x_{lk} \leq t$, be the failure time of k^{th} unit belonging to l^{th} subpopulation, $l = 1, 2, 3$ and $k = 1, 2, \ldots, r_l$. So the likelihood function of the 3-component mixture model for the random sample $\mathbf{x}$ is:

$$L(\Psi|\mathbf{x}) \propto \prod_{k=1}^{r_1} p_1 f_1(x_{1k}) + \prod_{k=1}^{r_2} p_2 f_2(x_{2k}) + \prod_{k=1}^{r_3} (1 - p_1 - p_2) f_3(x_3 k) \tag{7}$$

$$\begin{aligned} L(\Psi|\mathbf{x}) \propto & \prod_{k=1}^{r_1} p_1 \theta_1 \exp(-\theta_1 x_{1k}) + \prod_{k=1}^{r_2} p_2 \theta_2 \exp(-\theta_2 x_{2k}) \\ & + \prod_{k=1}^{r_3} (1 - p_1 - p_2) \theta_3 \exp(-\theta_3 x_{3k}) \end{aligned} \tag{8}$$

$$\begin{aligned} L(\Psi|\mathbf{x}) \propto & \; \theta_1^{r_1} \theta_2^{r_2} \theta_3^{r_3} \sum_{i=0}^{n-r} \sum_{j=0}^{i} \binom{n-r}{i} \binom{i}{j} \exp\left\{-\theta_1 (nt - rt - it + \sum_{k=1}^{r_1} x_{1k})\right\} \\ & \times \exp\left\{-\theta_2 (it - jt + \sum_{k=1}^{r_2} x_{2k})\right\} \exp\left\{-\theta_3 (jt + \sum_{k=1}^{r_3} x_{3k})\right\} \\ & \times p_1^{n-r-i+r_1} p_2^{i-j+r_2} (1 - p_1 - p_2)^{j+r_3} \end{aligned} \tag{9}$$

where $\Psi = (\theta_1, \theta_2, \theta_3, p_1, p_2)$ and $\mathbf{x} = (x_{11}, x_{12}, \ldots, x_{1r_1}, x_{21}, x_{22}, \ldots, x_{2r_2}, x_{31}, x_{32}, \ldots, x_{3r_3})$.

3.2. The Posterior Distribution Assuming the UP and the JP

When elicitation of hyperparameters is difficult and little or no prior information is given then we use the non-informative prior. The UP and the JP are the most common non-informative priors. Bayes (1763), de Laplace (1820) and Geisser (1984) proposed that may take the uniform distribution for the unknown parameters of interest. Following Bayes (1763), de Laplace (1820) and Geisser (1984), in this study, we assume the UP over the interval $(0, \infty)$ for the scale parameters θ_1, θ_2 and θ_3 of an Exponential distribution and the UP over the interval $(0, 1)$ for the mixing proportions p_1 and p_2. So the joint prior distribution of parameters $\theta_1, \theta_2, \theta_3, p_1$, and p_2 is:

$$\Pi_1(\Psi) \propto 1, \quad \theta_1, \theta_2, \theta_3 > 0, p_1, p_2 \geq 0, p_1 + p_2 \leq 1 \tag{10}$$

Jeffreys (1946), Jeffreys (1961), Bernardo (1979) and Berger (1985) discussed the JP. The JP is defined as: $p(\theta_m) \propto \sqrt{| I(\theta_m) |}$, $m = 1, 2, 3$, where $I(\theta_m) = -E\left[\frac{\partial^2 f(x|\theta_m)}{\partial \theta_m^2}\right]$ is the Fisher's information matrix. The prior distributions of the mixing proportions p_1 and p_2 are again assumed as the uniform distribution over the interval (0,1). Assuming the independence of parameters, the joint prior distribution of parameters θ_1, θ_2, θ_3, p_1, and p_2 is:

$$\Pi_2(\Psi) \propto \frac{1}{\theta_1\theta_2\theta_3}, \quad \theta_1, \theta_2, \theta_3 > 0, p_1, p_2 \geq 0, p_1 + p_2 \leq 1 \tag{11}$$

This prior will be considered as the JP of the parameters of 3-component of an Exponential distribution. The joint posterior distribution of parameters θ_1, θ_2, θ_3, p_1 and p_2 given data $\mathbf{x}$ assuming the UP and the JP is:

$$g_v(\Psi|\mathbf{x}) = \frac{L(\Psi|\mathbf{x})\Pi_v(\Psi)}{\int_\Psi L(\Psi|\mathbf{x})\Pi_v(\Psi)d\Psi} \tag{12}$$

$$g_v(\Psi|\mathbf{x}) = \frac{1}{E_v}\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \times \frac{\exp\{-\theta_1 B_{1v}\}\exp\{-\theta_2 B_{2v}\}\exp\{-\theta_3 B_{3v}\}p_1^{A_{0v}-1}p_2^{B_{0v}-1}(1-p_1-p_2)^{C_{0v}-1}}{\theta_1^{1-A_{1v}}\theta_2^{1-A_{2v}}\theta_3^{1-A_{3v}}}$$

where $A_{11} = r_1 + 1$, $A_{21} = r_2 + 1$, $A_{31} = r_3 + 1$, $A_{12} = r_1$, $A_{22} = r_2$, $A_{32} = r_3$, $B_{1v} = nt - rt - it + \sum_{k=1}^{r_1} x_{1k}$, $B_{2v} = it - jt + \sum_{k=1}^{r_2} x_{2k}$, $B_{3v} = jt + \sum_{k=1}^{r_3} x_{3k}$, $A_{0v} = n - r - i + r_1 + 1$, $B_{0v} = i - j + r_2 + 1$, $C_{0v} = j + r_3 + 1$, $E_v = \Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}B(A_{0v}, B_{0v}, C_{0v})B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}$ $B_{3v}^{-A_{3v}}$, $v = 1$ for the UP and $v = 2$ for the JP.

4. The Bayes Estimators and Posterior Risks Assuming the UP and the JP under SELF

If $L(\theta, d)$ is a loss function then the expected value of the loss function for a given decision with respect to the posterior distribution is posterior risk function and if $\widehat{d}$ is a Bayes estimator then $\rho(\widehat{d})$ is called posterior risk and is defined as: $\rho(\widehat{d}) = E_{\theta|\mathbf{x}}\{L(\theta,\widehat{d})\}$. The SELF $L(\theta,d) = (\theta - d)^2$ suggested by Legendre (1806) to develop the least square theory. Chin-Chuan, Liang-Yuh & Hsin-Lin (2003) discussed the SELF for process capability estimation under different priors. By using SELF the Bayes estimators and posterior risk are: $\widehat{d} = E_{\theta|\mathbf{x}}(\theta)$ and $\rho(\widehat{d}) = E_{\theta|\mathbf{x}}(\theta^2) - \{E_{\theta|\mathbf{x}}(\theta)\}^2$, respectively. So the Bayes estimators and posterior risks assuming the UP and the JP for parameters θ_1, θ_2, θ_3, p_1 and p_2 under SELF are obtained with respective marginal posterior distribution as:

$$\begin{aligned}\widehat{\theta}_{1v} =& \frac{1}{E_v}\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-(A_{1v}+1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\end{aligned} \tag{13}$$

$$\begin{aligned}\widehat{\theta}_{2v} =& \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\end{aligned} \tag{14}$$

$$\begin{aligned}\widehat{\theta}_{3v} =& \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\end{aligned} \tag{15}$$

$$\begin{aligned}\widehat{p}_{1v} =& \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})\end{aligned} \tag{16}$$

$$\begin{aligned}\widehat{p}_{2v} =& \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\end{aligned} \tag{17}$$

$$\begin{aligned}\rho(\widehat{\theta}_{1v}) =& \frac{1}{E_v}\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-(A_{1v}+2)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\\ & -\left\{\frac{1}{E_v}\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.\\ & \left. B_{1v}^{-(A_{1v}+1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2\end{aligned} \tag{18}$$

$$\rho(\widehat{\theta}_{2v}) = \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+2)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v}) - \left\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2 \tag{19}$$

$$\rho(\widehat{\theta}_{3v}) = \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+2)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v}) - \left\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^2 \tag{20}$$

$$\rho(\widehat{p}_{1v}) = \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v}) - \left\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})\right\}^2 \tag{21}$$

$$\rho(\widehat{p}_{2v}) = \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v}) - \left\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\right\}^2 \tag{22}$$

5. The Bayes Estimators and Posterior Risks Assuming the UP and the JP under PLF

Norstrom (1996) discussed an asymmetric precautionary loss function (PLF) and a special case of general class of precautionary loss functions is: $L(\theta, d) = \frac{(\theta-d)^2}{d}$. The Bayes estimators and posterior risk are: $\widehat{d} = \{E_{\theta|\mathbf{x}}(\theta^2)\}^{\frac{1}{2}}$ and $\rho(\widehat{d}) = 2\{E_{\theta|\mathbf{x}}(\theta^2)\}^{\frac{1}{2}} - 2E_{\theta|\mathbf{x}}(\theta)$, respectively. The respective marginal posterior distribution yields the Bayes estimators and posterior risks assuming the UP and the JP for parameters θ_1, θ_2, θ_3, p_1 and p_2 under PLF as:

$$\widehat{\theta}_{1v} = \Bigg\{ \frac{1}{E_v}\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v}) \sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-(A_{1v}+2)} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \Bigg\}^{\frac{1}{2}} \tag{23}$$

$$\widehat{\theta}_{2v} = \Bigg\{ \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v}) \sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-(A_{2v}+2)} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \Bigg\}^{\frac{1}{2}} \tag{24}$$

$$\widehat{\theta}_{3v} = \Bigg\{ \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2) \sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-(A_{3v}+2)} B(A_{0v}, C_{0v}) B(B_{0v}, A_{0v}+C_{0v}) \Bigg\}^{\frac{1}{2}} \tag{25}$$

$$\widehat{p}_{1v} = \Bigg\{ \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}) \sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(B_{0v}, C_{0v}) B(A_{0v}+2, B_{0v}+C_{0v}) \Bigg\}^{\frac{1}{2}} \tag{26}$$

$$\widehat{p}_{2v} = \Bigg\{ \frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}) \sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} B_{1v}^{-A_{1v}} B_{2v}^{-A_{2v}} B_{3v}^{-A_{3v}} B(A_{0v}, C_{0v}) B(B_{0v}+2, A_{0v}+C_{0v}) \Bigg\}^{\frac{1}{2}} \tag{27}$$

$$
\begin{aligned}
\rho(\widehat{\theta}_{1v}) =& 2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \\
& B_{1v}^{-(A_{1v}+2)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{\frac{1}{2}} \\
& -2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \\
& B_{1v}^{-(A_{1v}+1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\rho(\widehat{\theta}_{2v}) =& 2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \\
& B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+2)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{\frac{1}{2}} \\
& -2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \\
& B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\rho(\widehat{\theta}_{3v}) =& 2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \\
& B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+2)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{\frac{1}{2}} \\
& -2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \\
& B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
\rho(\widehat{p}_{1v}) =& 2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \\
& B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})\Bigg\}^{\frac{1}{2}} \\
& -2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j} \\
& B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})\Bigg\}
\end{aligned}
\tag{31}
$$

$$\begin{aligned}\rho(\widehat{p}_{2v}) =&2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v})\Bigg\}^{\frac{1}{2}}\\&-2\Bigg\{\frac{1}{E_v}\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\Bigg\}\end{aligned} \tag{32}$$

6. The Bayes Estimators and Posterior Risks Assuming the UP and the JP under DLF

DeGroot (2005) introduced the asymmetric loss function: $L(\theta,d) = (\frac{\theta-d}{d})^2$, which is DLF.The Bayes estimators and their posterior risk under DLF are: $\widehat{d} = \frac{E_{\theta|\mathbf{x}}(\theta^2)}{E_{\theta|\mathbf{x}}(\theta)}$ and $\rho(\widehat{d}) = 1-\frac{\{E_{\theta|\mathbf{x}}(\theta)\}^2}{E_{\theta|\mathbf{x}}(\theta^2)}$, respectively. The Bayes estimators and posterior risks assuming the UP and the JP for parameters θ_1, θ_2, θ_3, p_1 and p_2 under DLF are:

$$\begin{aligned}\widehat{\theta}_{1v} =&\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-(A_{1v}+2)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\\&\Bigg\{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-(A_{1v}+1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{-1}\end{aligned} \tag{33}$$

$$\begin{aligned}\widehat{\theta}_{2v} =&\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+2)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\\&\Bigg\{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{-1}\end{aligned} \tag{34}$$

$$\begin{aligned}\widehat{\theta}_{3v} =&\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+2)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\\&\left\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.\\&\left.B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{-1}\end{aligned} \tag{35}$$

$$\begin{aligned}\widehat{p}_{1v} =&\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})\\&\left\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.\\&\left.B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})\right\}^{-1}\end{aligned} \tag{36}$$

$$\begin{aligned}\widehat{p}_{2v} =&\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\&B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v})\\&\left\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.\\&\left.B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\right\}^{-1}\end{aligned} \tag{37}$$

$$\begin{aligned}\rho(\widehat{\theta}_{1v}) =&1-\left\{\Gamma(A_{1v}+1)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.\\&\left.B_{1v}^{-(A_{1v}+1)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{2}\\&\left\{E_v\Gamma(A_{1v}+2)\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.\\&\left.B_{1v}^{-(A_{1v}+2)}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\right\}^{-1}\end{aligned} \tag{38}$$

$$\begin{aligned}\rho(\widehat{\theta}_{2v}) =& 1-\Bigg\{\Gamma(A_{1v})\Gamma(A_{2v}+1)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+1)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{2}\\ & \Bigg\{E_v\Gamma(A_{1v})\Gamma(A_{2v}+2)\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-(A_{2v}+2)}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{-1}\end{aligned} \tag{39}$$

$$\begin{aligned}\rho(\widehat{\theta}_{3v}) =& 1-\Bigg\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+1)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+1)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{2}\\ & \Bigg\{E_v\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v}+2)\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-(A_{3v}+2)}B(A_{0v},C_{0v})B(B_{0v},A_{0v}+C_{0v})\Bigg\}^{-1}\end{aligned} \tag{40}$$

$$\begin{aligned}\rho(\widehat{p}_{1v}) =& 1-\Bigg\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+1,B_{0v}+C_{0v})\Bigg\}^{2}\\ & \Bigg\{E_v\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\\ & B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(B_{0v},C_{0v})B(A_{0v}+2,B_{0v}+C_{0v})\Bigg\}^{-1}\end{aligned} \tag{41}$$

$$\begin{aligned}\rho(\widehat{p}_{2v}) =& 1-\left\{\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.\\ &\left. B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+1,A_{0v}+C_{0v})\right\}^{2}\\ &\left\{E_v\Gamma(A_{1v})\Gamma(A_{2v})\Gamma(A_{3v})\sum_{i=0}^{n-r}\sum_{j=0}^{i}\binom{n-r}{i}\binom{i}{j}\right.\\ &\left. B_{1v}^{-A_{1v}}B_{2v}^{-A_{2v}}B_{3v}^{-A_{3v}}B(A_{0v},C_{0v})B(B_{0v}+2,A_{0v}+C_{0v})\right\}^{-1}\end{aligned} \quad (42)$$

7. Simulation Study

A simulation study is done to investigate the performance and highlight the properties of the Bayes estimates (BEs) of the 3-component mixture of an Exponential distribution using the UP and the JP in terms of different sample sizes, test termination times and loss functions. Samples of different sizes i.e., $n = 30, 100, 200, 500$ were generated from the 3-component mixture of an Exponential distribution with different combinations of parameters $(\theta_1, \theta_2, \theta_3) \in \{(2, 3, 4), (4, 3, 2), (3, 3, 3)\}$ and $(p_1, p_2) \in \{(0.3, 0.5), (0.5, 0.3), (0.4, 0.4)\}$. The $p_1 n$ observations were taken randomly from first component density $f_1(x; \theta_1)$, $p_2 n$ observations were chosen randomly from second component density $f_2(x; \theta_2)$ and remaining $(1 - p_1 - p_2)n$ observations were selected randomly from third component density $f_3(x; \theta_3)$. A sample censored at a fixed test termination time t such that $t = 0.5, 0.8$, is selected to check the impact of test termination time on BEs. The observations which are greater than t were considered as censored ones. Only failures can be identified as a member of subpopulation-I or subpopulation-II or subpopulation-III of the 3-component mixture model. By using Mathematica package, each time 1000 samples were generated and BEs and posterior risks (PRs) were computed under SELF, PLF and DLF.

It is observed from Tables 1-6 that the PRs of BEs using the UP and the JP under SELF, PLF and DLF are reduced with an increase in sample size at different test termination times. For large test termination time, the PRs of BEs assuming the UP and the JP under SELF, PLF and DLF are less than the PRs of BEs of small test termination time at different sample sizes because we get more information from samples. An important feature about selection of the UP and the JP under SELF, PLF and DLF based on PR, the JP due to less PR under SELF and PLF is more efficient and preferable as compared to the UP but we can't identify which prior is suitable under DLF due to the mix pattern for different sample sizes. The same observation is made for the different test termination times. While comparing three different loss functions, considered in this study, it is observed that the PRs of BEs of component parameters θ_1, θ_2 and θ_3 using the UP and the JP under PLF are smaller than SELF but larger than DLF at different sample sizes, so in that case DLF is more preferable. Again the same observation is made for different test termination times resulting in deciding DLF as the suitable loss function. However, the SELF is observed to be a more suitable choice for estimating the proportion parameters p_1 and p_2 when either the UP or the JP is taken as non-informative prior.

From Tables 1-6, it can be seen that the component parameters are over-estimated using the UP and the JP under SELF, PLF and DLF for different sample sizes at a fix test termination time. Similarly, for a fix sample size, the BEs provide an over-estimate of the component parameters at varying test termination times. If $\theta_1 < \theta_2 < \theta_3$, the extent of over-estimation of the BEs of first component parameter θ_1 is greater than θ_2 and θ_3 with a few exceptions. Similarly, the over-estimation of the BEs of third component parameter θ_3 is also greater than θ_1 and θ_2 with a few exceptions in both the cases when $\theta_1 > \theta_2 > \theta_3$ and $\theta_1 = \theta_2 = \theta_3$. Also, the extent of over-estimation of the BEs of component parameters is greater for small test termination time as compared to large test termination time. The extent of over-estimation reduces to zero when either sample size or test termination time is increased.

8. Conclusion

The importance and application of mixture models in real life problems is undeniable. There are many advantages of Bayesian analysis over classical analysis. The main advantage is that additional information is provided by prior distribution for making Bayesian inference. In this study, we have considered the Bayesian analysis of 3-componenten mixture of an Exponential distribution assuming non-informative priors under different loss functions. The closed form expressions of posterior distributions, Bayes estimators and posterior risks have been derived. To judge relative performance of the Bayes estimators, an extensive simulation study has been conducted. From simulated results, we observed that an increase in sample size or test termination time provides improved BEs. In particular, we observed that for different sample sizes at a fixed test termination time, component parameters θ_1, θ_2 and θ_3 using the UP and the JP under SELF, PLF and DLF are over-estimated and the same is the case with different test termination times at a fixed sample size. Furthermore, as sample size (test termination time) increases the PRs of BEs decrease for a fixed test termination time (sample size). Also, the DLF (SELF) is observed as a suitable choice for estimating component parameters (mixing proportions). Finally, we conclude that the JP is more suitable prior under SELF for estimating the proportion parameters. In this case, when DLF is considered, both the UP and the JP are suitable priors for component parameters but JP is more suitable when PLF is used to estimate the component parameters.

TABLE 1: Bayes estimates (BEs) and posterior risks (PRs) using the UP under SELF, PLF and DLF with $\theta_1 = 2$, $\theta_2 = 3$, $\theta_3 = 4$, $p_1 = 0.3$, $p_2 = 0.5$, and $t = 0.5, 0.8$.

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	4.22502	4.12726	6.1777	0.279064	0.47946
			PR	**8.15251**	**3.39503**	**15.10191**	**0.010472**	**0.012063**
		PLF	BE	5.1376	4.49203	7.18782	0.293622	0.493002
			PR	**1.39216**	**0.69481**	**1.77224**	**0.035649**	**0.024644**
		DLF	BE	5.6979	4.84539	8.29039	0.319303	0.50616
			PR	**0.26519**	**0.153**	**0.2519**	**0.122357**	**0.050965**
	100	SELF	BE	2.82534	3.41437	4.58215	0.28609	0.494412
			PR	**1.42727**	**0.87915**	**3.10053**	**0.004652**	**0.005043**
		PLF	BE	3.02404	3.55837	4.81943	0.293774	0.498282
			PR	**0.45041**	**0.24588**	**0.63001**	**0.016393**	**0.010315**
		DLF	BE	3.34182	3.65504	5.08054	0.297635	0.504747
			PR	**0.14063**	**0.06638**	**0.13312**	**0.055457**	**0.020375**
	200	SELF	BE	2.55291	3.2131	4.20087	0.287237	0.499641
			PR	**0.67698**	**0.43722**	**1.52285**	**0.002749**	**0.002939**
		PLF	BE	2.65133	3.25081	4.40783	0.29399	0.503553
			PR	**0.24553**	**0.13122**	**0.35194**	**0.009786**	**0.00596**
		DLF	BE	2.75563	3.38736	4.57304	0.297731	0.502229
			PR	**0.09035**	**0.04011**	**0.08346**	**0.033882**	**0.011995**
	500	SELF	BE	2.24891	3.08457	4.04449	0.293118	0.499637
			PR	**0.24**	**0.1869**	**0.63178**	**0.001376**	**0.001402**
		PLF	BE	2.32179	3.12392	4.16258	0.29429	0.502834
			PR	**0.10336**	**0.05903**	**0.156**	**0.004693**	**0.00278**
		DLF	BE	2.35893	3.148333	4.19699	0.29934	0.501041
			PR	**0.04223**	**0.01852**	**0.03862**	**0.01574**	**0.005498**
0.8	30	SELF	BE	3.26123	3.64985	5.59487	0.289478	0.486841
			PR	**2.74831**	**1.54244**	**7.90588**	**0.00731**	**0.008565**
		PLF	BE	3.5949	3.88067	6.19255	0.304998	0.493138
			PR	**0.65382**	**0.39054**	**1.11818**	**0.025117**	**0.017681**
		DLF	BE	3.91776	4.02417	6.73606	0.317094	0.505014
			PR	**0.17498**	**0.09895**	**0.18498**	**0.081482**	**0.035776**
	100	SELF	BE	2.44023	3.21209	4.39368	0.293743	0.496348
			PR	**0.52603**	**0.42506**	**1.64928**	**0.002678**	**0.003036**
		PLF	BE	2.54208	3.25658	4.55337	0.297071	0.500343
			PR	**0.20092**	**0.1266**	**0.35628**	**0.009122**	**0.00612**
		DLF	BE	2.66905	3.31082	4.72786	0.302239	0.504831
			PR	**0.07559**	**0.03782**	**0.07965**	**0.030513**	**0.012043**
	200	SELF	BE	2.24829	3.11467	4.10499	0.295224	0.498811
			PR	**0.23525**	**0.21217**	**0.76801**	**0.001428**	**0.001601**
		PLF	BE	2.27724	3.12934	4.27153	0.297688	0.500939
			PR	**0.09899**	**0.06669**	**0.18504**	**0.004857**	**0.00322**
		DLF	BE	2.35991	3.16481	4.30783	0.298987	0.502605
			PR	**0.04321**	**0.02132**	**0.04521**	**0.016486**	**0.006435**
	500	SELF	BE	2.09842	3.02853	4.04969	0.29741	0.500331
			PR	**0.08355**	**0.0863**	**0.30578**	**0.000608**	**0.000676**
		PLF	BE	2.1379	3.04763	4.10623	0.298096	0.501332
			PR	**0.03958**	**0.02838**	**0.07569**	**0.002049**	**0.001351**
		DLF	BE	2.13715	3.06005	4.14309	0.299792	0.501527
			PR	**0.01791**	**0.00919**	**0.01856**	**0.006828**	**0.002686**

Table 2: Bayes estimates (BEs) and posterior risks (PRs) using the JP under SELF, PLF and DLF with $\theta_1 = 2$, $\theta_2 = 3$, $\theta_3 = 4$, $p_1 = 0.3$, $p_2 = 0.5$, and $t = 0.5, 0.8$.

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	3.30251	3.80772	4.69855	0.285562	0.468126
			PR	**5.59252**	**2.81375**	**10.62742**	**0.01027**	**0.011437**
		PLF	BE	3.88392	4.24931	5.97612	0.305594	0.477112
			PR	**1.22263**	**0.67181**	**1.66652**	**0.035526**	**0.024567**
		DLF	BE	4.63851	4.60617	6.69907	0.322874	0.492464
			PR	**0.29363**	**0.15237**	**0.29622**	**0.116565**	**0.051428**
	100	SELF	BE	2.59094	3.37002	4.09863	0.287101	0.487014
			PR	**1.19822**	**0.82769**	**2.74483**	**0.004564**	**0.004843**
		PLF	BE	2.84556	3.46839	4.3205	0.291864	0.493622
			PR	**0.4179**	**0.22874**	**0.60506**	**0.015665**	**0.009879**
		DLF	BE	3.06	3.66652	4.74192	0.305363	0.496471
			PR	**0.14592**	**0.06575**	**0.1429**	**0.054722**	**0.020363**
	200	SELF	BE	2.39769	3.23173	4.04244	0.290167	0.493372
			PR	**0.58809**	**0.43129**	**1.4586**	**0.002736**	**0.002842**
		PLF	BE	2.54947	3.26191	4.12043	0.292388	0.498294
			PR	**0.23269**	**0.12746**	**0.34734**	**0.009536**	**0.005789**
		DLF	BE	2.6671	3.3241	4.28005	0.296589	0.503393
			PR	**0.08676**	**0.03834**	**0.08659**	**0.03239**	**0.011581**
	500	SELF	BE	2.20636	3.11246	3.96786	0.292763	0.498829
			PR	**0.22998**	**0.18495**	**0.6234**	**0.001345**	**0.00136**
		PLF	BE	2.26946	3.11048	4.036	0.294748	0.500866
			PR	**0.09823**	**0.0574**	**0.156**	**0.004582**	**0.002724**
		DLF	BE	2.33202	3.13397	4.06804	0.297384	0.502753
			PR	**0.04255**	**0.01837**	**0.04014**	**0.015857**	**0.005459**
0.8	30	SELF	BE	2.81825	3.45786	4.51884	0.290502	0.482349
			PR	**2.17628**	**1.41093**	**6.22287**	**0.007287**	**0.008496**
		PLF	BE	3.11885	3.59629	5.05967	0.306621	0.491692
			PR	**0.63554**	**0.37005**	**1.0956**	**0.025019**	**0.017593**
		DLF	BE	3.38868	3.86221	5.7383	0.316467	0.501535
			PR	**0.19306**	**0.10113**	**0.21529**	**0.081781**	**0.035712**
	100	SELF	BE	2.32222	3.14771	4.01524	0.292611	0.495313
			PR	**0.48271**	**0.4006**	**1.49651**	**0.002644**	**0.003003**
		PLF	BE	2.4096	3.23169	4.19547	0.297436	0.497527
			PR	**0.19095**	**0.12374**	**0.35394**	**0.009004**	**0.006057**
		DLF	BE	2.55192	3.24702	4.37088	0.302902	0.501224
			PR	**0.07781**	**0.03805**	**0.08686**	**0.030504**	**0.012113**
	200	SELF	BE	2.196	3.07891	4.00946	0.294894	0.498743
			PR	**0.22745**	**0.20678**	**0.76217**	**0.001427**	**0.0016**
		PLF	BE	2.22341	3.12594	4.09304	0.298617	0.499203
			PR	**0.09479**	**0.06558**	**0.1849**	**0.004789**	**0.003189**
		DLF	BE	2.30396	3.1319	4.14499	0.298829	0.50121
			PR	**0.04281**	**0.02103**	**0.04733**	**0.016299**	**0.006378**
	500	SELF	BE	2.0792	3.02399	4.00729	0.298329	0.499437
			PR	**0.08108**	**0.08547**	**0.3049**	**0.000606**	**0.000672**
		PLF	BE	2.1063	3.03813	4.00521	0.299169	0.499824
			PR	**0.03825**	**0.02798**	**0.07563**	**0.002031**	**0.001345**
		DLF	BE	2.1272	3.05488	4.04098	0.299277	0.501092
			PR	**0.01805**	**0.00918**	**0.01925**	**0.006844**	**0.002682**

TABLE 3: Bayes estimates (BEs) and posterior risks (PRs) using the UP under SELF, PLF and DLF with $\theta_1 = 4$, $\theta_2 = 3$, $\theta_3 = 2$, $p_1 = 0.5$, $p_2 = 0.3$, and $t = 0.5, 0.8$.

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	5.03127	4.81717	14.4058	0.49336	0.30599
			PR	**3.78733**	**7.04383**	**393.721**	**0.010409**	**0.009351**
		PLF	BE	5.18096	5.34651	7.10057	0.506304	0.321826
			PR	**0.65794**	**1.14487**	**2.26392**	**0.021169**	**0.0301**
		DLF	BE	5.69559	6.08516	32.996	0.517202	0.336103
			PR	**0.12627**	**0.21448**	**0.31318**	**0.042015**	**0.094862**
	100	SELF	BE	4.32408	3.48119	3.18176	0.501051	0.306898
			PR	**1.06177**	**1.4796**	**2.4949**	**0.004021**	**0.003927**
		PLF	BE	4.43269	3.75565	3.47246	0.50503	0.311397
			PR	**0.23514**	**0.39934**	**0.664**	**0.008038**	**0.012548**
		DLF	BE	4.5158	3.99458	3.8948	0.510777	0.317893
			PR	**0.05348**	**0.10395**	**0.18747**	**0.01599**	**0.039946**
	200	SELF	BE	4.15969	3.30336	2.65355	0.500567	0.304289
			PR	**0.54772**	**0.77122**	**1.068**	**0.002233**	**0.002223**
		PLF	BE	4.183	3.38534	2.9208	0.504513	0.30787
			PR	**0.12781**	**0.22162**	**0.37151**	**0.004457**	**0.007263**
		DLF	BE	4.29189	3.53659	3.03098	0.506274	0.309894
			PR	**0.03027**	**0.0645**	**0.12266**	**0.008705**	**0.02344**
	500	SELF	BE	4.05214	3.15383	2.35046	0.50027	0.301999
			PR	**0.22525**	**0.32251**	**0.39443**	**0.000954**	**0.000999**
		PLF	BE	4.10314	3.20109	2.38615	0.502247	0.303106
			PR	**0.05569**	**0.10072**	**0.15376**	**0.001896**	**0.003306**
		DLF	BE	4.09851	3.22349	2.50417	0.504551	0.306
			PR	**0.0136**	**0.03104**	**0.06324**	**0.003759**	**0.010796**
0.8	30	SELF	BE	4.60708	4.0673	3.9591	0.491784	0.305857
			PR	**2.06164**	**3.05704**	**6.71197**	**0.008108**	**0.007047**
		PLF	BE	4.78976	4.411	4.44132	0.499841	0.317388
			PR	**0.4109**	**0.64347**	**1.02895**	**0.016344**	**0.022641**
		DLF	BE	5.04824	4.71186	4.92032	0.50771	0.328947
			PR	**0.0847**	**0.14425**	**0.21952**	**0.0325**	**0.070789**
	100	SELF	BE	4.19187	3.34385	2.65896	0.498415	0.303129
			PR	**0.57004**	**0.73496**	**0.89049**	**0.002707**	**0.002426**
		PLF	BE	4.27433	3.4119	2.72112	0.499986	0.307765
			PR	**0.13347**	**0.20684**	**0.29326**	**0.005432**	**0.00797**
		DLF	BE	4.34097	3.53208	2.93405	0.503415	0.311406
			PR	**0.03143**	**0.06083**	**0.10652**	**0.010834**	**0.02589**
	200	SELF	BE	4.08408	3.13722	2.34034	0.499107	0.302543
			PR	**0.28609**	**0.35794**	**0.38474**	**0.001398**	**0.001282**
		PLF	BE	4.1427	3.17768	2.43561	0.500479	0.305603
			PR	**0.06892**	**0.10915**	**0.15321**	**0.00278**	**0.004201**
		DLF	BE	4.13182	3.26269	2.52608	0.502583	0.306298
			PR	**0.01687**	**0.0347**	**0.06245**	**0.005568**	**0.01385**
	50	SELF	BE	4.02646	3.05852	2.15061	0.500052	0.301697
			PR	**0.11448**	**0.1457**	**0.13524**	**0.000566**	**0.00053**
		PLF	BE	4.04934	3.08478	2.20076	0.500284	0.302363
			PR	**0.02834**	**0.04699**	**0.06098**	**0.00113**	**0.001751**
		DLF	BE	4.06012	3.1112	2.20327	0.501021	0.302973
			PR	**0.0071**	**0.0156**	**0.02807**	**0.002266**	**0.005845**

Table 4: Bayes estimates (BEs) and posterior risks (PRs) using the JP under SELF, PLF and DLF with $\theta_1 = 4$, $\theta_2 = 3$, $\theta_3 = 2$, $p_1 = 0.5$, $p_2 = 0.3$, and $t = 0.5, 0.8$.

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	4.68438	4.14788	4.1481	0.486255	0.307003
			PR	**3.28607**	**5.84447**	**112.213**	**0.010047**	**0.009113**
		PLF	BE	5.06787	4.50278	4.9724	0.492724	0.320652
			PR	**0.63799**	**1.05847**	**2.13408**	**0.020695**	**0.02965**
		DLF	BE	5.43172	5.13024	11.0016	0.505493	0.336744
			PR	**0.12292**	**0.2233**	**0.37228**	**0.041414**	**0.090771**
	100	SELF	BE	4.28903	3.44976	2.65577	0.496809	0.303804
			PR	**1.00298**	**1.45326**	**1.90329**	**0.003884**	**0.003721**
		PLF	BE	4.38009	3.58409	3.01029	0.50327	0.309635
			PR	**0.22362**	**0.37814**	**0.60703**	**0.007759**	**0.012199**
		DLF	BE	4.48458	3.75851	3.37003	0.504503	0.316679
			PR	**0.05133**	**0.10363**	**0.19409**	**0.015604**	**0.039164**
	200	SELF	BE	4.17641	3.21003	2.41831	0.497047	0.303295
			PR	**0.53203**	**0.72122**	**0.90182**	**0.002145**	**0.002173**
		PLF	BE	4.19861	3.37741	2.64199	0.502472	0.305365
			PR	**0.12219**	**0.21241**	**0.33145**	**0.004263**	**0.006915**
		DLF	BE	4.25922	3.47091	2.83717	0.503951	0.310409
			PR	**0.02966**	**0.06356**	**0.12632**	**0.008578**	**0.022924**
	500	SELF	BE	4.05903	3.09132	2.26161	0.501055	0.302517
			PR	**0.22268**	**0.3103**	**0.36579**	**0.000935**	**0.000999**
		PLF	BE	4.08058	3.15479	2.32977	0.501268	0.304104
			PR	**0.05472**	**0.09782**	**0.15093**	**0.001872**	**0.003278**
		DLF	BE	4.09179	3.21856	2.42694	0.503379	0.305254
			PR	**0.01337**	**0.03068**	**0.06262**	**0.003715**	**0.01067**
0.8	30	SELF	BE	4.39979	3.52249	3.04594	0.486904	0.306358
			PR	**1.9416**	**2.57549**	**4.29295**	**0.008017**	**0.007075**
		PLF	BE	4.60389	3.91436	3.79902	0.495912	0.316711
			PR	**0.40382**	**0.6224**	**0.98201**	**0.016328**	**0.022631**
		DLF	BE	4.82216	4.34275	3.93999	0.504094	0.327098
			PR	**0.08604**	**0.15228**	**0.25101**	**0.032569**	**0.070579**
	100	SELF	BE	4.13305	3.21866	2.38451	0.498135	0.302131
			PR	**0.55222**	**0.69769**	**0.76144**	**0.002695**	**0.002421**
		PLF	BE	4.2013	3.26838	2.54904	0.499784	0.307532
			PR	**0.13079**	**0.2023**	**0.2869**	**0.005401**	**0.007958**
		DLF	BE	4.22792	3.43384	2.7499	0.502684	0.310692
			PR	**0.03143**	**0.06125**	**0.10953**	**0.010812**	**0.025775**
	200	SELF	BE	4.06307	3.10621	2.24022	0.498726	0.302046
			PR	**0.28071**	**0.35039**	**0.35378**	**0.001388**	**0.001271**
		PLF	BE	4.09626	3.13286	2.34996	0.499902	0.305688
			PR	**0.06796**	**0.10693**	**0.14942**	**0.002775**	**0.004169**
		DLF	BE	4.14565	3.22203	2.39826	0.501271	0.306324
			PR	**0.01668**	**0.03474**	**0.06291**	**0.005545**	**0.013783**
	500	SELF	BE	4.00818	3.0361	2.11174	0.499893	0.301017
			PR	**0.11417**	**0.14553**	**0.1327**	**0.000565**	**0.00053**
		PLF	BE	4.02705	3.07224	2.14838	0.500585	0.301972
			PR	**0.0283**	**0.04686**	**0.06064**	**0.001122**	**0.001745**
		DLF	BE	4.05063	3.09485	2.17791	0.501232	0.302493
			PR	**0.00701**	**0.01538**	**0.02751**	**0.002253**	**0.005804**

TABLE 5: Bayes estimates (BEs) and posterior risks (PRs) using the UP under SELF, PLF and DLF with $\theta_1 = 3$, $\theta_2 = 3$, $\theta_3 = 3$, $p_1 = 0.4$, $p_2 = 0.4$, and $t = 0.5, 0.8$.

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	4.40313	4.53307	5.95249	0.387659	0.38919
			PR	**4.4799**	**5.76279**	**37.4921**	**0.010711**	**0.010679**
		PLF	BE	4.83228	4.79493	6.34263	0.402791	0.401776
			PR	**0.84004**	**0.83197**	**1.72601**	**0.026691**	**0.02691**
		DLF	BE	5.18784	5.26827	7.88663	0.417248	0.416545
			PR	**0.1708**	**0.17043**	**0.26622**	**0.066822**	**0.066982**
	100	SELF	BE	3.60591	3.55661	3.71869	0.392043	0.393056
			PR	**1.13949**	**1.12571**	**2.4805**	**0.004235**	**0.004262**
		PLF	BE	3.61581	3.70818	4.20398	0.402343	0.397494
			PR	**0.28894**	**0.2975**	**0.61595**	**0.010935**	**0.010925**
		DLF	BE	3.82067	3.81653	4.48226	0.405382	0.40486
			PR	**0.07889**	**0.07995**	**0.14895**	**0.027385**	**0.027594**
	200	SELF	BE	3.29032	3.34384	3.46155	0.397624	0.394252
			PR	**0.54156**	**0.56044**	**1.26117**	**0.002457**	**0.002438**
		PLF	BE	3.40206	3.43576	3.64011	0.399253	0.398434
			PR	**0.16153**	**0.16227**	**0.34739**	**0.006222**	**0.006195**
		DLF	BE	3.47577	3.49754	3.76741	0.402404	0.401539
			PR	**0.04622**	**0.04634**	**0.09585**	**0.015387**	**0.015422**
	500	SELF	BE	3.15504	3.17065	3.15331	0.397938	0.397102
			PR	**0.22607**	**0.22703**	**0.51076**	**0.001117**	**0.001112**
		PLF	BE	3.16717	3.21328	3.24307	0.400645	0.399019
			PR	**0.06933**	**0.0708**	**0.15869**	**0.002823**	**0.002816**
		DLF	BE	3.23192	3.22077	3.32458	0.39972	0.400583
			PR	**0.02164**	**0.02164**	**0.04926**	**0.007001**	**0.006998**
0.8	30	SELF	BE	3.86322	3.93524	4.69523	0.391706	0.391028
			PR	**2.02831**	**2.11956**	**7.25194**	**0.007871**	**0.007856**
		PLF	BE	4.17602	4.02812	5.22036	0.401521	0.402691
			PR	**0.46656**	**0.45329**	**0.99891**	**0.019777**	**0.019816**
		DLF	BE	4.39234	4.42823	5.85265	0.412837	0.411405
			PR	**0.11258**	**0.11216**	**0.19054**	**0.04902**	**0.049234**
	100	SELF	BE	3.29736	3.30604	3.50951	0.396915	0.39625
			PR	**0.51465**	**0.5177**	**1.19762**	**0.00272**	**0.002718**
		PLF	BE	3.37232	3.36145	3.64555	0.400763	0.399836
			PR	**0.14863**	**0.14903**	**0.31634**	**0.00682**	**0.006836**
		DLF	BE	3.47538	3.46255	3.71255	0.40264	0.403458
			PR	**0.04403**	**0.04415**	**0.08843**	**0.01708**	**0.017056**
	200	SELF	BE	3.15255	3.17165	3.2727	0.398564	0.39771
			PR	**0.25008**	**0.25334**	**0.57695**	**0.001423**	**0.001421**
		PLF	BE	3.21215	3.21529	3.31454	0.399578	0.399438
			PR	**0.07817**	**0.07801**	**0.16873**	**0.003572**	**0.003568**
		DLF	BE	3.25981	3.22158	3.3948	0.400824	0.402019
			PR	**0.02423**	**0.02431**	**0.05179**	**0.008936**	**0.008931**
	500	SELF	BE	3.0636	3.0904	3.0855	0.399523	0.398409
			PR	**0.0995**	**0.10148**	**0.22632**	**0.000591**	**0.000589**
		PLF	BE	3.09489	3.09174	3.10797	0.399754	0.399125
			PR	**0.03251**	**0.03262**	**0.07271**	**0.00148**	**0.001482**
		DLF	BE	3.10608	3.10378	3.17514	0.400447	0.400649
			PR	**0.0105**	**0.01049**	**0.02349**	**0.003702**	**0.003702**

Table 6: Bayes estimates (BEs) and posterior risks (PRs) using the JP under SELF, PLF and DLF with $\theta_1 = 3$, $\theta_2 = 3$, $\theta_3 = 3$, $p_1 = 0.4$, $p_2 = 0.4$, and $t = 0.5, 0.8$.

t	n	Loss Functions		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
0.5	30	SELF	BE	3.89959	3.901	4.22869	0.384535	0.382467
			PR	**3.70095**	**3.69599**	**11.9292**	**0.010412**	**0.010356**
		PLF	BE	4.3734	4.37845	5.28244	0.395482	0.393811
			PR	**0.80087**	**0.816**	**1.71702**	**0.026299**	**0.026589**
		DLF	BE	4.87943	4.77165	6.57426	0.412955	0.412053
			PR	**0.18071**	**0.18089**	**0.31939**	**0.0681**	**0.067773**
	100	SELF	BE	3.46461	3.50344	3.38452	0.392303	0.389735
			PR	**1.03652**	**1.06684**	**2.23219**	**0.004152**	**0.004119**
		PLF	BE	3.57439	3.60956	3.5893	0.396227	0.394865
			PR	**0.2778**	**0.28172**	**0.56935**	**0.010546**	**0.010559**
		DLF	BE	3.73504	3.72565	3.95071	0.401945	0.40307
			PR	**0.07768**	**0.07743**	**0.15679**	**0.026922**	**0.026651**
	200	SELF	BE	3.28978	3.28116	3.23675	0.394694	0.393651
			PR	**0.53153**	**0.53052**	**1.16381**	**0.002403**	**0.002389**
		PLF	BE	3.36164	3.40011	3.39762	0.398786	0.39571
			PR	**0.1538**	**0.15695**	**0.33629**	**0.005984**	**0.006017**
		DLF	BE	3.43647	3.43024	3.58179	0.399134	0.401394
			PR	**0.04598**	**0.04584**	**0.09893**	**0.015318**	**0.015244**
	500	SELF	BE	3.14877	3.12995	3.11748	0.396248	0.398366
			PR	**0.21966**	**0.2169**	**0.50064**	**0.001093**	**0.001099**
		PLF	BE	3.16797	3.16917	3.19564	0.398918	0.398742
			PR	**0.06801**	**0.06781**	**0.15499**	**0.002755**	**0.002752**
		DLF	BE	3.23925	3.19993	3.23796	0.399442	0.400421
			PR	**0.02118**	**0.02108**	**0.04875**	**0.006867**	**0.006842**
0.8	30	SELF	BE	3.59335	3.60542	3.71188	0.389038	0.390301
			PR	**1.86968**	**1.87405**	**5.29144**	**0.007819**	**0.007833**
		PLF	BE	3.8047	3.86789	4.22313	0.399092	0.398715
			PR	**0.44894**	**0.44658**	**0.96622**	**0.019698**	**0.019802**
		DLF	BE	3.98683	4.05734	4.84202	0.41045	0.410331
			PR	**0.11805**	**0.11755**	**0.22071**	**0.049373**	**0.049349**
	100	SELF	BE	3.23911	3.21045	3.23147	0.395215	0.395792
			PR	**0.4914**	**0.48372**	**1.07248**	**0.002684**	**0.002692**
		PLF	BE	3.28734	3.27828	3.3668	0.399167	0.398653
			PR	**0.14836**	**0.14675**	**0.31384**	**0.00682**	**0.006819**
		DLF	BE	3.37426	3.3912	3.58094	0.403311	0.402756
			PR	**0.04416**	**0.04423**	**0.09255**	**0.016958**	**0.016995**
	200	SELF	BE	3.14486	3.13195	3.08487	0.396773	0.397297
			PR	**0.25002**	**0.2471**	**0.53228**	**0.001419**	**0.001417**
		PLF	BE	3.16184	3.20174	3.17914	0.399149	0.398523
			PR	**0.07717**	**0.07799**	**0.16716**	**0.003563**	**0.003561**
		DLF	BE	3.22993	3.18969	3.27041	0.400146	0.401571
			PR	**0.02444**	**0.02435**	**0.05332**	**0.008952**	**0.008916**
	500	SELF	BE	3.05307	3.04486	3.06066	0.398639	0.399306
			PR	**0.09935**	**0.0986**	**0.2257**	**0.000589**	**0.000588**
		PLF	BE	3.06541	3.07431	3.11049	0.400063	0.399434
			PR	**0.03196**	**0.03216**	**0.07249**	**0.001474**	**0.001477**
		DLF	BE	3.09913	3.09457	3.11135	0.39991	0.400345
			PR	**0.01049**	**0.01047**	**0.0237**	**0.003702**	**0.003695**

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