

# Optimization of Spearman's Rho

## Optimización de Rho de Spearman

SAIKAT MUKHERJEE<sup>1,a</sup>, FARHAD JAFARI<sup>2,b</sup>, JONG-MIN KIM<sup>3,c</sup>

<sup>1</sup>DEPARTMENT OF MATHEMATICS, NATIONAL INSTITUTE OF TECHNOLOGY MEGHALAYA,  
SHILLONG, INDIA

<sup>2</sup>DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WYOMING, LARAMIE, USA

<sup>3</sup>DIVISION OF SCIENCE AND MATHEMATICS, UNIVERSITY OF MINNESOTA-MORRIS, MORRIS,  
USA

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### Abstract

This paper proposes an approximation method to achieve optimum possible values of Spearman's rho for a special class of copulas.

**Key words:** Approximation, Copula, Kendall's Tau, Spearman's Rho.

### Resumen

El artículo propone un método de aproximación para alcanzar los valores óptimos posibles del coeficiente rho de Spearman para algunas clases especiales de cópulas.

**Palabras clave:** aproximación, cópula, tau de Kendall, rho de Spearman.

## 1. Introduction

A study of dependence by using copulas has been getting more attention in the areas of finance, actuarial science, biomedical studies and engineering because a copula does not require a normal distribution and independent, identical distribution assumptions. Furthermore, the invariance property of copula has been attractive in the area of finance. But most copulas including the Archimedean copula family are symmetric functions so that fitting these copulas to asymmetric data is not appropriate. Recently, Liebscher (2008) and Durante (2009) have studied several methods for the construction of asymmetric multivariate copulas. Ambard & Girard (2009) have proposed a generalized FGM copula family and

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<sup>a</sup>Assistant Professor. E-mail: [saikat.mukherjee@nitm.ac.in](mailto:saikat.mukherjee@nitm.ac.in)

<sup>b</sup>Professor. E-mail: [fjafari@uwyo.edu](mailto:fjafari@uwyo.edu)

<sup>c</sup>Professor. E-mail: [jongmink@morris.umn.edu](mailto:jongmink@morris.umn.edu)

discussed the range of Spearman's rho. Rodríguez-Lallena & Úbeda-Flores (2004) and Kim, Sungur, Choi & Heo (2011) investigated new classes of bivariate copulas and studied different measures of association.

The two most important non-parametric measures of association between two random variables are Spearman's rho ( $\rho$ ) and Kendall's tau ( $\tau$ ). In this paper we study bivariate copulas with copula densities of the form  $c(u, v) = 1 + g(u)h(v)$ . We approximate these copulas through a two-parameter family of copulas, in general asymmetric in nature, and show that the Spearman's rho values of these copulas have a range of  $(-\frac{3}{4}, \frac{3}{4})$ . We also extend these results to more general case where  $c(u, v)$  takes the form as  $1 + \sum_{i=1}^n g_i(u)h_i(v)$ .

## 2. Definition and Preliminary

In this section we recall some definitions and results that are necessary to understand a (bivariate) copula. A copula is a multivariate distribution function defined on  $\mathbb{I}^n$ , where  $\mathbb{I} := [0, 1]$ , with uniformly distributed marginals. In this paper, we focus on bivariate (two-dimensional,  $n = 2$ ) copulas.

**Definition 1.** A bivariate copula is a function  $C : \mathbb{I}^2 \rightarrow \mathbb{I}$ , which satisfies the following properties:

$$(P1) \quad C(0, v) = C(u, 0) = 0, \quad \forall u, v \in \mathbb{I}$$

$$(P2) \quad C(1, u) = C(u, 1) = u, \quad \forall u \in \mathbb{I}$$

$$(P3) \quad C \text{ is 2-increasing, i.e., } \forall u_1, u_2, v_1, v_2 \in \mathbb{I} \text{ with } u_1 \leq u_2, v_1 \leq v_2,$$

$$C(u_2, v_2) + C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) \geq 0.$$

The importance of copulas has been growing because of their applications in several fields of research. Their relevance primarily comes from Sklar's Theorem (see (Sklar 1959) and (Sklar 1973) for details): *If  $X$  and  $Y$  are two continuous random variables with joint distribution function  $H$  and marginal distribution functions  $F$  and  $G$ , respectively, then there exists a unique copula  $C$  such that  $H(x, y) = C(F(x), G(y))$  for all  $(x, y) \in \mathbb{R}^2$  and conversely, given a copula  $C$  and two univariate distribution functions  $F$  and  $G$ , the function  $H$  defined above is a joint distribution function with margins  $F$  and  $G$ .* Sklar's theorem clarifies the role that copulas play in the relationship between multivariate distribution functions and their univariate margins. A proof of this theorem can be found in (Schweizer & Sklar 1983).

**Definition 2.** Suppose  $X$  and  $Y$  are two random variables with marginal distribution functions  $F$  and  $G$ , respectively. Then *Spearman's rho* is the ordinary (Pearson) correlation coefficient of the transformed random variables  $F(X)$  and  $G(Y)$ , while *Kendall's tau* is the difference between the probability of concordance  $Pr[(X1 - X2)(Y1 - Y2) > 0]$  and the probability of discordance  $Pr[(X1 - X2)(Y1 - Y2) < 0]$  for two independent pairs  $(X1, Y1)$  and  $(X2, Y2)$  of observations drawn from the distribution.

In terms of dependence properties, Spearman's rho is a measure of average quadrant dependence, while Kendall's tau is a measure of average likelihood ratio dependence (see Nelsen 2006, for details). If  $X$  and  $Y$  are two continuous random variables with copula  $C$ , then Spearman's rho and Kendall's tau of  $X$  and  $Y$  are given by,

$$\rho = 12 \int \int_{\mathbb{I}^2} C(u, v) \, du \, dv - 3 \tag{1}$$

$$\tau = 4 \int \int_{\mathbb{I}^2} C(u, v) \, dC(u, v) - 1 \tag{2}$$

**Definition 3.** A copula  $C$  is called absolutely continuous if, when considered as a joint distribution function,  $C(u, v)$  has a joint density function given by  $c(u, v) := \frac{\partial^2 C}{\partial u \partial v}$  and in that case  $dC(u, v) = \frac{\partial^2 C}{\partial u \partial v} \, du \, dv$ .

Denoting  $c(u, v) - 1$  as  $h(u, v)$ , the following theorem gives a characterization of absolutely continuous copulas (see De la Peña, Ibragimov & Sharakhmetov 2006).

**Theorem 1.** A function  $C : \mathbb{I}^2 \rightarrow \mathbb{I}$  is an absolutely continuous bivariate copula if and only if there exists a function  $h : \mathbb{I}^2 \rightarrow \mathbb{I}$ , satisfying the following conditions,

1. Integrability:  $\int \int_{\mathbb{I}^2} |h(x, y)| \, dx \, dy < \infty$ ,
2. Degeneracy:  $\int_{\mathbb{I}} h(x, \xi) \, d\xi = \int_{\mathbb{I}} h(\xi, y) \, d\xi = 0 \quad \forall x, y \in \mathbb{I}$ ,
3. Positive Definiteness:  $h(x, y) \geq -1 \quad \forall (x, y) \in \mathbb{I}^2$ ,

and such that

$$C(u, v) = \int_0^v \int_0^u (1 + h(x, y)) \, dx \, dy.$$

A copula  $C$  is called *symmetric* if  $C(u, v) = C(v, u)$  for all  $u, v \in \mathbb{I}$ , otherwise *asymmetric*.

Let us denote the independent copulas as  $\Pi(u, v) := uv$ .

### 3. Optimization of Rho

We will assume that the function  $h$ , mentioned in Theorem 1, has the form  $h(u, v) = \varphi(u)\psi(v)$ , where  $\varphi$  and  $\psi$  are continuous real-valued functions on  $\mathbb{I}$ . Therefore  $h$  is continuous on  $\mathbb{I}^2$ . For non-triviality, we assume  $h$  is not identically equal to zero. Then from Theorem 1, integrability of  $h$  becomes obvious, degeneracy of  $h$  implies  $\int_{\mathbb{I}} \varphi(x) \, dx = \int_{\mathbb{I}} \psi(x) \, dx = 0$ , and positive definiteness simplifies to  $\min_{(u,v) \in \mathbb{I}^2} \varphi(u)\psi(v) \geq -1$  and hence

$$C(u, v) = \Pi(u, v) + \int_0^u \varphi(s) \, ds \int_0^v \psi(t) \, dt$$

forms a copula.

Notice that in this special case,  $\rho$  can be simplified into the following form,

$$\rho = 12 \int_0^1 \int_0^u \varphi(s) ds du \int_0^1 \int_0^v \psi(t) dt dv.$$

This suggests that optimizing  $\rho$  is equivalent to optimizing both  $\int_0^1 \int_0^u \varphi(s) ds du$  and  $\int_0^1 \int_0^v \psi(t) dt dv$ .

Define  $G(u) := \int_0^u \varphi(s) ds$  and  $H(v) := \int_0^v \psi(t) dt$ . Then for some positive  $\alpha_1, \alpha_2, \beta_1, \beta_2$ , the optimization problems become,

$$\begin{array}{ll} \max/\min & I_1 := \int_0^1 G(u) du \\ \text{subject to} & G(0) = G(1) = 0 \\ & -\alpha_1 \leq G'(u) \leq \beta_1, \end{array} \quad \begin{array}{ll} \max/\min & I_2 := \int_0^1 H(v) dv \\ \text{subject to} & H(0) = H(1) = 0 \\ & -\alpha_2 \leq H'(v) \leq \beta_2. \end{array}$$

Although it apparently looks like these two optimization problems can be solved independently, they are related by the fact that  $G'(u)H'(v) = \varphi(u)\psi(v) \geq -1$  for all  $(u, v) \in \mathbb{I}^2$ . This implies  $\max\{\alpha_1\beta_2, \alpha_2\beta_1\} \leq 1$ . For the optimal possibility, we choose,  $\beta_2 = (\alpha_1)^{-1}$  and  $\alpha_2 = (\beta_1)^{-1}$ . This is evident from the fact that both  $I_1$  and  $I_2$  can be positive or negative,  $\rho_{\max}$  will occur either if both  $I_1$  and  $I_2$  are maximum or if both are minimum and  $\rho_{\min}$  will occur if one of  $I_1$  and  $I_2$  is maximum and the other is minimum.

Since  $G$  is continuous on  $\mathbb{I}$ , assuming  $-\alpha_1 \leq G' \leq \beta_1$  whenever  $G'$  exists, geometrically,  $I_1$  will be maximum if  $G$  has the form as in Figure 1 and will be minimum if  $G$  has the form as in Figure 2.

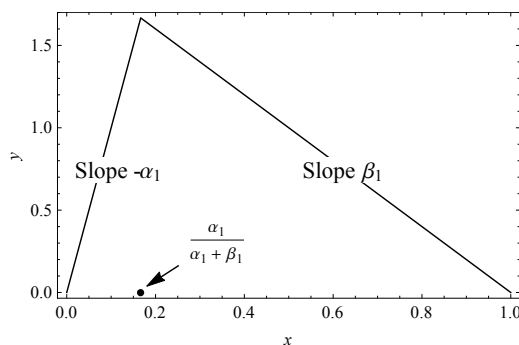


FIGURE 1:  $G$ , Maximizing  $I_1$

One can easily prove that, in order to optimize  $I_1$ ,  $\beta_1$  must be equal to  $\alpha_1$ . For convenience, now onwards we will write  $\alpha$  for  $\alpha_1$ ,  $GM$  for the  $G$  that maximizes  $I_1$  and  $Gm$  for the  $G$  that minimizes  $I_1$ . This suggests that if  $GM(x) = -\alpha|x - 0.5| + 0.5\alpha$  and  $Gm(x) = \alpha|x - 0.5| - 0.5\alpha$ , then  $I_1$  will be maximum and minimum, respectively. But in either case,  $G$  is not differentiable at  $x = 0.5$ , and hence  $\varphi$  is not continuous. To avoid this, we will approximate  $GM$  and  $Gm$  by smooth functions as follows: for arbitrarily small  $\varepsilon_1 > 0$ , define

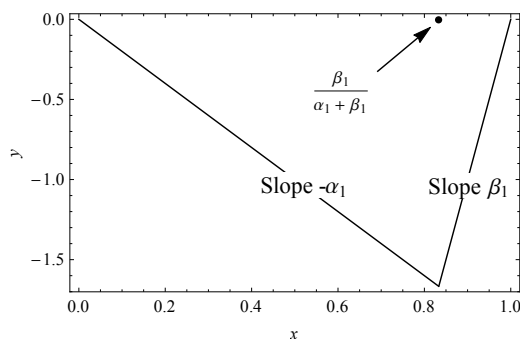


FIGURE 2:  $G$ , Minimizing  $I_1$

$$\widetilde{GM}(x) = -\widetilde{Gm}(x) = \frac{\alpha}{2} \left( \sqrt{1 + 4\varepsilon_1^2} - \sqrt{(1 - 2x)^2 + 4\varepsilon_1^2} \right).$$

It is worth noting that  $\sup_{x \in \mathbb{I}} \{ |\widetilde{GM}(x) - GM(x)|, |\widetilde{Gm}(x) - Gm(x)| \} \rightarrow 0$  as  $\varepsilon_1 \rightarrow 0$  and  $-\alpha \leq \widetilde{GM}'(x), \widetilde{Gm}'(x) \leq \alpha$ . Figures 3 and 4 validate this fact.

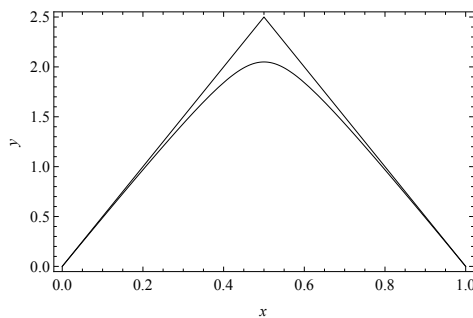


FIGURE 3:  $GM, \widetilde{GM}$  for  $\alpha = 5, \varepsilon_1 = 0.1$

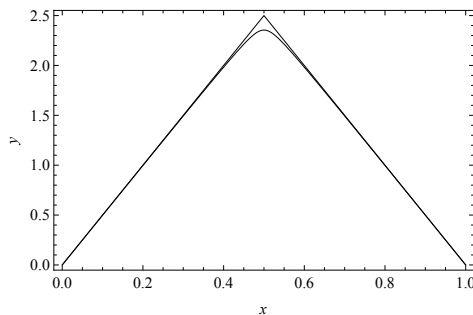


FIGURE 4:  $GM, \widetilde{GM}$  for  $\alpha = 5, \varepsilon_1 = 0.03$

We can similarly, for  $\varepsilon_2 > 0$ , optimize  $I_2$  by approximating maximum and minimum of  $H$  by the following functions

$$\widetilde{HM}(x) = -\widetilde{Hm}(x) = \frac{1}{2\alpha} \left( \sqrt{1 + 4\varepsilon_2^2} - \sqrt{(1 - 2x)^2 + 4\varepsilon_2^2} \right).$$

Hence optimum values of  $\rho$  will be obtained by approximating  $h$  by the following functions,

$$\begin{aligned} h_{\max}^\varepsilon(x, y) &= \widetilde{GM}'(x)\widetilde{HM}'(y) = \widetilde{Gm}'(x)\widetilde{Hm}'(y) \\ &= \frac{(1 - 2x)(1 - 2y)}{\sqrt{(1 - 2x)^2 + 4\varepsilon_1^2}\sqrt{(1 - 2y)^2 + 4\varepsilon_2^2}}, \end{aligned}$$

$$\begin{aligned} h_{\min}^\varepsilon(x, y) &= \widetilde{GM}'(x)\widetilde{Hm}'(y) = \widetilde{Gm}'(x)\widetilde{HM}'(y) \\ &= -\frac{(1 - 2x)(1 - 2y)}{\sqrt{(1 - 2x)^2 + 4\varepsilon_1^2}\sqrt{(1 - 2y)^2 + 4\varepsilon_2^2}}, \end{aligned}$$

where  $\varepsilon = (\varepsilon_1, \varepsilon_2)$ . Notice that each of  $h_{\max}^\varepsilon$  and  $h_{\min}^\varepsilon$  will generate a copula as it satisfies all the hypothesis of Theorem 1 and the corresponding copulas are given by,

$$\begin{aligned} C_{\max}^\varepsilon(u, v) &= \Pi(u, v) \\ &\quad + \frac{1}{4} \left( \sqrt{1 + 4\varepsilon_1^2} - \sqrt{(1 - 2u)^2 + 4\varepsilon_1^2} \right) \left( \sqrt{1 + 4\varepsilon_2^2} - \sqrt{(1 - 2v)^2 + 4\varepsilon_2^2} \right), \\ C_{\min}^\varepsilon(u, v) &= \Pi(u, v) \\ &\quad - \frac{1}{4} \left( \sqrt{1 + 4\varepsilon_1^2} - \sqrt{(1 - 2u)^2 + 4\varepsilon_1^2} \right) \left( \sqrt{1 + 4\varepsilon_2^2} - \sqrt{(1 - 2v)^2 + 4\varepsilon_2^2} \right). \end{aligned}$$

Figures 5 and 6 show the asymmetric behavior of these copulas.

Then corresponding Spearman's rho and Kendall's tau are given by,

$$\begin{aligned} \rho_{\max}^\varepsilon &= \frac{3}{4} \prod_{i=1}^2 \left[ \sqrt{1 + 4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1} \left( \sqrt{1 + 4\varepsilon_i^2} \right) \right] \\ \rho_{\min}^\varepsilon &= -\frac{3}{4} \prod_{i=1}^2 \left[ \sqrt{1 + 4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1} \left( \sqrt{1 + 4\varepsilon_i^2} \right) \right] \\ \tau_{\max}^\varepsilon &= \frac{1}{2} \prod_{i=1}^2 \left[ \sqrt{1 + 4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1} \left( \sqrt{1 + 4\varepsilon_i^2} \right) \right] \\ \tau_{\min}^\varepsilon &= -\frac{1}{2} \prod_{i=1}^2 \left[ \sqrt{1 + 4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1} \left( \sqrt{1 + 4\varepsilon_i^2} \right) \right] \end{aligned}$$

The optimal values of  $\rho$  and corresponding  $\tau$  will be obtained by letting  $(\varepsilon_1, \varepsilon_2) \rightarrow (0, 0)$ . Table 1 shows how the values of  $\rho$  approach the optimal values as  $(\varepsilon_1, \varepsilon_2) \rightarrow (0, 0)$  and it is clear that  $-0.75 \leq \rho \leq 0.75$  and  $-0.5 \leq \tau \leq 0.5$ .

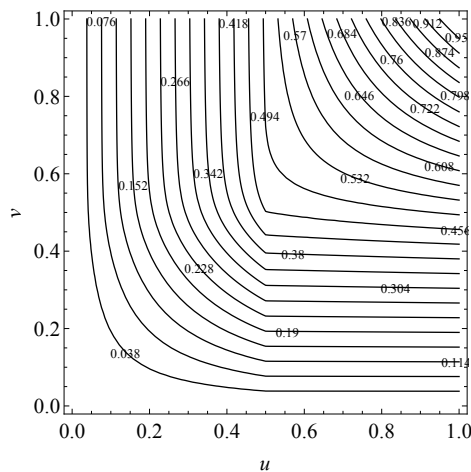


FIGURE 5: Contour plot of  $C_{\max}^\epsilon$  for  $\epsilon = (0.001, 0.1)$

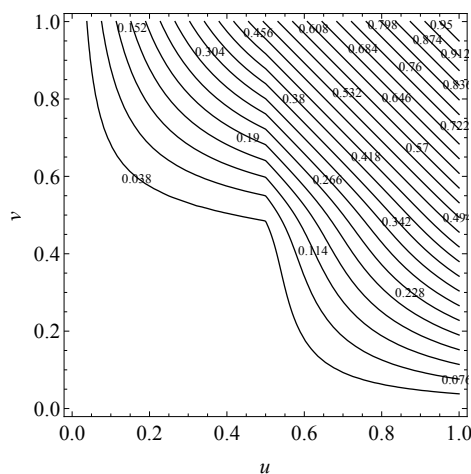


FIGURE 6: Contour plot of  $C_{\min}^\epsilon$  for  $\epsilon = (0.001, 0.1)$

TABLE 1:  $\rho$  and  $\tau$  values as  $\epsilon$  changes.

$\epsilon = (\epsilon_1, \epsilon_2)$	$\rho_{\max}^\epsilon$	$\rho_{\min}^\epsilon$	$\tau_{\max}^\epsilon$	$\tau_{\min}^\epsilon$
(1, 1)	0.0726437	-0.0726437	0.0484292	-0.0484292
(0.1, 0.1)	0.644923	-0.644923	0.429949	-0.429949
(0.01, 0.01)	0.747539	-0.747539	0.498359	-0.498359
(0.001, 0.001)	0.749962	-0.749962	0.499974	-0.499974
(0.0001, 0.0001)	0.749999	-0.749999	0.5	-0.5

### 3.1. General Case: $h(u, v) = \sum_{i=1}^n \varphi_i(u)\psi_i(v)$

The above optimization method can be generalized to the case when the function  $h$  has the form  $h(u, v) = \sum_{i=1}^n \varphi_i(u)\psi_i(v)$ , where, as before,  $\varphi_i, \psi_i$  are continu-

ous real valued functions on  $\mathbb{I}$ . Then  $h$  is automatically integrable. The degeneracy of  $h$ , from Theorem 1, implies

$$\sum_{i=1}^n A_i \varphi_i(x) = 0 = \sum_{i=1}^n B_i \psi_i(x) \quad \forall x \in \mathbb{I}, \quad (3)$$

where  $A_i = \int_0^1 \psi_i(\xi) d\xi$  and  $B_i = \int_0^1 \varphi_i(\xi) d\xi$ ,  $i = 1, 2, \dots, n$ . Since  $\varphi_i$  and  $\psi_i$  are arbitrary, we have from equation (3)

$$\int_0^1 \varphi_i(\xi) d\xi = 0 = \int_0^1 \psi_i(\xi) d\xi \quad \text{for } i = 1, 2, \dots, n.$$

Positive definiteness of  $h$  implies that

$$\min_{(u,v) \in \mathbb{I}^2} \sum_{i=1}^n \varphi_i(u) \psi_i(v) \geq -1.$$

In this case, the copula and the corresponding Spearman's rho will take the following forms,

$$\begin{aligned} C(u, v) &= \Pi(u, v) + \sum_{i=1}^n \int_0^u \varphi_i(s) ds \int_0^v \psi_i(t) dt, \\ \rho &= 12 \sum_{i=1}^n \int_0^1 G_i(u) du \int_0^1 H_i(v) dv, \end{aligned}$$

where  $G_i(u) = \int_0^u \varphi_i(s) ds$  and  $H_i(v) = \int_0^v \psi_i(t) dt$ ,  $i = 1, 2, \dots, n$ . Hence optimization of  $\rho$  leads towards the problems of optimizing the following quantities,

$$I_1^{(i)} := \int_0^1 G_i(u) du, \quad I_2^{(i)} := \int_0^1 H_i(v) dv, \quad i = 1, 2, \dots, n.$$

Then for some positive constants  $\alpha_i, \beta_i, k_i$ , with  $\sum_{i=1}^n k_i \leq 1$ , the optimization problems for  $i = 1, 2, \dots, n$ , become,

$$\begin{array}{ll} \max/\min & I_1^{(i)} \\ \text{subject to} & G_i(0) = G_i(1) = 0 \\ & -\alpha_i \leq G_i'(u) \leq \beta_i, \end{array} \quad \begin{array}{ll} \max/\min & I_2^{(i)} \\ \text{subject to} & H_i(0) = H_i(1) = 0 \\ & -k_i/\beta_i \leq H_i'(v) \leq k_i/\alpha_i. \end{array}$$

Again, as before, the optimal values will occur if  $\alpha_i = \beta_i$ . Since, for every  $i$ ,  $G_i$  and  $H_i$  have similar forms as  $G$  and  $H$  of special case, by a similar approximation



method, as mentioned in the special case, we obtain,

$$\begin{aligned} \rho_{\max}^{\varepsilon} &= \frac{3}{4} \prod_{i=1}^2 \left[ \sqrt{1 + 4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1} \left( \sqrt{1 + 4\varepsilon_i^2} \right) \right] \sum_{j=1}^n k_j \\ \rho_{\min}^{\varepsilon} &= -\frac{3}{4} \prod_{i=1}^2 \left[ \sqrt{1 + 4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1} \left( \sqrt{1 + 4\varepsilon_i^2} \right) \right] \sum_{j=1}^n k_j \\ \tau_{\max}^{\varepsilon} &= \frac{1}{2} \prod_{i=1}^2 \left[ \sqrt{1 + 4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1} \left( \sqrt{1 + 4\varepsilon_i^2} \right) \right] \sum_{j=1}^n k_j \\ \tau_{\min}^{\varepsilon} &= -\frac{1}{2} \prod_{i=1}^2 \left[ \sqrt{1 + 4\varepsilon_i^2} - 4\varepsilon_i^2 \coth^{-1} \left( \sqrt{1 + 4\varepsilon_i^2} \right) \right] \sum_{j=1}^n k_j \end{aligned}$$

Since  $\sum_{i=1}^n k_i \leq 1$ , by taking  $(\varepsilon_1, \varepsilon_2) \rightarrow (0, 0)$  we have  $-0.75 \leq \rho \leq 0.75$  and  $-0.5 \leq \tau \leq 0.5$ .

## 4. Conclusion

We proposed an optimization method to increase the range of Spearman's rho for a special class of copulas and by doing so we generated a two-parameter family of asymmetric copulas.

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