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WIENER MEASURE ON $P_e(G)$

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ABSTRACT Nonstandard methods allow a flat integral representation of de Wiener measure on $P_0(\mathbf{R})$. A representation of the Wiener measure on $P_0(\mathbf{R}^d)$ is given, allowing us to give a nonstandard representation of the Wiener measure on $P_e(G)$ by using Ito map.

0. PRELIMINARIES

For a good introduction of nonstandard analysis we can see (Albeverio, S. (1986)). The main features that we need in our work are the following.

We assume the existence of a set ${}^*\mathbf{R} \supseteq \mathbf{R}$, called the set of the nonstandard real numbers and a mapping $* : V(\mathbf{R}) \to V({}^*\mathbf{R})$, (where $V_1(S) = S$, $V_{n+1}(S) = V_n(S) \cup \mathcal{P}(V_n(S))$ and $V(S) = \cup_{n\in\mathbf{N}} V_n(S)$) with three basic properties. To state the properties we give the following notions.

An elementary statement is a statement Φ built up from "$=$", "$\in$", relations: $u = v$, $u \in v$, the conectives "*and*", "*or*", "*not*", and "*implies*", bounded quantifiers $(\forall u \in v)$, $(\exists u \in v)$.

An internal object A is an element of $V({}^*\mathbf{R})$ such that $A =^* S$, $S \in V(\mathbf{R})$. A set in $V({}^*\mathbf{R})$ which is not internal is called external.

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(1) Extension Principle. $^*\mathbf{R}$ is a proper extension of $\mathbf{R}$ and $* : V(\mathbf{R}) \to V(^*\mathbf{R})$ is an embedding such that $^*r = r$ for all $r \in \mathbf{R}$.

(2) The Saturation Property: Let $\{R_n : n \in \mathbf{N}\}$ be a sequence of internal objects and $\{S_m : m \in \mathbf{N}\}$ be a sequence of internal sets. If for each $m \in \mathbf{N}$ there is an $N_m \in \mathbf{N}$ such that for all $n \geq N_m$, $R_n \in S_m$, then $\{R_n : n \in \mathbf{N}\}$ can be extended to an internal sequence $\{R_\eta : \eta \in {}^*\mathbf{N}\}$ such that $R_\eta \in \cap_m S_m$ for every $\eta \in {}^*\mathbf{N} - \mathbf{N}$.

(2') General Saturation Principle: Let κ be an infinite cardinal. A nonstandard extension is called κ-saturated if for every family $\{X_i\}_{i \in I}$, $card(I) < \kappa$, with the infinite intersection property, the intersection $\cap_{i \in I} X_i$ is nonempty, i.e. this intersection contains some internal object.

(3) Transfer Principle: Let $\Phi(X_1, ..., X_m, x_1, ..., x_n)$ be an elementary statement in $V(\mathbf{R})$. Then, for any $A_1, ..., A_m \subseteq \mathbf{R}$ and $r_1, ..., r_n \in \mathbf{R}$,

$$\Phi(A_1, ..., A_m, r_1, ..., r_n)$$

is true in $V(\mathbf{R})$ if and only if

$$\Phi(^*A_1, ..., ^*A_m, ^*r_1, ..., ^*r_n)$$

is true in $V(^*\mathbf{R})$.

$(^*\mathbf{R}, ^*+, ^*\cdot, ^*\leq)$ extends $\mathbf{R}$ as an ordered field, in general we will omit the * for the operation and the order relation.

In $\mathbf{R}$ we can distinguish three kinds for numbers:

(a) $x \in {}^*\mathbf{R}$ is infinitesimal, if $|x| < r$ for each $r \in \mathbf{R}^+$.

(b) $x \in {}^*\mathbf{R}$ is finite, if there is a real number $r \in \mathbf{R}^+$ such that $|x| < r$.

(c) $x \in {}^*\mathbf{R}$ is infinite number, if $|x| > r$ for each $r \in \mathbf{R}^+$

For each finite number $x \in {}^*\mathbf{R}$ we can associate a unique real $r := st(x) := {}^\circ x$, such that $x = r + \epsilon$, where ϵ is infinitesimal. We say that x is infinitely closed to y, denoted by $x \approx y$ if and only if $x - y$ is infinitesimal.

In general we use capital letters H, F, X, etc. for internal functions and processes, while h, f, x etc. are used for standard ones. For stopping times we will always use capital letters, and specify whether standard or nonstandard is meant.

For given set A, ${}^*A$ stands for the elementary extension of A, and $ns({}^*A)$ denotes the nearstandard points in ${}^*A$. If s is an element in $ns({}^*A)$, the standard part of s is written as $st(s)$, or ${}^\circ s$. For given function f, ${}^*f$ means the elementary extension of f.

We say that the set T is S-dense if $\{{}^\circ\underline{t} : \underline{t} \in T,\ {}^\circ\underline{t} < \infty\} = [0, \infty)$, and $ns(T) = \{\underline{t} \in T :\ {}^\circ\underline{t} < \infty\}$. With T we denote an internal S-dense subset of ${}^*[0, \infty)$. The elements of T, or more generally, of ${}^*[0, \infty)$, are denoted with $\underline{s}$, $\underline{t}$, $\underline{u}$, etc... . The real numbers in $[0, \infty)$ are denoted by s, t, u, etc... We will work with different sets T, so will always specify the definition of such T.

With $\mathbf{N}$ we denote the set of nonzero natural numbers $\{1, 2, 3, ...\}$, and $\mathbf{N}_o = \mathbf{N} \cup \{0\}$. Elements of $\mathbf{N}_o$ are denoted with n, m, l, etc... while, elements in ${}^*\mathbf{N} - \mathbf{N}$ will be denoted with η, $\mathbf{N}$, etc... .

When we say that $F : A \to B$ is an internal function, mean that the domain, the range and the graph of the function are internal concepts.

1. Definition. A subset $A \subseteq {}^*\mathbf{R}$ which is internal and for which there exists $N \in {}^*\mathbf{N}$ and an internal bijection $F : A \to \{0, 1, 2, \ldots, N-1\}$ is called hyperfinite set. In such case A is said to have hyperfinite internal cardinality N, and we write $|A| = N$.

Hyperfinite sets are to the nonstandard universe what the finite sets are to the standard one.

2. Proposition. Let A and B be hyperfinite sets with internal cardinalities H and N, respectively. Then:

i) $A \times B$ is hyperfinite, with $|A \times B| = HN$

ii) $A^B = \{F : B \to A : F \text{ is an internal function}\}$ is a hyperfinite set and its cardinality is H^N.

iii) $A \cup B$, $A \cap B$ are hyperfinite.

iv) If A is hyperfinite and $C \subseteq A$ is an internal set, also C is hyperfinite.

Let ${}^*\bar{\mathbf{R}}_+ = {}^*\mathbf{R} \cup \{0, \infty\}$ be the extended nonnegative hyperreals. An internal finitely additive measure on the internal algebra $\mathcal{U}$ is an internal set function $\mu : \mathcal{U} \to {}^*\bar{\mathbf{R}}_+$, such that

(i) $\mu(\phi) = 0$

(ii) For $A, B \in \mathcal{U}$ with $A \cap B = \phi$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Since μ is internal, the finite additivity extends to hyperfinite unions. Let Ω be a hyperfinite set and let $\mathcal{U}$ be the class of all internal subsets of Ω. Let us define a finitely additive measure ${}^\circ\mu : \mathcal{U} \to {}^*\bar{\mathbf{R}}_+$ by ${}^\circ\mu(A) = {}^\circ(\mu(A))$, where ${}^\circ r = \infty$ when r is an infinitely large element of ${}^*\bar{\mathbf{R}}_+$.

A countable union of sets can be written as a countable disjoint union of sets of the same kind. As have seen in Corollary A2.8 (Muñoz de Özak, M. (1995)), a countable union of disjoint internal sets is not internal. Then, ${}^\circ\mu$ is a σ-additive measure on the algebra of internal hyperfinite subsets of Ω. The Loeb measure is basically the extension v of ${}^\circ\mu$ to the σ-algebra generated by $\mathcal{U}$ by means of the Carathéodory's Extension Theorem.

3. Theorem (Loeb). The extended real valued function $v = L(\mu)$ has a standard σ-additive extension to the smallest (external) σ-algebra $\mathcal{M}$ on Ω containing $\mathcal{U}$. For each $B \in \mathcal{M}$, the value of this extension is given by $v(B) = \inf_{A \in \mathcal{U}, B \subseteq A} {}^\circ\mu(A)$. This extension is unique if $\mu(\Omega) < +\infty$, in which case, for each $B \in \mathcal{M}$, $v(B) = \sup_{A \in \mathcal{U}, B \supseteq A} {}^\circ\mu(A)$ and there is $A \in \mathcal{U}$ with $v(B \Delta A) = v((B - A) \cup (A - B)) = 0$.

For the proof see (Loeb, P. (1975)).

We say that A is Loeb measurable if

$$P_{ex}(B) = \inf_{A \in \mathcal{U}, B \subseteq A} {}^\circ\mu(A) = \sup_{A \in \mathcal{U}, B \supseteq A} {}^\circ\mu(A) = P_{in}(B),$$

and we denote this common value by $L(\mu)$. The collection of all measurable sets is denoted with $L(\Omega)$.

4. Theorem. $(\Omega, L(\Omega), L(\mu))$ is a complete probability space which extends $(\Omega, \mathcal{U}, \mu)$. It is called the Loeb space associated with $(\Omega, \mathcal{U}, \mu)$.

For the proof see A3.2 in the appendix in (Muñoz de Özak, M (1995)).

5. Theorem. (Fubini type) Let $(\Omega_1, \mathcal{U}_1, P_1)$ and $(\Omega_2, \mathcal{U}_2, P_2)$ be hyperfinite probability spaces and let $F : \Omega_1 \times \Omega_2 \to \mathbf{R}$ be a Loeb integrable function. Then:

(i) $f(w_1, \cdot)$ is Loeb integrable for almost all $w_1 \in \Omega_1$.

(ii) $g(w_1) = \int f(w_1, w_2)\, dL(P_2)$ is Loeb integrable on Ω_1.

(iii) $\int f(w_1, w_2)\, dL(P_1 \times P_2) = \int \left(\int f(w_1, w_2)\, dL(P_2)\right) dL(P_1)$.

The proof is due to Keisler. See (Keisler, H.J. (1984)), Theorem 1.14.b)

1. INTRODUCTION

We extend the one dimensional definition of N. Cutland (1990) of the Wiener measure on $P_o(\mathbf{R})$ to $P_o(\mathbf{R}^d)$. This allows to give a nonstandard definition of Wiener measure on Lie algebras. Then by means of Ito's map, we obtain the notion of a nonstandard representation of the Wiener measure on $P_e(G)$, where G is a Lie group.

2. WIENER MEASURE ON $P_e(G)$

Let

$$P_o(\mathbf{R}) = \{x : [0,1] \to \mathbf{R} \mid x \text{ is continuous and } x_o = 0\}$$

and let $\mathcal{C}$ the Borel σ-algebra on $P_o(\mathbf{R})$ ($P_o(\mathbf{R})$ is given with the uniform convergence norm). The Wiener measure μ_o over $(P_o(\mathbf{R}), \mathcal{C})$ is a probability measure such that, for $0 = t_0 < t_1 < \cdots < t_n = 1$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$,

$$\mu_0(x_{t_i} \leq \alpha_i, 1 \leq i \leq n) = \int\limits_{y \leq \alpha} \prod_{i=0}^{n-1} (2\pi(t_{i+1} - t_i))^{-1/2} \exp\left(-\frac{(y_{i+1} - y_i)^2}{2(t_{i+1} - t_i)}\right) dy$$

where $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, $y_0 = 0$ and dy the Lebesgue measure on $\mathbf{R}^n$. μ_0 can be also described as a probability on $(P_o(\mathbf{R}), \mathcal{C})$ making the increments $\left(X_{t_{i+1}} - X_{t_i}\right)_{0 \leq i \leq n-1}$ independent and $N(0, t_{i+1} - t_i)$ distributed. The canonical continuous process given by μ_0 is a Brownian motion.

Let $T = \{0, \Delta t, 2\Delta t, \ldots, 1\}$ be the hyperfinite unit interval. Following Cutland we can make a nonstandard construction of the Brownian motion that gives us an adequate definition of the Wiener measure on $(P_o(\mathbf{R}), \mathcal{C})$ as follows:

Fix an internal probability space $(\Omega, \mathcal{U}, \bar{P})$ carrying independent $N(0,t)$ random variables $(\eta_{\underline{t}})_{\underline{t}\in T}$. Define a process $B : T \times \Omega \to {}^*\mathbf{R}$ by

$$B(0,w) = 0$$

$$\Delta B(\underline{t}, w) = B(\underline{t}, w) - B(\underline{t} - \Delta t, w) = \eta_{\underline{t}}, \qquad \underline{t} \in T.$$

Let $P = L(\bar{P})$. Cutland obtains the following result:

(i) For P-a.a. w, $B(\cdot, w)$ is S-continuous.

(ii) The process $b(\cdot . w) = {}^{o}B(\cdot, w)$ is a brownian motion.

Cutland also shows that this construction of b gives rise to a construction of the Wiener measure that can be expressed as follows: Let Γ be the internal measure on ${}^*\mathbf{R}^T$ induced by B, i.e., for $A \in \mathcal{D}$, where $\mathcal{D}$ is the Borel σ-algebra in ${}^*\mathbf{R}^T$,

$$\begin{aligned}\Gamma(A) &= \bar{P}(B(\cdot, w) \in A) \\ &= (2\pi\Delta t)^{-N/2} \int_A \prod_{\underline{t}\in T} \exp\left(-\frac{(X_{\underline{t}} - X_{\underline{t}-\Delta t})^2}{2\Delta t}\right) dX_{\Delta t} dX_{2\Delta t} \ldots dX_1\end{aligned}$$

with dX_t denoting the *Lebesgue measure over ${}^*\mathbf{R}$. Writting dX for the *Lebesgue measure on ${}^*\mathbf{R}^T$, and

$$\dot{X}_{\underline{t}} = \frac{X_{\underline{t}} - X_{\underline{t}-\Delta t}}{\Delta t} = \frac{\Delta X_{\underline{t}}}{\Delta t},$$

we have

$$\Gamma(A) = (2\pi\Delta t)^{-N/2} \int_A \exp\left(-\frac{1}{2} \sum_{\underline{t}\in T} \dot{X}_{\underline{t}}^2 \Delta t\right) dX$$

and is follows that, with respect to $L(\Gamma)$, X is S-continuous for almost all $X \in {}^*\mathbf{R}^T$, and the Wiener measure on $(P_o(\mathbf{R}), \mathcal{C})$ is given by

$$\mu_0(D) = L(\Gamma)\left(st^{-1}(D)\right), \quad D \in \mathcal{C},$$

where $st^{-1}(D) = \left\{X \in {}^*\mathbf{R}^T : {}^{\circ}X \in D\right\}$.

Now consider

$$P_o\left(\mathbf{R}^d\right) = \left\{x : [0,1] \to \mathbf{R}^d \mid x \text{ continuous and } x_o = 0\right\}$$

and denoted with $\mathcal{C}^d$ the Borel σ-algebra on $P_o\left(\mathbf{R}^d\right)$. The Wiener measure on $\left(P_o\left(\mathbf{R}^d\right), \mathcal{C}^d\right)$ is defined by

$$\mu_0\left(x_{t_i} \in A_i, 1 \le i \le n\right) =$$
$$\int_{A_1} \cdots \int_{A_n} \prod_{i=0}^{n-1} \left(2\pi\left(t_{i+1} - t_i\right)\right)^{-d/2} \exp\left(-\frac{\left\|y_{i+1} - y_i\right\|^2}{2\left(t_{i+1} - t_i\right)}\right) dy_1 \cdots dy_n$$

where $\{t_i : 1 \le i \le n\}$ is a partition of $[0,1]$, $A_i \in \mathcal{B}\left(\mathbf{R}^d\right)$, $\|\alpha\|$ is the length of α and dy_i is the Lebesgue measure on $\mathbf{R}^d$.

Generalizing Cutland's constructions for the Brownian motion, we can construct d independent $B^i(\cdot, w)$ processes such that $b^i(\cdot, w) = {}^{\circ}B^i(\cdot, w)$. Then

$${}^{\circ}B(\cdot, w) = \left(b^1(\cdot . w), \cdots, b^d(\cdot, w)\right)$$

is an $\mathbf{R}^d$ valued Brownian motion. Similarly as for the one dimensional Brownian motion, we can construct a Wiener measure that can be expressed as follows:

$$\begin{aligned}\Gamma^d(D) &= \bar{P}(B(\cdot,w)\in D)\\ &= (2\pi\Delta t)^{-Nd/2}\int_D \exp\left(-\frac{1}{2}\sum_{\underline{t}\in T}\left\|\dot{X}_{\underline{t}}\right\|^2\Delta t\right) dX_{\Delta t}dX_{2\Delta t}\ldots dX_1\end{aligned}$$

Where $D\in\mathcal{D}\times\cdots\times\mathcal{D}$ (d-times), $dX_{\underline{t}}$ denotes the *Lebesgue measure over ${}^*\mathbf{R}^d$, and $\dot{X}_{\underline{t}} = \dfrac{\Delta X_{\underline{t}}}{\Delta t}\in {}^*\mathbf{R}^T$.

Now let $D = D_1\times\cdots\times D_d$, where D_i is an internal Borel set in ${}^*\mathbf{R}^T$. For $i=1,\ldots,d$. This class of sets generates $\mathcal{D}^d$. For $X\in {}^*\left(\mathbf{R}^d\right)^T$, $X=(X^1,\ldots,X^d)$, with $X_i\in {}^*\mathbf{R}^T$. $i=1,\ldots,d$. Applying Theorem 5. (Keisler-Fubini Theorem) we have

$$\begin{aligned}\Gamma(D_1)\cdots\Gamma(D_d) &= (2\pi\Delta t)^{-Nd/2}\left[\int_{D_1}\exp\left(-\frac{1}{2}\sum_{\underline{t}\in T}\left(\dot{X}^1_{\underline{t}}\right)^2\Delta t\right)dX^1_{\Delta t}dX^1_{2\Delta t}\ldots dX^1_1\right]\cdots\\ &\left[\int_{D_d}\exp\left(-\frac{1}{2}\sum_{\underline{t}\in T}\left(\dot{X}^d_{\underline{t}}\right)^2\Delta t\right)dX^d_{\Delta t}dX^d_{2\Delta t}\ldots dX^d_1\right]\\ &= (2\pi\Delta t)^{-Nd/2}\left[\int_{D_1}\cdots\int_{D_d}\exp\left(-\frac{1}{2}\sum_{\underline{t}\in T}\left(\dot{X}^1_{\underline{t}}\right)^2\Delta t\right)\cdots\right.\\ &\left.\exp\left(-\frac{1}{2}\sum_{\underline{t}\in T}\left(\dot{X}^d_{\underline{t}}\right)^2\Delta t\right)dX^1_{\Delta t}\cdots dX^d_{\Delta t}\ldots dX^1_1\cdots dX^d_1\right]\\ &= (2\pi\Delta t)^{-Nd/2}\int_D\exp\left(-\frac{1}{2}\sum_{\underline{t}\in T}\left\|\dot{X}_{\underline{t}}\right\|^2\Delta t\right)dX_{\Delta t}\ldots dX_1\end{aligned}$$

so that for $D = D_1\times\cdots\times D_d$, $D_i\in\mathcal{D}$,

$$\Gamma^d(D) = \Gamma(D_1)\cdots\Gamma(D_d)$$

and for $A = A_1 \times \cdots \times A_d$, with $A_i \in \mathcal{C}$, $i = 1, 2, \cdots, d$,

$$\mu_0^d(A) = \mu_0(A_1) \cdots \mu_0(A_d) = L(\Gamma)\left(st^{-1}(A_1)\right) \cdots L(\Gamma)\left(st^{-1}(A_d)\right)$$

Since the sets $A = A_1 \times \cdots \times A_d$, with $A_i \in \mathcal{C}$, $i = 1, 2, \cdots, d$, generate the Borel σ-algebra $\mathcal{C}^d$, we can extend the definition of μ_0^d to $\mathcal{C}^d$.

Let G be a compact, connected Lie group, and let g be the corresponding Lie algebra. Let us take an Euclidean metric on g which is $Ad(g)$ invariant. This metric induces a Riemannian metric on G. Suppose $\dim G = d$. Using and orthonormal basis,

$$P_o(g) = \{x : [0,1] \to g \mid x \text{ is continuous and } x_o = 0\}$$

is isomorphic to $P_o(\mathbf{R}^d)$. let $P_e(G)$ be the set of $x : [0,1] \to G$ which are continuous , $x_o = e$ and x_t is invertible with respect to the group operation for all $t \in [0,1]$. From Wiener's Theorem we can assume the existence of a Wiener measure on $(P_e(G), \mathcal{B}(P_e(G)))$, where $\mathcal{B}(P_e(G))$ is the Borel σ-algebra on $P_e(G)$, we want to give a nonstandard construction of this Wiener measure.

Following P.Malliavin and M.Malliavin (1990), given $x \in P_o(g)$ and a partition $S = \{t_o, \cdots, t_n\}$ of $[0,1]$, we define $\exp_s(x) = \gamma$ as follows:

$$\begin{aligned} &\gamma(0) = e \\ &\gamma(t) = \gamma(t_{j-1}) \exp\left(\left(\tfrac{t - t_{j-1}}{t_j - t_{j-1}}\right)\left(x(t_j) - x(t_{j-1})\right)\right), \quad t \in [t_{j-1}, t_j] \end{aligned}$$

It is known that when the mesh of S tends to zero μ_o^d a.e., then , the following limit exists in the metric space $P_e(G)$:

$$\lim \exp_s(x) = I(x) \ .$$

The map $x \to I(x)$ is called the Ito map and is a measurable map.

Now consider the space ${}^*g^T$. We know that the nearstandard elements of this space are the S-continuous functions, and also that with respect to $L(\Gamma^d)$, X is S-continuous for almost all $X \in {}^*g^T$. With no loss of generality we can assume that for all $X \in {}^*g^T$, X is S-continuous.

For $X \in {}^*g^T$ define the internal function $Y \in {}^*G$ as follows:

$$Y(0) = e$$
$$Y(\underline{t}) = \prod_{j=0}^{k-1} \exp\left(X_{\underline{t}_{j+1}} - X_{\underline{t}_j}\right)$$

where, $\underline{t} = \underline{t}_k = k\delta t$, $\underline{t} \in T_\eta = T$. Considering ${}^*\gamma$, the elementary extension of γ, defined above, we see that ${}^*\gamma|_T = Y$; and since ${}^*\gamma$ is S-continuous, then Y is S-continuous and so $Y \in {}^*G^T$. Thus, Y is nearstandard in ${}^*G^T$. Also $Y(\underline{t})$ is invertible for all $\underline{t} \in T$, and we can define a map $\overline{\mathcal{I}} : {}^*g^T \to {}^*G^T$, such that $\overline{\mathcal{I}}(X) = Y$.

From the above nonstandard construction of the Wiener measure on $P_o(\mathbf{R}^d)$ and the $\mathbf{R}^d$ valued Brownian motion, we have that

$${}^o\overline{\mathcal{I}}(B(\cdot, w)) = \mathcal{E}({}^oB(\cdot, w)) = I(b(\cdot, w)) ,$$

where $\mathcal{E}$ is the stochastic exponential function defined in Theorem 1.3.8.in (Muñoz de Özak, M. (1995)). Since I is a measurable map, $\overline{\mathcal{I}}$ is a *Borel measurable map. We can define an internal measure on $({}^*G^T, \mathcal{B}({}^*G^T))$ by

$$\nu(A) = \Gamma^d\left(\overline{\mathcal{I}}^{-1}(A)\right)$$

for A Borel subset of ${}^*G^T$.

6. Theorem. For a Borel set B in $P_e(G)$, we can define the Wiener measure $\mu_{P_e(G)}(B)$ as

$$\mu_{P_e(G)}(B) = L(\nu)\left(st^{-1}(B)\right).$$

proof. For B a Borel set in $P_e(G)$ we have

$$\begin{aligned} st^{-1}\left(I^{-1}(B)\right) &= \left\{X \in {}^*g^T : {}^\circ X \in I^{-1}(B)\right\} \\ &= \left\{X \in {}^*g^T : I({}^\circ X) \in B\right\} \end{aligned}$$

and

$$\begin{aligned} \overline{\mathcal{I}}^{-1}\left(st^{-1}(B)\right) &= \overline{\mathcal{I}}^{-1}\left(\left\{Y \in {}^*G^T : {}^\circ Y \in B\right\}\right) \\ &= \left\{X \in {}^*g^T : {}^\circ\overline{\mathcal{I}}(X) \in B\right\} \\ &= \left\{X \in {}^*g^T : I({}^\circ X) \in B\right\} \end{aligned}$$

so that, $st^{-1}\left(I^{-1}(B)\right) = \overline{\mathcal{I}}^{-1}\left(st^{-1}(B)\right)$. Since $\mu_{P_e(G)}(B) = \mu_o^d\left(I^{-1}(B)\right)$ from the nonstandard definition of μ_o^d, we then have

$$\begin{aligned} \mu_{P_e(G)}(B) &= \mu_o^d\left(I^{-1}(B)\right) = L\left(\Gamma^d\right)\left(st^{-1}\left(I^{-1}(B)\right)\right) \\ &= L\left(\Gamma^d\right)\left(\overline{\mathcal{I}}^{-1}\left(st^{-1}(B)\right)\right) = L(\nu)\left(st^{-1}(B)\right) \end{aligned}$$

□

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