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Robust Post-Hoc Multiple Comparisons: Skew t Distributed Error Terms

Compararciones múltiples a posteriori robustas con errores siguendo una t-student sesgada

Nuri Celik[1,a]

[1]Department of Mathematics, Gebze Technical University, Kocaeli, Turkey

Abstract

The pairwise comparisons or post-hoc methods are used for determining the source of the difference of group means in one-way ANOVA. These methods are mostly depend on normality assumption. However, nonnormal distributions are more prevalent than normal distribution. Therefore, robust estimation methods become very important tools in statistical analysis. In this paper, we assume that the distribution of the error terms is Azzalini's skew t and obtain the robust estimators in order to make post-hoc tests in one-way ANOVA. We use maximum likelihood (ML) methodology and compare this methodology with some of robust estimators like M estimator, Wave estimator, trimmed mean and modified maximum likelihood (MML) methodology with Monte Carlo simulation study. Simulation results show that the proposed methodology is more preferable. We also compare power values of the test statistics and conclude that the test statistics based on the ML estimators are more powerful than the test statistics based on other methods.

***Key words*:** One-way ANOVA; Post-Hoc Comparison; Skew t distribution; Robustness.

Resumen

Las comparaciones por pares o métodos post-hoc se utilizan para determinar la fuente de la diferencia de medias de grupo en ANOVA unidireccional. Estos métodos dependen principalmente de la suposición de normalidad. Sin embargo, no normales distribuciones son más frecuentes que la distribución normal. Por lo tanto, los métodos robustos de estimación se convierten en herramientas muy importantes en el análisis estadístico. En este artículo, asumimos que la distribución de los términos de error

[a]Ph.D. E-mail: nuricelik@gtu.edu.tr

es la de Azzalini sesgar t y obtener los estimadores robustos para realizar pruebas post-hoc en ANOVA de una vía. Utilizamos la metodología de máxima verosimilitud (ML) y comparamos esta metodología con algunos de los estimadores robustos como el estimador M, estimador de onda, media recortada y máxima verosimilitud modificada (MML) metodología con estudio de simulación Monte Carlo. Los resultados de la simulación muestran que la metodología propuesta es más preferible. También comparamos potencia valores de las estadísticas de prueba y concluimos que las estadísticas de prueba basadas en los estimadores ML son más poderosos que las estadísticas de prueba basadas en otros métodos.

***Palabras clave*:** ANOVA unidireccional; Comparación post-hoc; Distribución t sesgada; Robustez.

1. Introduction

The analysis of variance (ANOVA) is used for comparing the means of three or more groups. Comparing group means is processed with the help of F statistics obtained by partition of total variance. ANOVA concludes that whether there is a difference between group means, however, the source of the difference cannot be determined. Therefore, multiple comparisons or post-hoc methods are used for understanding the source of the difference of group means. There are many methods in statistics literature such as Least Significant Difference (LSD) method, Tukey's method and linear contrast method.

Most of post-hoc tests depend on normality assumption. If the assumption of normality is not satisfied, the least square (LS) estimators lose their efficiency and the test statistics based on them have lower power values. Therefore, it is very important to develop robust estimators of the parameter of interest. There are many studies about robust multiple comparison test, for example, (Tukey & McLaughlin, 1963) proposed T-methodology based on studentized range distribution with equal sample sizes. Tukey & McLaughlin (1963) and Dunnett (1982) developed this methodology with unequal sample sizes. Ringland (1983) use M estimators, Dunnett (1982) compared many robust estimators like Huber's wave (W24), bisqure (BS82), Hampel (H22), trimmed mean with winsorized standard deviation using 10% trimming (TM10), modified maximum likelihood (MML) estimator with 10% censoring (MML10) and sample mean in a modified T-method for pairwise multiple comparisons. Balci & Akkaya (2015) used MML estimators with using short tailed symmetric distributions and resulted that to avoid W24 and TM10 methodology under short tailed symmetric distributions. In this study we extend the comparisons with using Azzalini's skew t (St) distribution (Azzalini & Capitanio, 2003).

The reason for assuming the St distribution as an error distribution is that it is flexible for modelling the data sets having both skewness and heavy tails Celik (2012), since it reduces to the well known normal, skew normal and Student's t distribution. Additionally, St distribution has a very wide application areas, such as finance, engineering, quality control and medicine, see for example Nakajima (2017); Ahmad Radi et al. (2017); Tagle et al. (2019) and Beranger et al. (2021).

The rest of the paper is organized as follows. In the following section, a brief description of robust post-hoc multiple comparison method is given. In section 3, skew t distribution and its some statistical properties are summarized. In section 4, alternative robust methods and ML methodology is described. In simulation study section, these estimators are compared by using relative efficiency (RE) criterion for some representative values of the distribution parameters. Also, power values of these test statistics are compared. A real data set is analyzed to interpretation of the proposed method. Conclusion is given at the end of this paper.

2. Robust Post-Hoc Multiple Comparisons

Consider the one-way ANOVA model

$$y_{ij} = \mu + \alpha_i + +\epsilon_{ij}, \quad i = 1, 2, \ldots, a; \, j = 1, 2, \ldots, n \tag{1}$$

where y_{ij} is the response corresponding to the jth observation in the ith treatment, μ is the overall mean, α_i is the effect of ith treatment and ϵ_{ij} are the independently and identically distributed (iid) error terms. In one-way ANOVA, our aim is to test the treatment effects. After rejecting the null hypothesis ($\mu_1 = \mu_2 = \cdots = \mu_a$), the post-hoc multiple comparisons methods are used to determine the difference between pairwise treatments. Therefore, the following null hypothesis

$$H_0 = \mu_i = \mu_j, i, j = 1, 2, \ldots, a; \; i \neq j \tag{2}$$

is tested. As mentioned in the previous section, many methods have been used for testing the hypothesis when the normality assumption is not satisfied. [4] proposed the modified T-method with robust estimators of location and scale parameters. The test statistic $\max|\tilde{t}_{ij}|$ is defined as

$$\tilde{t}_{ij} = \frac{\tilde{\mu}_i - \tilde{\mu}_i - (\mu_i - \mu_j)}{\sqrt{\tilde{S}_i^2/n_i + \tilde{S}_j^2/n_j}} \tag{3}$$

where $\tilde{\mu}_i$ is the robust location estimator of the ith treatment, $\tilde{S}_i^2$ is the robust scale estimator of the corresponding treatment and n_i is the sample size. If $\max|\tilde{t}_{ij}| > A^*_{ij,\alpha,a}$ the null hypothesis given in equation (2) is rejected at the level of significance α. $A^*_{ij,\alpha,a}$ represents a constant which is to be chosen so that the desired joint confidence coefficient $1-\alpha$ is achieved as closely as possible (Balci & Akkaya, 2015). In this study the value of $A^*_{ij,\alpha,a}$ is simulated in order to achieve true α value.

3. Skew t Distribution

The probability density function (pdf) of the St distribution has the following form

$$f_{St}(x;\lambda) = 2t_{\nu}(x)T_{v+1}\left(\lambda x\sqrt{\frac{v+1}{v+x^2}}\right) \tag{4}$$

where $t_{\nu}(.)$ is pdf of Student's t distribution with degrees of freedom ν, $T_{v+1}(.)$ is cumulative distribution function (cdf) of Student's t distribution with degrees of freedom $v+1$. The location-scale form of 4 can be written as follows

$$f_{St}(x;\mu,\sigma,\lambda) = \frac{2}{\sigma}t_{\nu}\Big(\frac{x-\mu}{\sigma}\Big)T_{v+1}\left\{\lambda\Big(\frac{x-\mu}{\sigma}\Big)\sqrt{\frac{v+1}{v+(\frac{x-\mu}{\sigma})^2}}\right\} \tag{5}$$

where μ is the location parameter, σ is the scale parameter, ν is the degrees of freedom and λ is the skewness parameter. We will use the notation $X \sim St_{\nu}(\mu,\sigma,\lambda)$ if the random variable X has St distribution with the parameters μ, σ, ν and λ. If $X \sim St_{\nu}(\mu,\sigma,\lambda)$, then the mean and the variance of X are given by

$$E(X) = \sqrt{\frac{\nu}{\pi}}\frac{\lambda\Gamma\left(\frac{v-1}{2}\right)}{\sqrt{\lambda^2+1}\Gamma\left(\frac{\nu}{2}\right)}, \quad \nu > 1 \tag{6}$$

and

$$V(X) = \frac{\nu}{\nu-2} - \frac{\nu}{\pi}\frac{\lambda^2}{1+\lambda^2}\left(\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}\right)^2, \quad \nu > 2, \tag{7}$$

respectively.

It should be also noted that $St_{\nu}(0,1,\lambda)$ reduces to the well-known Student's t distribution when $\lambda = 0$ and to the standard normal distribution when $v \to \infty$ and $\lambda = 0$. For $\nu \to \infty$ and $\lambda > 0$, St distribution reduces to skew normal distribution. This property gains us flexibility for modelling various types of data with symmetric, skewed and heavy tailed. Therefore, St distribution is used in many areas such as finance, medicine, engineering and geography.

4. Alternative Robust Estimators

In this paper, we use alternative robust estimators in order to compare the performances of test statistics given in equation (3). We use sample mean and variance, W24 estimator, trimmed mean with winsorized standard deviation, M estimators and ML estimators (proposed by Ringland, 1983).

The location and scale estimators of Wave method are given as

$$\hat{\mu_W} = T_0 + (hS_0)\tan^{-1}\left[\frac{\sum \sin(zi)}{\sum \cos(zi)}\right] \tag{8}$$

and

$$\hat{\sigma_W} = (hS_0)\tan^{-1}\left[n\frac{\sum \sin^2(zi)}{(\sum \cos(zi))^2}\right]^{1/2} \tag{9}$$

respectively, where $z_i = \frac{y_i - T_0}{hS_0}$, $T_0 = median(y_i)$, $S_0 = median(|y_i - T_0|)$ and $h = 2.4$. The trimmed mean with winsorized standard deviation is defined by (Tukey & McLaughlin, 1963) as follows:

$$\hat{\mu_T} = \frac{\sum_{i=r+1}^{n-r} y_{(i)} + (r - \alpha n)(y_{(r)} + y_{(n-r+1)})}{n(1-2\alpha)} \tag{10}$$

and

$$\hat{\sigma_T} = \frac{\sum_{i=r+1}^{n-r}(y_{(i)} - \hat{\mu_T})^2 + r(y_{(r)} - \hat{\mu_T})^2 + r(y_{(n-r+1)} - \hat{\mu_T})^2}{n(1-2\alpha)^2} \tag{11}$$

where $r = [\alpha n] + 1$, ([.] is the greatest integer), $y_{(i)}$ is i^{th} order statistics and α is the trimming proportion.

An M-estimator of location T_n is defined as the solution of the equation

$$\sum_{i=1}^{n} \phi\Big(\frac{y_i - \mu}{\hat{\sigma_0}}\Big) = 0 \tag{12}$$

where $\hat{\sigma_0}$ is mean absolute devaition, and an M-estimator of scale S_n is defined as the solution of the equation

$$\frac{1}{n}\sum_{i=1}^{n} \rho\Big(\frac{y_i - \mu}{\hat{\sigma}}\Big) = k, 0 < k < \rho(\infty) \tag{13}$$

where the loss function ρ is even, differentiable, and non-decreasing on the positive numbers.

To obtain the ML estimators of the unknown parameters we maximize the following log-likelihood ($\ln L$) function according to the parameters of interest

$$\ln L = n\ln 2 - n\ln\sigma + n\ln\left(\frac{\Gamma(\frac{\nu+1}{2})\nu^{\nu/2}}{\sqrt{\pi}\Gamma(\frac{v}{2})}\right) - \frac{\nu+1}{2}\sum_{i=1}^{n}\ln(\nu + y^2) + \sum_{i=1}^{n}\ln\left(T_{\nu+1}\left(\lambda y\sqrt{\frac{\nu+1}{\nu+y^2}}\right)\right) \tag{14}$$

Taking the derivatives of the $\ln L$ with respect to the parameters of interest and setting them equal to 0, we obtain the following likelihood equations

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \mu} &= \sum_{i=1}^{n} w_i y_i - \lambda \frac{\nu}{\nu+1} \sum_{i=1}^{n} \frac{t_{\nu+1}(\lambda y_i \sqrt{w_i})}{T_{\nu+1}(\lambda y_i \sqrt{w_i})} w_i^{3/2} = 0 \\
\frac{\partial \ln L}{\partial \sigma} &= -n + \sum_{i=1}^{n} w_i y_i - \lambda \frac{\nu}{\nu+1} \sum_{i=1}^{n} \frac{t_{\nu+1}(\lambda y_i \sqrt{w_i})}{T_{\nu+1}(\lambda y_i \sqrt{w_i})} y_i w_i^{3/2} = 0
\end{aligned} \tag{15}
$$

and

$$
\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n} \frac{t_{\nu+1}(\lambda y_k \sqrt{w_i})}{T_{\nu+1}(\lambda y_i \sqrt{w_i})} y_i \sqrt{w_i} = 0.
$$

Here, $w_i = \frac{\nu+1}{\nu+y_i^2}$ and $\psi(x) = \frac{d}{dx}\Gamma(x)$. Because of the intractable functions in the likelihood equations, it is not possible to find the closed form expressions for the ML estimators. Therefore, we resort to iterative methods to solve them numerically. In order to obtain the ML estimates of the unknown parameters in model (1), we use the EM algorithm which is an iterative computational method. The solutions of the likelihood functions and the steps for the iteration method are given in Celik (2012).

5. Simulation Study

In this section, we compare sample mean, W24, TM10, M and ML estimators of location in terms of efficiencies for some representative values of skewness parameter λ and the degrees of freedom ν. To compare the efficiencies of the related estimators under St distribution, we simulated the means and the variances of the maximum difference of $\tilde{\mu_i} - \tilde{\mu}_i,\ i,j = 1,2,\ldots,a; i \neq j$ with $[|100,000/n|]$ times via MATLAB programming. $[|.|]$ denotes rounding a decimal to the nearest integer number. It should be also noted that the RE values are calculated based on LS estimators.

As can be seen from Table 1 that the efficiencies of the ML estimators increases when the sample size (n) and the skewness parameter (λ) increase. On the other hand the efficiencies decreases when the degrees of freedom of St distribution increases. It can be also seen that; the efficiencies of ML estimators are higher than other methods as expected. The efficiencies of W24 and TM10 estimators are lower than efficiencies of the traditional LS methodology under skew t distribution.

We now compare the power of the test statistics for various values of the skewness parameter and the degrees of freedom. Simulation results showing the power comparisons of the proposed tests with the traditional tests are given in Table 2. Table 2 indicates that, the power values of the test statistics obtained by ML methodology has the highest values among other alternatives for all values of λ and ν. On the other hand, the power values of the test statistics based on M methodologies are higher than other robust alternatives under the St distributions.

Table 1: RE values for the ML, W24, TM10 and M estimators.

		ν	3	5	10
$n = 10$	$\lambda = 0$	ML	89	89	90
		$W24$	117	117	118
		$TM10$	115	116	117
		M	98	98	98
$n = 20$		ML	85	86	86
		$W24$	115	116	116
		$TM10$	113	114	115
		M	98	98	99
$n = 10$	$\lambda = 0.5$	ML	85	85	86
		$W24$	115	116	116
		$TM10$	116	116	116
		M	98	98	98
$n = 20$		ML	82	82	83
		$W24$	112	112	112
		$TM10$	110	110	111
		M	97	97	97
$n = 10$	$\lambda = 1.0$	ML	82	83	83
		$W24$	113	113	113
		$TM10$	112	112	113
		M	95	95	95
$n = 20$		ML	77	78	80
		$W24$	109	110	110
		$TM10$	108	109	110
		M	95	95	96

6. Numerical Example

In this section, Acylated Steryl Glucosides (ASG) values of patients are examined after giving three types of blood serum in order to obtain whether there is a difference between serum types with respect to ASG values of patients. Table 3 shows the data.

To clarify the distribution of the error terms, we fit some selected statistical distributions to the data then calculate $\ln L$ and the Akaike information criterion (AIC) values for these distribution. The St distribution provides better fitting to the data then the other distributions with the maximum $\ln L$ and the minimum AIC values. Estimates of the degrees of freedom ν and the skewness parameter λ are obtained to be 7.02 and 0.74 respectively. After processing robust ANOVA assuming that the distribution of error terms as St distribution, we reject the hypothesis saying that there is no difference between blood serum types with respect to ASG values. On the other hand, when the traditional ANOVA process is used, the hypothesis is not rejected, see detailed calculations and explanations at Celik (2012).

To test the null hypothesis $H_0 = \mu_2 = \mu_3$ against $H_0 = \mu_2 \neq \mu_3$, we obtain the values of the test statistic based on LS, ML, M, W24 and TM10 estimators as follows:

Table 2: Power of the test statistics obtained by using alternative methods.

	λ		0					0.4		
					$\nu = 3$					
d	LS	ML	$W24$	$TM10$	M	LS	ML	$W24$	$TM10$	M
0	0.048	0.049	0.041	0.042	0.047	0.047	0.051	0.043	0.045	0.049
1	0.11	0.13	0.10	0.10	0.11	0.12	0.15	0.11	0.12	0.13
2	0.32	0.35	0.31	0.29	0.32	0.31	0.37	0.32	0.28	0.33
3	0.58	0.67	0.54	0.55	0.59	0.51	0.69	0.50	0.49	0.52
4	0.75	0.82	0.70	0.71	0.74	0.72	0.84	0.69	0.68	0.73
5	0.88	0.94	0.87	0.86	0.89	0.81	0.92	0.82	0.81	0.82
6	0.93	0.99	0.91	0.92	0.94	0.91	0.98	0.90	0.89	0.92
	λ			0.7				1.0		
0	0.048	0.049	0.041	0.042	0.047	0.047	0.051	0.043	0.045	0.049
1	0.10	0.14	0.11	0.10	0.12	0.11	0.15	0.11	0.12	0.13
2	0.30	0.35	0.32	0.31	0.33	0.29	0.36	0.30	0.29	0.32
3	0.57	0.68	0.56	0.57	0.59	0.50	0.71	0.51	0.51	0.51
4	0.73	0.83	0.71	0.72	0.73	0.69	0.85	0.68	0.69	0.70
5	0.87	0.94	0.88	0.87	0.88	0.79	0.92	0.82	0.82	0.83
6	0.90	0.99	0.92	0.93	0.93	0.90	0.99	0.91	0.90	0.91
	λ		0					0.4		
					$\nu = 5$					
0	0.048	0.049	0.041	0.042	0.047	0.047	0.051	0.043	0.045	0.049
1	0.12	0.14	0.10	0.10	0.11	0.12	0.14	0.11	0.12	0.12
2	0.32	0.35	0.31	0.29	0.32	0.30	0.38	0.32	0.29	0.32
3	0.55	0.66	0.55	0.55	0.60	0.52	0.70	0.51	0.50	0.54
4	0.75	0.82	0.70	0.71	0.74	0.71	0.85	0.67	0.67	0.72
5	0.87	0.95	0.88	0.86	0.89	0.80	0.93	0.81	0.81	0.83
6	0.92	0.99	0.90	0.90	0.93	0.91	0.98	0.90	0.90	0.93
	λ		0.7					1.0		
0	0.048	0.049	0.041	0.042	0.047	0.047	0.051	0.043	0.045	0.049
1	0.10	0.13	0.10	0.10	0.11	0.12	0.15	0.11	0.12	0.13
2	0.29	0.36	0.28	0.29	0.31	0.31	0.36	0.32	0.28	0.33
3	0.55	0.67	0.50	0.52	0.55	0.51	0.67	0.50	0.49	0.51
4	0.71	0.81	0.63	0.66	0.70	0.72	0.85	0.72	0.71	0.77
5	0.81	0.93	0.79	0.80	0.83	0.80	0.91	0.79	0.79	0.82
6	0.89	0.98	0.95	0.85	0.89	0.92	0.90	0.89	0.89	0.91

Table 3: ASG Data.

Serum I	Serum 2	Serum 3
1.04	1.11	0.73
0.90	0.99	0.71
0.94	1.08	1.06
1.31	1.09	1.00
1.08	0.91	0.88
1.08	1.05	1.03
0.98	1.13	1.05

Since the distribution of the test statistic is not known, the simulated p-values are obtained and given as follows.

Table 4: Estimation of the parameters and the test statistics.

	LS	ML	M	W24	TM10
μ_1	1.04	1.08	1.05	1.02	1.05
μ_2	1.05	1.09	1.08	1.07	1.10
μ_3	0.92	0.95	0.94	0.94	0.96
$\max\lvert t_{13}\rvert$	2.002	2.895	2.711	2.471	2.032

As can be seen from Table 5, the p-value obtained by ML methods is lower than 0.05, the p-value obtained by M method is approximately 0.05 and on the other hand, the others are greater than 0.05. Therefore, we reject the null hypothesis $H_0 = \mu_2 = \mu_3$ with ML methodology, however we do not reject with other methodologies. The p-value obtained by using ML estimators is smallest which implies that ML estimators are the most efficient.

Table 5: p-values of the test statistics.

	LS	ML	M	W24	TM10
p-value	0.114	0.041	0.051	0.072	0.083

7. Conclusion

In this study, robust post-hoc (modified-T) methodology is applied by using LS, ML, W24, TM10 and M estimators under St distribution. Efficiencies and power values of these estimators are compared using Monte Carlo simulation study. The efficiencies based on ML methodology has the greatest values among other alternatives. Similarly, the power values obtained by using ML methodology are the highest than other robust alternatives.

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