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Representing Matrices, M-ideals and Tensor Products of L_1-predual Spaces

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Abstract: Motivated by Bratteli diagrams of Approximately Finite Dimensional (AF) C^*-algebras, we consider diagrammatic representations of separable L_1-predual spaces and show that, in analogy to a result in AF C^*-algebra theory, in such spaces, every M-ideal corresponds to directed sub diagram. This allows one, given a representing matrix of a L_1-predual space, to recover a representing matrix of an M-ideal in X. We give examples where the converse is true in the sense that given an M-ideal in a L_1-predual space X, there exists a diagrammatic representation of X such that the M-ideal is given by a directed sub diagram and an algorithmic way to recover a representing matrix of M-ideals in these spaces. Given representing matrices of two L_1-predual spaces we construct a representing matrix of their injective tensor product.

Key words: representing matrix, generalized diagram, directed sub diagram, M-ideals, tensor products.

AMS *Subject Class.* (2010): 46B25, 46B20.

1. Introduction

In 1971 Lazar and Lindenstrauss (see [3]) introduced notion of representing matrices for separable L_1-predual spaces. The idea to construct representing matrix of a L_1-predual space depends on following result in [3, Theorem 3.2], which essentially says that any separable L_1-predual space is built up by putting together increasing union of ℓ_∞^n, $n = 1, 2, \ldots \infty$'s.

Theorem 1.1. *Let X be a separable infinite dimensional Banach space such that X^* is isometric to $L_1(\mu)$ for some positive measure μ. Let F be a finite dimensional space whose unit ball is a polytope. Then there exists a sequence $\{E_n\}_{n=1}^\infty$ of finite dimensional subspaces of X such that $E_1 \supset F$, $E_{n+1} \supset E_n$ and $E_n = \ell_\infty^{m_n}$ for every n and $X = \overline{\cup_{n=1}^\infty E_n}$.*

We now describe the notion of representing matrices. By Theorem 1.1 any separable L_1-predual space is $\overline{\cup_{n=1}^\infty \ell_\infty^n}$ and different such spaces are constructed depending on how one embeds $\ell_\infty^n \to \ell_\infty^{n+1}$.

Let $\{e_i\}_{i=1}^n$ denote the standard unit vector basis of ℓ_∞^n. By admissible basis of ℓ_∞^n we mean a basis of the form $\{\theta_i e_{\pi(i)}\}_{i=1}^n$ where $\theta_i = \pm 1$ and π is a permutation of $\{1, \dots, n\}$.

It is easy to see that if $\{u_i\}$ is an admissible basis of ℓ_∞^n then for any $m > n$ a linear operator $T : \ell_\infty^n \to \ell_\infty^m$ is an isometry if and only if there exists an admissible basis $\{v_i\}_{i=1}^m$ of ℓ_∞^m such that

$$Tu_i = v_i + \Sigma_{j=n+1}^m a_j^i v_j$$

with $\Sigma_{i=1}^n |a_j^i| \leq 1$ for every $n+1 \leq j \leq m$.

Now for any separable L_1-predual space with the representation $X = \overline{\cup_{n\in\mathbb{N}} E_n}$ where $E_n \subseteq E_{n+1}$ and each E_n is isometric to ℓ_∞^n we may choose admissible basis $\{e_n^i\}_{i=1}^n$ of E_n such that, after relabelling,

$$T_n e_n^i = e_{n+1}^i + a_n^i e_{n+1}^{n+1}$$

with $\Sigma_{i=1}^n |a_n^i| \leq 1$.

A triangular matrix $A = \left(a_n^i\right)_{n\geq 1}^{1\leq i\leq n}$ associated with X in this manner is called a representing matrix of X.

The construction of the representing matrix is best understood in the context of $C(K)$, K is totally disconnected. For use in the later part of this paper, we illustrate this with an example by constructing of representing matrix for such a space.

Let K be a totally disconnected compact metric space. Then there exists a sequence $\{\prod_n\}_{n=1}^\infty$ of partitions of K into disjoint closed sets so that for every n, $\{\prod_n\}$ has n elements, $\{\prod_{n+1}\}$ is a refinement of $\{\prod_n\}$ and

$$\varrho_n = \max_{A\in\prod_n} \mathrm{d}(A) \to 0$$

where $\mathrm{d}(A)$ denotes diameter of A.

Let E_n be the linear span of the characteristic functions of the sets in $\prod_n$. Then it follows trivially that each E_n is isometric to ℓ_∞^n, $E_n \subseteq E_{n+1}$ and $C(K) = \overline{\bigcup_{n=1}^\infty E_n}$. Let us denote $\prod_n = \{K_n^1, K_n^2, \dots, K_n^n\}$ for all $n \in \mathbb{N}$. We may write $1_{K_1^1} = 1_{K_2^1} + 1_{K_2^2}$. Now $\prod_3 = \{K_3^1, K_3^2, K_3^3\}$, $1_{K_2^1} = 1_{K_3^1} + 1_{K_3^3}$ and $1_{K_2^2} = 1_{K_3^2}$. We continue this procedure to get a representing matrix of $C(K)$ which is 0, 1-valued [3, Theorem 5.1].

A L_1-predual space X has a rich collection of structural subspaces of X, namely M-ideals. M-ideals in a L_1-predual space are themselves L_1-preduals and in some sense deterministic for the isometric properties of the space, meaning, any isometric property of a L_1-predual space can be read off from some isometric properties of its M-ideals. On the other hand, representing matrices 'encode' every possible information of the structure of a L_1-predual space.

A separable predual X of L_1 may be thought of as an isometric version (commutative, where *-isomorphism is replaced by linear isometry) of Approximately Finite Dimensional (AF) real C^*-algebras. Two sided norm closed ideals in an AF C^*-algebra are completely determined by hereditary directed sub diagrams of its Bratteli diagram (see [1]). The analogous notion of closed two sided ideals in a C^*-algebra in Banach space category is M-ideals. Here we present a representing diagram of a separable L_1-predual space, the diagram itself arise out of representing matrix of such a space. We show that every directed sub diagram of a representing diagram represents an M-ideal in the corresponding space. Since by definition of representing diagram, it is always hereditary, this is an exact analogy to the corresponding result for AF C^*-algebras. We believe the converse is also true and we establish it in some cases.

We now briefly describe the plan of this paper. In section 2 we present our main idea of diagrammatic representation of a separable L_1-predual space X and directed sub diagram. We show any directed sub diagram corresponds to an M-ideal in X and the residual diagram corresponds to X/M. If M is an M-summand then we show the diagram for X splits into two directed sub diagram. This recovers the result in [7]. We believe that the converse, that any M-ideal in a L_1-predual space X is represented by a directed sub diagram of some diagram is true. However there is a problem here. There are M-ideals which have empty sub diagram. Nevertheless we present converse for $C(K)$ spaces (with extra assumption for general K). We also observe that for $A(K)$ -the space of affine continuous function on K, where K is a separable Poulsen simplex (note that $A(K)$ is isometric to the Gurariy space in this case) given any M-ideal, there exists a diagrammatic representation of corresponding space such that the given M-ideal is represented by a directed sub diagram.

In Section 3 we describe a 'Fill in the Gaps' algorithm for construction of representing matrix from information that $X = \overline{\cup_{n=1}^{\infty} \ell_{\infty}^{m_n}}$. This in one hand provides way to construct representing matrix for an M-ideal given by directed sub diagram and on the other, allows one to write down representing matrix of $X \check{\otimes} Y$, X, Y L_1-preduals, knowing the representing matrix of X and Y. We also show that for $C[0,1]$, given an M-ideal, there exists a diagrammatic representation of $C[0,1]$ such that the given M-ideal is represented by a directed sub diagram.

Through out this work we only consider separable L_1-predual spaces. Recall that a subspace M of Banach space is called an M-ideal if there exists a projection (called L-projection) $P : X^* \to X^*$ such that $\ker P = M^\perp$ and $X^* = \operatorname{Range} P \oplus_1 \ker P$, where $\oplus_1$ denote the ℓ_1-sum. In this case $\operatorname{Range} P$ is isometric to M^*. M is said to be an M-summand in X if $X = M \oplus_\infty N$. Trivially any M-summand is an M-ideal.

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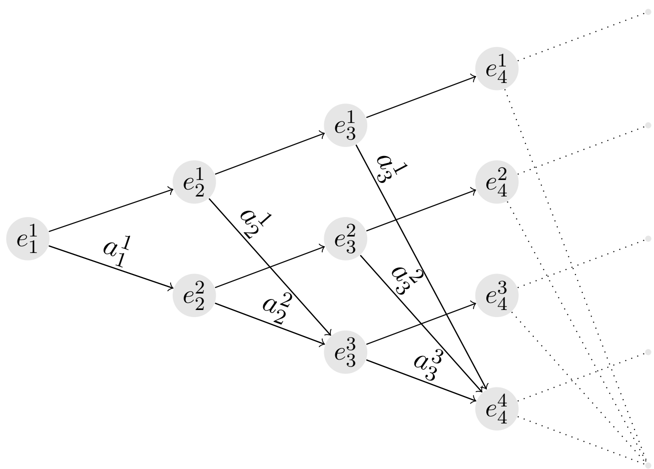
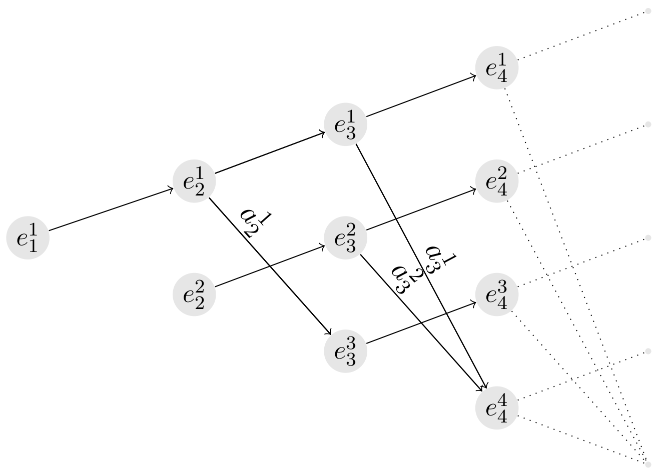
2. Directed diagrams and M-ideals

For a L_1-predual space X with a representing matrix $A = \left(a_n^i\right)_{n\geq 1}^{1\leq i\leq n}$ we will consider the following diagrammatic representation of X.

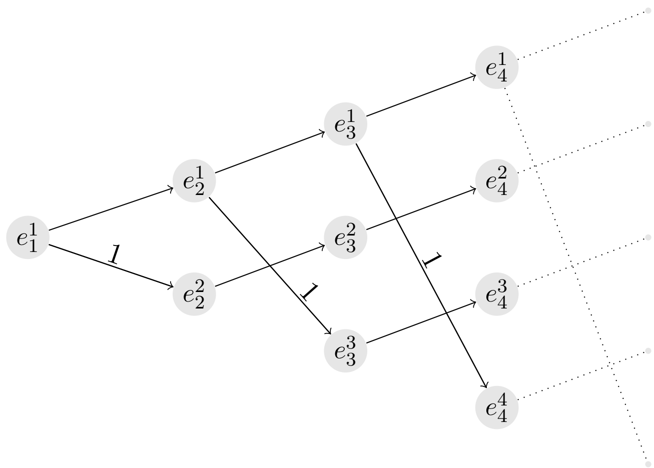
A diagram $\mathcal{D}$ of a L_1-predual space $X = \overline{\cup_{n=1}^\infty \ell_\infty^n}$, and representing matrix $A = \left(a_n^i\right)_{n\geq 1}^{1\leq i\leq n}$ consists of nodes and weighted arrows. The nodes at the n-th level of the diagram are $\{e_n^i : 1 \leq i \leq n\}$ where $\operatorname{span}\{e_n^i : 1 \leq i \leq n\}$ is isometric to ℓ_∞^n, $n \in \mathbb{N}$. For a node e_n^i, there can be at most two arrows from e_n^i one reaching to e_{n+1}^i and another to e_{n+1}^{n+1}. Any arrow from e_n^i to e_{n+1}^i has weight 1 and there is an arrow from e_n^i to e_{n+1}^{n+1}, then it has a weight a_n^i. For example if all $a_n^i \neq 0$ then we have the following diagram:



In case some a_n^i's are zero we do not put arrows from e_n^i to e_{n+1}^{n+1}. For example diagram for a space with $a_1^1, a_2^2, a_3^3 = 0$, will look like the following:



In the following we describe the diagram for the space c with representing matrix A such that $a_n^1 = 1,\ n \geq 1$ and $a_n^j = 0, j \neq 1$ (see [3]):



Note that every representing matrix of a L_1-predual space corresponds to a unique diagram $\mathcal{D}$ and vice-versa. For a given diagram $\mathcal{D}$ we will denote the corresponding space by $X_{\mathcal{D}}$.

Now we will introduce the notion of generalized diagram for a L_1-predual space X, where $X = \overline{\cup X_n}$ and X_n is isometric to $\ell_\infty^{m_n}$ for an increasing sequence (m_n). Let $\{e^1_{m_n}, \ldots, e^{m_n}_{m_n}\}$ be the admissible basis of X_n. Any isometry $T_{m_n} : \ell_\infty^{m_n} \to \ell_\infty^{m_{n+1}}$ is uniquely specified by scalars $(a^i_{m_n+j})$, $1 \leq j \leq m_{n+1} - m_n$, $1 \leq i \leq m_n$ such that

$$T_{m_n} e^i_{m_n} = e^i_{m_{n+1}} + a^i_{m_n+1} e^{m_n+1}_{m_{n+1}} + \cdots + a^i_{m_{n+1}} e^{m_{n+1}}_{m_{n+1}}, \qquad i = 1, 2, \ldots, m_n.$$

For a node $e^i_{m_n}$, there will be one arrow from $e^i_{m_n}$ to $e^i_{m_{n+1}}$. If $a^i_{m_n+j} \neq 0$, then there will be a weighted arrow from $e^i_{m_n}$ to $e^{m_n+j}_{m_{n+1}}$, $1 \leq j \leq m_{n+1} - m_n$ with weight $a^i_{m_n+j}$.

Definition 2.1. A sub diagram $\mathcal{S}$ of $\mathcal{D}$ will be called a directed sub diagram if whenever $e^i_n \in \mathcal{S}$ for some $n, i \in \mathbb{N}$, $i \leq n$ then

(a) $e^i_{n+1} \in \mathcal{S}$,
(b) if $a^i_n \neq 0$, $e^{n+1}_{n+1} \in \mathcal{S}$.

A sub diagram $\mathcal{S} \subseteq \mathcal{D}$ is directed if whenever $e^i_n \in \mathcal{S}$ for some $n, i \in \mathbb{N}$, $i \leq n$ and there is an arrow from e^i_n to e^j_{n+1} then $e^j_{n+1} \in \mathcal{S}$.

We define directed sub diagram of a generalized diagram similarly.

If we take $\mathcal{S} \subseteq \mathcal{D}$, and, $\mathcal{S}$ is directed then the original isometric embedding of X_n into X_{n+1} is preserved (see introduction). Hence $X_{\mathcal{S}}$ will be an isometric subspace of $X_{\mathcal{D}}$. Moreover there exists a norm one projection $P : X^* \to X^*$ with $\ker P = X^\perp_{\mathcal{S}}$. To see this observe that $X_{\mathcal{S}} = \overline{\cup_{n=1}^\infty \ell_\infty^{m_n}}$, hence $X_{\mathcal{S}}$ is itself a L_1-predual space which is an isometric subspace of $X_{\mathcal{D}}$. We prove that for any directed sub diagram $\mathcal{S}$ the space $X_{\mathcal{S}}$ is an M-ideal in $X_{\mathcal{D}}$ and the diagram $\mathcal{D} \setminus \mathcal{S}$ represents the space $X_{\mathcal{D}}/X_{\mathcal{S}}$.

Theorem 2.2. *Let X be a L_1-predual space with a given diagram $\mathcal{D}$. Then for any directed sub diagram $\mathcal{S}$ of $\mathcal{D}$ the subspace $X_{\mathcal{S}}$ is an M-ideal in X.*

Proof. Let $X = \overline{\cup X_n}$, where $X_n \subset X_{n+1}$, X_n is isometric to ℓ^n_∞ for each n. Let $P : X^* \to X^*$ be a norm one projection with $\ker P = {X_{\mathcal{S}}}^\perp$, that is, $X^* = {X_{\mathcal{S}}}^\perp \oplus F$ where $F = \operatorname{Range} P$. We need to prove that $X^* = {X_{\mathcal{S}}}^\perp \oplus_1 F$.

Let $M_n = \operatorname{span}\{e^i_n : e^i_n \in \mathcal{S},\ 1 \leq i \leq n\}$ and $F_n = \operatorname{span}\{e^i_n : e^i_n \notin \mathcal{S},\ 1 \leq i \leq n\}$ for each $n \in \mathbb{N}$.

Then $X_n = M_n \oplus_\infty F_n$ and $X^*_n = M^\perp_n \oplus_1 F^\perp_n$. For any $x^* \in X^*$ we can write $x^* = x^*_1 + x^*_2$ where $x^*_1 \in {X_{\mathcal{S}}}^\perp$ and $x^*_2 \in F$. Then $x^*|_{X_n} = x^*_1|_{X_n} + x^*_2|_{X_n}$ and $||x^*|_{X_n}|| = ||x^*_1|_{X_n}|| + ||x^*_2|_{X_n}||$. For given $\epsilon > 0$, we can choose some $m \in \mathbb{N}$

such that $||x^*|_{X_n}|| \geq ||x^*|| - \epsilon$, $||x_1^*|_{X_n}|| \geq ||x_1^*|| - \epsilon$ and $||x_2^*|_{X_n}|| \geq ||x_2^*|| - \epsilon$ for all $n \geq m$. Now

$$||x_1^*|| + ||x_2^*|| \geq ||x^*|| \geq ||x^*|_{X_n}|| = ||x_1^*|_{X_n}|| + ||x_2^*|_{X_n}|| \geq ||x_1^*|| + ||x_2^*|| - 2\epsilon.$$

Thus it follows that $||x^*|| = ||x_1^*|| + ||x_2^*||$ for all $x^* \in X^*$. From this we can conclude that $X^* = X_{\mathcal{S}}{}^{\perp} \oplus_1 F$. ∎

Remark 2.3. Let X be a L_1-predual space with a given generalized diagram $\mathcal{D}$. Same proof as in Theorem 2.2 shows that directed sub diagram $\mathcal{S}$ of $\mathcal{D}$ represents the subspace $X_{\mathcal{S}}$ which is an M-ideal in X.

Next Theorem is analogous to [1, Theorem III.4.4]) in the case of L_1-predual spaces.

THEOREM 2.4. *Let X be a L_1-predual space with a given diagram $\mathcal{D}$ and $\mathcal{S}$ a directed sub diagram of $\mathcal{D}$. Then the diagram $\mathcal{D} \setminus \mathcal{S}$ represents the space $X/X_{\mathcal{S}}$.*

Proof. Let $X = \overline{\cup X_n}$ where X_n is isometric to ℓ_∞^n. As before, let $M_n = \operatorname{span}\{e_n^i : e_n^i \in \mathcal{S},\ 1 \leq i \leq n\}$ and $F_n = \operatorname{span}\{e_n^i : e_n^i \notin \mathcal{S},\ 1 \leq i \leq n\}$ for each $n \in \mathbb{N}$. Then $X_{\mathcal{S}} = \overline{\cup M_n}$, $M_n = \ell_\infty^m$ for some $m \leq n$, is the M-ideal corresponding to the directed diagram $\mathcal{S}$ and $X_n = M_n \oplus_\infty F_n$. Consider the norm one projection $P_n : X_n \to F_n$ where

$$P_n\left(\sum_{i=1}^n a_i e_n^i\right) = \sum_{e_n^i \notin \mathcal{S}} a_i e_n^i.$$

Let $i_n : F_n \to F_{n+1}$ be the isometry determined by arrows of the diagram $\mathcal{D} \setminus \mathcal{S}$, that is, for $e_n^i \in \mathcal{D} \setminus \mathcal{S}$,

$$\begin{aligned}
&i_n(e_n^i) = e_{n+1}^i + a_n^i e_{n+1}^{n+1} \qquad \text{if } \ e_{n+1}^i,\ e_{n+1}^{n+1} \in \mathcal{D} \setminus \mathcal{S},\\
&i_n(e_n^i) = e_{n+1}^i \qquad \text{if } \ e_{n+1}^i \in \mathcal{D} \setminus \mathcal{S},\ e_{n+1}^{n+1} \notin \mathcal{D} \setminus \mathcal{S},\\
&i_n(e_n^i) = a_n^i e_{n+1}^{n+1} \qquad \text{if } \ e_{n+1}^{n+1} \in \mathcal{D} \setminus \mathcal{S},\ e_{n+1}^i \notin \mathcal{D} \setminus \mathcal{S},\\
&i_n(e_n^i) = 0 \qquad \text{if } \ e_{n+1}^i,\ e_{n+1}^{n+1} \notin \mathcal{D} \setminus \mathcal{S}.
\end{aligned}$$

It is straightforward to verify that $P_{n+1}|_{X_n} = i_n \circ P_n$.

We now define $P : \cup X_n \to \cup F_n$ by $Px = P_n x$ if $x \in X_n$. It follows that P is well defined and extends as a quotient map from X to the space determined by $\overline{\cup F_n}$ which is the space determined by the diagram $\mathcal{D} \setminus \mathcal{S}$. This completes the proof. ∎

We now investigate the converse of Theorem 2.2. Explicitly stated the problem is the following.

PROBLEM 2.5. Let X be a L_1-predual space and M an M-ideal in X. Then there exists a diagram $\mathcal{D}$ representing X and a directed sub diagram $\mathcal{S}$ of $\mathcal{D}$ such that $M = X_{\mathcal{S}}$.

We believe the answer to Problem 2.5 is affirmative. We will present evidences towards this for M-summands in general and M-ideals in some class of L_1-predual spaces.

The following proposition shows that any M-summand in a L_1-predual space is represented by a directed sub diagram.

PROPOSITION 2.6. *Let X be a L_1-predual space and M be an M-summand in X. Then there exists a diagram $\mathcal{D}$ representing X such that M corresponds to some directed sub diagram $\mathcal{S}$ of $\mathcal{D}$.*

Proof. Let N be the complement of M in X, that is, $X = M \oplus_\infty N$. Then by [7, Proposition 2.4] it follows that X has a representing matrix of the form

$$A = \begin{bmatrix} 0 & a_2^1 & 0 & a_4^1 & 0 & a_6^1 & \dots \\ & 0 & a_3^2 & 0 & a_5^2 & 0 & \dots \\ & & 0 & a_4^3 & 0 & a_6^3 & \dots \\ & & & 0 & a_5^4 & 0 & \dots \\ & & & & 0 & a_6^5 & \dots \\ & & & & & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where $B_A = (b_n^i)$ with $b_n^i = a_{2n}^{2i-1}$, $C_A = (c_n^i)$ with $c_n^i = a_{2n}^{2i}$, $n \in \mathbb{N}$, $1 \leq i \leq n$, are the matrices for M and N respectively. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the diagrams corresponding to matrices B_A and C_A respectively. Now it follows that $\mathcal{S}_1$ and $\mathcal{S}_2$ are directed sub diagrams of the diagram of X corresponding to the representing matrix A. ∎

Remark 2.7. Directed sub diagrams $\mathcal{S}_1$ and $\mathcal{S}_2$ considered in Proposition 2.6 are disjoint in the sense that no arrows of $\mathcal{S}_1$ enters into $\mathcal{S}_2$ and vice-versa.

We now consider M-ideals in $C(K)$-spaces. We need to recall few notation and a result from [7].

Let X be a L_1-predual space with $X = \overline{\cup_{n=1}^{\infty} \ell_\infty^n}$ and $\{e_n^i : 1 \leq i \leq n\}$ are admissible bases of ℓ_∞^n, $n \in \mathbb{N}$. Define $\phi_j \in X^*$, $j \in \mathbb{N}$, by

$$\phi_j(e_n^i) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j;\ i = 1, \dots, n;\ j \leq n;\ n \in \mathbb{N}. \end{cases}$$

By $\operatorname{ext} B_{X^*}$ we will denote the extreme point of B_{X^*}.

LEMMA 2.8. [7, Lemma 1.2] *Let X and $\{\phi_j\}$ be as above. Then*

(a) $\phi_j \in \operatorname{ext} B_{X^*}$ *for all* $j \in \mathbb{N}$, *and*

(b) $\overline{\{\pm\phi_i : i \in \mathbb{N}\}} = \overline{\operatorname{ext} B_{X^*}}$, *where closure is taken in weak***-topology of* B_{X^*}.

Remark 2.9. For each i, $\ker \phi_i$ represents the space $X_{\mathcal{S}i}$ for some directed sub diagram $\mathcal{S}_i$ of a given diagram $\mathcal{D}$ of X where the line passing through e_i^i is a part of the diagram $\mathcal{D} \setminus \mathcal{S}_i$.

The idea of the proof for the following result is to use the flexibility provided by Lemma 2.8 for the choice of ϕ_i in a totally disconnected compact metric space K. Recall that for any $C(K)$ space where K is a compact metric space, an M-ideal is given by $J_D = \{f \in C(K) : f|_D = 0\}$, where D is some closed subset of K.

PROPOSITION 2.10. *Let K be a totally disconnected compact metric space and D a closed subset of K. Then there exists a diagram $\mathcal{D}$ representing $C(K)$ and a directed sub diagram $\mathcal{D} \subseteq \mathcal{S}$ such that $J_D = X_{\mathcal{S}}$.*

Proof. Since K be a totally disconnected we can get a sequence $\{\prod_n\}_{n=1}^{\infty}$, $\prod_n = \{K_n^1, K_n^2, \dots, K_n^n\}$ of partitions of K into disjoint closed sets, $\prod_{n+1}$ is a refinement of $\prod_n$ and $\varrho_n = \max_{A \in \prod_n} \operatorname{diam}(A) \to 0$ (see introduction).

Let $D_0 = \{d_n : n \in \mathbb{N}\}$ be a countable dense set in D. Choose $\phi_1 = \delta_{d_1}$. For $n \geq 2$ by renaming the elements in $\prod_n$ we assume that $d_1 \in K_n^1$.

For $n = 2$ if

(a) $D_0 \cap K_2^2 \neq \emptyset$, we find the least n_0 such that $d_{n_0} \in D_0 \cap K_2^2$ and choose $\phi_2 = \delta_{d_{n_0}}$. We will assume for all $n \geq 3$, $d_{n_0} \in K_n^2$, again by possibly renaming the members of $\prod_n$,

(b) otherwise choose and fix any $k \in K_2^2$ and take $\phi_2 = \delta_k$. We will assume for all $n \geq 3$, $k \in K_n^2$.

We will follow the same procedure for $n \geq 3$.

We need to ensure that each d_n will be chosen. Let N be the least number among all k's such that $d_k \in K_n^i$ for some i, n. Let $m \neq N$ and $d_m \in K_n^i$ as well. Since $\operatorname{diam}(K_n^i) \to 0$ we can choose some suitable large $M \in \mathbb{N}$ such that $d_m \in K_M^M$ and m is the least among all k's such that $d_k \in K_M^M$.

So following the algorithm above we define $\phi_M = \delta_{d_m}$.

Let $\mathcal{D}$ be the diagram representing $C(K)$ given by the partition $\{\prod_n\}$ after renaming the elements of $\{\prod_n\}$ as considered above. Since D_0 is dense in D, we have

$$J_D = \bigcap_{d \in D} \ker \delta_d = \bigcap_{d \in D_0} \ker \delta_d.$$

Thus $J_D = X_{\mathcal{S}}$, where $\mathcal{S}$ is the intersection of directed diagrams corresponding to kernel of $\phi_i = \delta_{d_i}$, $d_i \in D_0$. ∎

Next result shows affirmative answer to Problem 2.5 for general $C(K)$ space with additional assumption on an M-ideal. By $\operatorname{int} D$ we mean interior of a set D.

PROPOSITION 2.11. *Let K be any compact metric space and D a closed subset of K such that $D = \overline{\operatorname{int} D}$. Then the M-ideal J_D corresponds to the space $X_{\mathcal{S}}$ for some directed sub diagram $\mathcal{S}$ of given diagram $\mathcal{D}$ of $C(K)$, provided, $\mathcal{S}$ is not an empty diagram.*

Proof. Let $\phi_j = \delta_{k_j}$, $k_j \in K$. Since $\{\phi_j\}$ are weak*-dense in extreme points of the dual unit ball of $C(K)$ and $D = \overline{\operatorname{int} D}$, we have a sub collection $\phi_{j_i} \subseteq \operatorname{int} D$ such that $\phi_{j_i} = \delta_{k_{j_i}}$ and k_{j_i} is dense in D. It follows $J_D = \bigcap \ker \phi_{j_i}$ and hence J_D is represented by the directed sub diagram $\mathcal{S}$ of $\mathcal{D}$ which is generated by intersection of directed sub diagram representing $\ker \phi_{j_i}$, where $\phi_{j_i} = \delta_{k_{j_i}}$. ∎

Remarks 2.12. (1) If we assume K to be a 'nice' compact metric space, then given D a closed subset in K, we can construct a diagrammatic representation of $C(K)$ such that J_D corresponds to a directed sub diagram. We will do it in next section as we need algorithm to construct representing matrix of a L_1-predual space X when it is given in the form $X = \overline{\cup_{n \geq 1} \ell_\infty^{m_n}}$.

(2) Let K be (the) separable Poulsen simplex. Then the space $A(K)$ - the space of real valued affine continuous functions on K is the separable

Gurariy space. It was proved in ([8]) that any infinite dimensional M-ideal in separable Gurariy space is isometric to itself. Thus any representing diagram of the Gurariy space represents M-ideals in it and Problem 2.5 has affirmative solution for the Gurariy space.

We note that an empty diagram is always a directed sub diagram of any given diagram $\mathcal{D}$. It may be the case that an M-ideal in a L_1-predual space corresponds to an empty diagram. We give an easy example towards this.

EXAMPLE 2.13. Consider the matrix A such that $a_n^1 = 1$ for all n and $a_n^i = 0$ for all $i > 1$. It is proved in [3] that A represents c. Consider the M-ideal $J = \{(x_n) \in c : x_n = 0,\ n \geq 2\}$. Then $J = \cap \ker \phi_n$, $\phi_n = \delta_n$, $n \geq 2$. In second figure on page 5, except the line segment starting from the node e_n^n and the line segment that starts from the node e_1^1 and ends at e_{n-1}^1, all the diagram represents the space $\ker \phi_n$. It is straightforward to verify that $\cap_{n \geq 2} \ker \phi_n$ is empty.

Another difficulty in solving Problem 2.5 affirmatively in general is empty diagram may represent a space which is not an M-ideal. We give an example of this in a typical non G-space. Note that an empty diagram is always directed.

EXAMPLE 2.14. Let $X = \left\{ f \in C[1, \omega_0] : f(\omega_0) = \frac{f(1)+f(2)}{2} \right\}$. Then X is a L_1 -predual space which is not a G-space (see [5]). We will consider the following admissible basis for X (see [2]):

$$
\begin{aligned}
&e_1^1 = \left(1, 1, 1, 1, \dots\right), \qquad e_2^1 = \left(1, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots\right), \\
&e_2^2 = \left(0, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots\right), \qquad e_3^1 = \left(1, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots\right), \\
&e_3^2 = \left(0, 1, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots\right), \qquad e_3^3 = \left(0, 0, 0, 1, 0, \dots\right), \qquad \dots .
\end{aligned}
$$

For $n \in [1, \omega_0]$, we denote by J_n the M-ideal $\{f \in C[1, \omega_0] : f(n) = 0\}$. Each J_n is of codimension 1 in $C[1, \omega_0]$. We consider the M-ideal in $C[1, \omega_0]$, $J_{[3,\omega_0]} = \{f \in C[1, \omega_0] : f(n) = 0,\ n \geq 3\}$.

Now consider the subspace $J_{[3,\omega_0]} \cap X = \cap_{n \geq 3} J_n \cap X$ of X. As in Example 2.13 it is easy to check that the intersection of corresponding directed sub diagrams of $J_n \cap X$ for $n \geq 3$ is empty diagram.

However, $J_{[3,\omega_0]} \cap X$ is not an M-ideal in X. To see this we observe that $J_{[3,\omega_0]} \cap X$ is the range of norm one projection $P : X \to X$ given by $P(f) = (f(1), -f(1), 0, 0, \dots)$. Thus if $J_{[3,\omega_0]} \cap X$ is an M-ideal then it is an M-summand as well. So for any $f \in X$, $\|f\| = \max\{\|Pf\|, \|(I - P)f\|\}$.

However if we consider the element $f \in X$ where $f(1) = 1$, $f(2) = 0$ and $f(n) = 1/2$ for all $n \geq 3$, i.e, $f = (1, 0, 1/2, 1/2, 1/2, \dots)$ then

$$\left(1, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots\right) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, 0, \dots\right) + \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots\right)$$

and the norm of both side will not match. Thus $J_{[3,\omega_0]} \cap X$ is not an M-summand in X.

3. Fill in the Gaps

In this section we provide an algorithm to construct representing matrix of a L_1-predual space X where X is given by $X = \overline{\cup X_n}$ and X_n is isometric to $\ell_\infty^{m_n}$ for an increasing sequence (m_n). This construction is implicit in the description of representing matrix given in [3]. However we fix an algorithm (there may be several as seen below) and use it for finding representing matrix of $X \check{\otimes} Y$ - the injective tensor product of two separable L_1-predual spaces, knowing the representing matrices of X and Y.

First we need to provide following justification to our construction.

FACT: Let X be a L_1-predual space such that $X = \overline{\cup X_n}$, where $X_n \subseteq X_{n+1}$ and X_n is isometric to $\ell_\infty^{m_n}$ for some increasing sequence (m_n). If Z is a L_1-predual space with $Z = \overline{\cup Z_n}$, where $Z_n \subseteq Z_{n+1}$, Z_n is isometric to ℓ_∞^n, $Z_{m_n} = X_n$ and the isometry $T_n : X_n \to X_{n+1}$ is same as composition of isometries of Z_{m_n} to Z_{m_n+1}, Z_{m_n+1} to $Z_{m_n+2}, \dots, Z_{m_{n+1}-1}$ to $Z_{m_{n+1}}$ given by the representing matrix of Z, then Z is isometric to X.

We now describe the proposed algorithm.

Let $X = \overline{\cup X_n}$ where for each n, $X_n = \ell_\infty^{m_n}$ with admissible basis $\{e_{m_n}^i\}_{i=1}^{m_n}$. Any isometry T_{m_n} from $\ell_\infty^{m_n}$ to $\ell_\infty^{m_{n+1}}$ in terms of admissible basis is given by

$$T_{m_n} e_{m_n}^i = e_{m_{n+1}}^i + a_{m_n+1}^i e_{m_{n+1}}^{m_n+1} + \ldots + a_{m_{n+1}}^i e_{m_{n+1}}^{m_{n+1}}, \quad i = 1, 2, \dots, m_n . \quad (1)$$

Hence given X as above and isometric embeddings $\ell_\infty^{m_n} \to \ell_\infty^{m_{n+1}}$ we know exactly $(m_{n+1} - m_n)m_n$ numbers of

$$\left(a_j^i\right), \qquad i = 1, \dots, m_n, \quad j = m_n + 1, \dots, m_{n+1} - m_n .$$

Let us assume $C = \left(c_n^i\right)_{n \geq 1}^{1 \leq i \leq n}$ is a representing matrix for X. We will write

$\{e^i_{m_n}\}$ in terms of $\{e^i_{m_{n+1}}\}$ according to isometries given by C:

$$\begin{aligned} e^i_{m_n} &= e^i_{m_n+1} + c^i_{m_n} e^{m_n+1}_{m_n+1} \\ &= e^i_{m_n+2} + c^i_{m_n+1} e^{m_n+2}_{m_n+2} + c^i_{m_n}(e^{m_n+1}_{m_n+2} + c^{m_n+1}_{m_n+1} e^{m_n+2}_{m_n+2}) \\ &= e^i_{m_n+2} + c^i_{m_n} e^{m_n+1}_{m_n+2} + (c^i_{m_n+1} + c^i_{m_n} c^{m_n+1}_{m_n+1}) e^{m_n+2}_{m_n+2} \; = \; \dots \; . \end{aligned}$$

This way we will have $\frac{(m_{n+1}-m_n)}{2}(m_{n+1}+m_n-1)$ numbers of (c^j_n) unknowns. We will put

$$c^{m_n+i}_{m_n+j} = 0\,, \qquad i = 1, \dots, m_{n+1}-m_n\,, \quad 1 \le j \le i\,.$$

It is a straight forward verification that this way we will have $\frac{(m_{n+1}-m_n)}{2}(m_{n+1} -m_n)$ of c^i_n's zero. Thus remaining $(m_{n+1}-m_n)m_n$ of c^i_n's equal the number of known variables a^i_n's and can be expressed in terms of linear equations.

We emphasize that the above way of choosing (c^i_n) is not unique and different ways will give us different representing matrices. Note that here we can not recover first $m_1 - 1$ columns of the representing matrix by the above algorithm so it can be chosen arbitrarily (see [6, Theorem 4.7]).

Remark 3.1. Let the admissible basis of X_n is $\{e^i_{m_n} : 1 \le i \le m_n\}$. If we follow the above algorithm of 'Fill in the Gaps' from X_n to X_{n+1} where X_n is isometric to $\ell^{m_n}_\infty$ and X_{n+1} is isometric to $\ell^{m_{n+1}}_\infty$ then the basis elements $e^{m_n+i}_{m_n+1}, e^{m_n+i}_{m_n+2}, \dots, e^{m_n+i}_{m_{n+1}-1}$ are same as $e^{m_n+i}_{m_{n+1}}$ for all $i \ge 1$.

We illustrate this procedure by considering two special cases. First one is simple trial case with $m_n = 2n$ and our second example provides us with representing matrix of $C[0,1]$ with entries 0 and $\frac{1}{2}$.

EXAMPLE 3.2. Let $C = \left(c^i_n\right)^{1\le i\le n}_{n\ge 1}$ be a representing matrix of X and $X_n = \operatorname{span}\left\{e^1_{2n}, \dots, e^{2n}_{2n}\right\}$ and $T_n : X_n \to X_{n+1}$ is an isometric embedding with

$$e^i_{2n} = e^i_{2(n+1)} + a^i_{2n+1} e^{2n+1}_{2(n+1)} + a^i_{2(n+1)} e^{2(n+1)}_{2(n+1)}\,, \qquad 1 \le i \le 2n\,,\ n \ge 1\,.$$

If we write the expression for e^i_{2n} according to the matrix C then we get

$$e^i_{2n} = e^i_{2(n+1)} + c^i_{2n} e^{2n+1}_{2(n+1)} + \left(c^i_{2n+1} + c^i_{2n} c^{2n+1}_{2n+1}\right) e^{2(n+1)}_{2(n+1)}\,.$$

From above two expressions for e^i_{2n} we have $a^i_{2n+1} = c^i_{2n}$ and $a^i_{2(n+1)} = c^i_{2n+1} + c^i_{2n} c^{2n+1}_{2n+1}$.

Now if we proceed by above algorithm and put $c_{2n+1}^{2n+1} = 0$, $n \in \mathbb{N}$ we get $a_{2n+1}^{i} = c_{2n}^{i}$, $a_{2(n+1)}^{i} = c_{2n+1}^{i}$, $1 \le i \le 2n$, $n \ge 1$, and, we have the following representing matrix for X,

$$C = \begin{bmatrix} - & a_3^1 & a_4^1 & a_5^1 & a_6^1 & \dots \\ \vdots & a_3^2 & a_4^2 & a_5^2 & a_6^2 & \dots \\ \vdots & \vdots & 0 & a_5^3 & a_6^3 & \dots \\ \vdots & \vdots & \vdots & a_5^4 & a_6^4 & \dots \\ \vdots & \vdots & \vdots & \vdots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The Fact stated above indeed justifies that the resulting matrix is a representing matrix of X.

EXAMPLE 3.3. Consider the function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(t) = 1 + t$ for $t \in [-1, 0]$, $\phi(t) = 1 - t$ for $t \in [0, 1]$, and $\phi(t) = 0$ for $t \notin [-1, 1]$. Define $g_{k,2^n} = \phi(2^n t - k)$, $t \in [0, 1]$. We can write $C[0, 1] = \overline{\cup X_n}$, $X_n = \operatorname{span}\{g_{k,2^n} : k = 0, 1, \dots, 2^n\}$ where $\{g_{k,2^n} : k = 0, 1, \dots, 2^n\}$ is an admissible basis of X_n. Then for all $n = 0, 1, \dots$ and $k = 1, 2, \dots, 2^n - 1$ we have (see [4])

$$\begin{aligned} g_{k,2^n} &= \frac{1}{2} g_{2k-1,2^{n+1}} + g_{2k,2^{n+1}} + \frac{1}{2} g_{2k+1,2^{n+1}} \, , \\ g_{0,2^n} &= g_{0,2^{n+1}} + \frac{1}{2} g_{1,2^{n+1}} \, , \\ g_{2^n,2^n} &= \frac{1}{2} g_{2^{n+1}-1,2^{n+1}} + g_{2^{n+1},2^{n+1}} \, . \end{aligned}$$

Let $C = \left(c_n^i\right)_{n \ge 1}^{1 \le i \le n}$ be a representing matrix of $C[0, 1]$. First we have to write the expression for $g_{k,2^n}$ according to C. Now comparing the equations with the above and put $c_{2^n+i}^{2^n+j} = 0$, $1 \le i \le 2^{n+1} - 2^n - 1$, $1 \le j \le i$, we will get a representing matrix of $C[0, 1]$ with entries 0 and $\frac{1}{2}$ only.

We now answer Problem 2.5 in affirmative for $C[0, 1]$.

THEOREM 3.4. *Let D be a closed subset of* $[0, 1]$. *Then there exists a diagram $\mathcal{D}$ representing $C[0, 1]$ such that the M-ideal J_D corresponds to the space $X_{\mathcal{S}}$ for some directed sub diagram $\mathcal{S}$ of $\mathcal{D}$, provided $\mathcal{S}$ is not an empty diagram.*

Proof. Let $D_0 = \{d_n : n \in \mathbb{N}\}$ be a countable dense set in D. We can extend D_0 to a set $M = \{k_i : i \in \mathbb{N}\}$ such that $\overline{M} = [0,1]$. Consider $e_2^1 = 1-t$, $t \in [0,1]$ and $e_2^2 = t$, $t \in [0,1]$. Without loss of generality choose an element $k_1 \in [0,1]$ and consider

$$\begin{aligned}
&e_3^1 = 1 - \frac{1}{k_1}t \quad \text{if } t \in [0,k_1]\,, && e_3^1 = 0 \quad \text{if } t \in [k_1,1]\,;\\
&e_3^2 = 0 \quad \text{if } t \in [0,k_1]\,, && e_3^2 = \frac{t-k_1}{1-k_1} \quad \text{if } t \in [k_1,1] \quad \text{and}\\
&e_3^3 = \frac{1}{k_1}t \quad \text{if } t \in [0,k_1]\,, && e_3^3 = \frac{1-t}{1-k_1} \quad \text{if } t \in [k_1,1]\,.
\end{aligned}$$

Here e_2^1, e_2^2, e_3^1, e_3^2, e_3^3 satisfy the following equations; $e_2^1 = e_3^1 + (1-k_1)e_3^3$, $e_2^2 = e_3^2 + k_1 e_3^3$. Now with out loss of generality choose $k_2 \in [0,k_1]$ and $k_3 \in [k_1,1]$. Consider

$$\begin{aligned}
&e_5^1 = 1 - \frac{1}{k_2}t \quad \text{if } t \in [0,k_2]\,, && e_5^1 = 0 \quad \text{if } t \in [k_2,1]\,;\\
&e_5^2 = 0 \quad \text{if } t \in [0,k_3]\,, && e_5^2 = \frac{t-k_3}{1-k_3} \quad \text{if } t \in [k_3,1]\,;\\
&e_5^3 = 0 \quad \text{if } t \in [0,k_2]\,, && e_5^3 = \frac{t-k_2}{k_1-k_2} \quad \text{if } t \in [k_2,k_1]\,,\\
&e_5^3 = \frac{k_3-t}{k_3-k_1} \quad \text{if } t \in [k_1,k_3]\,, && e_5^3 = 0 \quad \text{if } t \in [k_3,1]\,;\\
&e_5^4 = \frac{1}{k_2}t \quad \text{if } t \in [0,k_2]\,, && e_5^4 = \frac{k_1-t}{k_1-k_2} \quad \text{if } t \in [k_2,k_1]\,,\\
&e_5^4 = 0 \quad \text{if } t \in [k_1,1] \quad \text{and} && e_5^5 = 0 \quad \text{if } t \in [0,k_1]\,,\\
&e_5^5 = \frac{t-k_1}{k_3-k_1} \quad \text{if } t \in [k_1,k_3]\,, && e_5^5 = \frac{1-t}{1-k_3} \quad \text{if } t \in [k_3,1]\,.
\end{aligned}$$

By the construction e_3^1, e_3^2, e_3^3, e_5^1, e_5^2, e_5^3, e_5^4, e_5^5 satisfy the following equations:

$$e_3^1 = e_5^1 + \frac{k_1-k_2}{k_2}e_5^4\,, \quad e_3^2 = e_5^2 + \frac{k_3-k_1}{1-k_1}e_5^5 \quad \text{and} \quad e_3^3 = e_5^3 + \frac{k_2}{k_1}e_5^4 + \frac{1-k_3}{1-k_1}e_5^5\,.$$

Similarly we can construct $e_{2^n+1}^i$, $1 \le i \le 2^n+1$. Take an element $f \in C[0,1]$. Define a sequence $(p_n)_{n=0}^\infty$ in the following way. Let $p_0 = f(0)e_2^1$,

$$\begin{aligned}
&p_1 = p_0 + (f(1) - p_0(1))e_2^2\,, && p_2 = p_1 + (f(k_1) - p_1(k_1))e_3^3\,,\\
&p_3 = p_2 + (f(k_2) - p_2(k_2))e_5^4\,, && p_4 = p_3 + (f(k_3) - p_3(k_3))e_5^5\,,
\end{aligned}$$

and so on. Here p_0 and f takes the same value at 0 while p_1 and f takes the same value at 0 and 1 and interpolates linearly in between, p_2 and f takes same value at 0, 1 and k_1 and interpolates linearly in between, and so on. It is straightforward to check that $\lim_n ||p_n - f||_\infty = 0$. Therefore we can write $C[0,1] = \overline{\cup E_n}$, where

$$E_n = \operatorname{span}\left\{e^i_{2^n+1} : 1 \le i \le 2^n + 1\right\}$$

and E_n is isometric to $\ell_\infty^{2^n+1}$. We know that the support of $e^i_{2^n+1}$ is going to zero as n approaches to infinity and it consists a single element of $\{k_i : i \in \mathbb{N}\}$. Each $i \in \mathbb{N}$, k_i will be in some $T_{j_i} = \cap_{n=1}^\infty \operatorname{supp}\{e^j_{2^n+1}\}$ and any two T_{j_i}'s are disjoint. Here we consider the generalized diagram of $C[0,1]$ with respect to above basis and from n-th to $(n+1)$-th step we choose 2^{n-1} of k_i's and these k_i's lie in the support of exactly one of the basis elements of $e^{2^n+1}_{2^{n+1}}, \ldots, e^{2^{n+1}}_{2^{n+1}}$. Now if we follow the algorithm for 'Fill in the Gaps' from n-th to $(n+1)$-th step and consider $k_i \in \operatorname{supp}(e^{2^n+j}_{2^{n+1}})$ chosen above, then $k_i \in \operatorname{supp}(e^{2^n+j}_m)$ and $k_i \notin \operatorname{supp}(e^l_m)$, $l \ne 2^n + j$ for $2^n + 1 \le m \le 2^{n+1} - 1$, $j \ge 1$ (see Remark 3.1). So by following the same procedure of choosing ϕ_i as in Proposition 2.10 we will get for any k_i there exists a ϕ_m such that $\phi_m = \delta_{k_i}$ and the set $\{k_i\}_{i=1}^\infty$ is dense in $[0,1]$ (see [2, Lemma 2]). Given that D_0 is dense in D so $J_D = \cap_{d\in D} \ker \delta_d = \cap_{d\in D_0} \ker \delta_d$. Thus $J_D = X_{\mathcal{S}}$, where $\mathcal{S}$ is the intersection of directed diagrams corresponding to kernel of $\phi_i = \delta_{d_i}$, $d_i \in D_0$. This completes the proof. ∎

Representing matrix for $X \check{\otimes} Y$: If X and Y are separable L_1-predual spaces, then it is known that $X\check{\otimes}Y$ is also a separable L_1-predual space. We adopt the above algorithm to find a representing matrix for $X\check{\otimes}Y$. Let X and Y has representing matrices $\left(a^i_n\right)^{1\le i\le n}_{n\ge 1}$ and $\left(b^i_n\right)^{1\le i\le n}_{n\ge 1}$ respectively corresponding to the admissible basis $\{e^i_n : 1 \le i \le n\}$ and $\{f^i_n : 1 \le i \le n\}$. Then $X\check{\otimes}Y = \overline{\cup_{n=1}^\infty E_{n^2}}$, where E_{n^2} is isometric to $\ell_\infty^{n^2}$ with admissible bases $\{e^i_n \otimes f^j_n,\ i = 1, \ldots, n;\ j = 1, \ldots, n\}$. We will denote this collection as $\left\{E^i_{n^2} : 1 \le i \le n^2\right\}$ with the following convention:

(a) First n^2 terms of the admissible basis of $E_{(n+1)^2}$ is same as the admissible basis of E_{n^2}. For example if $E^i_{n^2} = e^k_{(n-1)^2} \otimes e^l_{(n-1)^2}$ then $E^i_{(n+1)^2} = e^k_{n^2} \otimes e^l_{n^2}$.

(b) We will choose $\left\{E^{n^2+i}_{(n+1)^2} : 1 \le i \le (n+1)^2 - n^2\right\}$ by the following way. Take $E^{n^2+1}_{(n+1)^2} = e^1_{n+1} \otimes f^{n+1}_{n+1}$, $E^{n^2+2}_{(n+1)^2} = e^{n+1}_{n+1} \otimes f^1_{n+1}$. For $i = 2k + 1$, $k \in \mathbb{N}$, $E^{n^2+2k+1}_{(n+1)^2} = e^k_{n+1} \otimes f^{n+1}_{n+1}$ and for $i = 2k + 2$, $k \in \mathbb{N}$, $E^{n^2+2k+2}_{(n+1)^2} = e^{n+1}_{n+1} \otimes f^k_{n+1}$.

We will now follow algorithm for 'Fill in the Gaps' described above. Let us illustrate this with first few steps.

Let $C = \left(c_n^i\right)_{n\geq 1}^{1\leq i\leq n}$ be the representing matrix of $X\check{\otimes}Y$. According to the above convention $E_1^1 = e_1^1 \otimes f_1^1$ and $E_4^1 = e_2^1 \otimes f_2^1$, $E_4^2 = e_2^1 \otimes f_2^2$, $E_4^3 = e_2^2 \otimes f_2^1$, $E_4^4 = e_2^2 \otimes f_2^2$. By expanding E_1^1 in terms of $\{E_4^i\}_{i=1}^4$ according to the given representing matrix of X and Y we get

$$E_1^1 = E_4^1 + b_1^1 E_4^2 + a_1^1 E_4^3 + a_1^1 b_1^1 E_4^4\,.$$

Similarly expansion of E_1^1 in terms of $\{E_4^i\}_{i=1}^4$ according to the representing matrix C of $X\check{\otimes}Y$,

$$E_1^1 = E_4^1 + c_1^1 E_4^2 + (c_2^1 + c_1^1 c_2^2) E_4^3 + (c_2^1 c_3^3 + c_1^1 c_3^2 + c_3^1 + c_1^1 c_2^2 c_3^3) E_4^4.$$

By following the algorithm we will get $c_1^1 = b_1^1$, $c_2^1 = a_1^1$, $c_2^2 = 0$, $c_3^1 = a_1^1 b_1^1$, $c_3^2 = 0$, $c_3^3 = 0$. By expanding $\{E_4^i\}_{i=1}^4$ in terms of $\{E_9^i\}_{i=1}^9$ according to given representing matrices for X, Y and matrix C we will get

$$\begin{aligned}
&c_4^1 = b_2^1\,, \quad c_4^2 = b_2^2\,, \quad c_4^3 = 0\,, \quad c_4^4 = 0\,,\\
&c_5^1 = a_2^1\,, \quad c_5^2 = 0\,, \quad c_5^3 = a_2^2\,, \quad c_5^4 = 0\,, \quad c_5^5 = 0\,,\\
&c_6^1 = 0\,, \quad c_6^2 = 0\,, \quad c_6^3 = b_2^1\,, \quad c_6^4 = b_2^2\,, \quad c_6^5 = 0\,, \quad c_6^6 = 0\,,\\
&c_7^1 = 0\,, \quad c_7^2 = a_2^1\,, \quad c_7^3 = 0\,, \quad c_7^4 = a_2^2\,, \quad c_7^5 = 0\,, \quad c_7^6 = 0\,, \quad c_7^7 = 0\,,\\
&c_8^1 = a_2^1 b_2^1\,, \quad c_8^2 = a_2^1 b_2^2\,, \quad c_8^3 = a_2^2 b_2^1\,, \quad c_8^4 = a_2^2 b_2^2\,, \quad c_8^5 = 0\,,\\
&\qquad\qquad c_8^6 = 0\,, \quad c_8^7 = 0\,, \quad c_8^8 = 0\,.
\end{aligned}$$

Proceeding as above we will get representing matrix of $X\check{\otimes}Y$ as

$$C = \begin{bmatrix}
b_1^1 & a_1^1 & a_1^1 b_1^1 & b_2^1 & a_2^1 & 0 & 0 & a_2^1 b_2^1 & \dots\\
\vdots & 0 & 0 & b_2^2 & 0 & 0 & a_2^1 & a_2^1 b_2^2 & \dots\\
\vdots & \vdots & 0 & 0 & a_2^2 & b_2^1 & 0 & a_2^2 b_2^1 & \dots\\
\vdots & \vdots & \ddots & 0 & 0 & b_2^2 & a_2^2 & a_2^2 b_2^2 & \dots\\
\vdots & \vdots & \ddots & \ddots & 0 & 0 & 0 & 0 & \dots\\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 & 0 & \dots\\
\vdots & \vdots & \vdots & \vdots & & \ddots & 0 & 0 & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots
\end{bmatrix}.$$

Remark 3.5. From above description of representing matrix for $X\check{\otimes}Y$ we can actually read off representing matrices $\left(a_n^i\right)_{n\geq 1}^{1\leq i\leq n}$ and $\left(b_n^i\right)_{n\geq 1}^{1\leq i\leq n}$ for X and Y respectively. For example representing matrix of Y is given by

$$B=\begin{bmatrix} c_1^1 & c_4^1 & c_9^1 & c_{4^2}^1 & \cdots & & & c_{n^2}^1 & \cdots \\ \vdots & c_4^2 & c_9^2 & c_{4^2}^2 & \cdots & & & c_{n^2}^2 & \cdots \\ \vdots & \vdots & c_9^5 & c_{4^2}^5 & \cdots & \ddots & & c_{n^2}^{2^2+1} & \cdots \\ \vdots & \vdots & \ddots & c_{4^2}^{10} & \ddots & & \ddots & c_{n^2}^{3^2+1} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & c_{n^2}^{(n-1)^2+1} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}.$$

Thus if a L_1-predual space has a representing matrix like C, it is actually tensor product of two L_1-predual spaces with representing matrices A and B.

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