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# On some Inequalities for Strongly Reciprocally Convex Functions

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*Abstract*: We establish some Hermite-Hadamard and Fejér type inequalities for the class of strongly reciprocally convex functions.

*Key words*: strongly reciprocally convex functions, Hermite-Hadamrd, Fejér.

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## 1. Introduction

Due to its important role in mathematical economics, engineering, management science, and optimization theory, convexity of functions and sets has been studied intensively; see [1, 5, 7, 8, 9, 11, 13, 15, 16] and the references therein. Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ be a interval. A function $f : I \to \mathbb{R}$ is said to be convex in the classical sense if it satisfies the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. We say that $f$ is concave if $-f$ is convex.

In recent years several extensions and generalizations have been considered for classical convexity, and the theory of inequalities has made essential contributions in many areas of mathematics. A significant subclass of convex functions is that of strongly convex functions introduced by B.T. Polyak [20]. Strongly convex functions are widely used in applied economics, as well as in nonlinear optimization and other branches of pure and applied mathematics. In this paper we present a new class of strongly convex functions, mainly the class of *strongly harmonically convex functions*. Our investigation is devoted to the classical results related to convex functions due to Charles Hermite, Jaques Hadamard [10] and Lipót Fejér [8]. The Hermite-Hadamard inequalities and Fejér inequalities have been the subject of intensive research, and many applications, generalizations and improvements of them can be found in the literature (see, for instance, [1, 7, 15, 18, 19, 21, 24] and the references therein).

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, this asserts that the mean value of a continuous convex functions $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ lies between the value of $f$ at the midpoint of the interval $[a, b]$ and the arithmetic mean of the values of $f$ at the endpoints of this interval, that is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\,. \tag{1.1}$$

Moreover, each side of this double inequality characterizes convexity in the sense that a real-valued continuous function $f$ defined on an interval $I$ is convex if its restriction to each compact subinterval $[a, b] \subseteq I$ verifies the left hand side of (1.1) (equivalently, the right hand side on (1.1)). See [17].

In [8], Lipót Fejér established the following inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1): If $f : [a, b] \to \mathbb{R}$ is a convex function, then the inequality

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,\mathrm{d}x &\leq \frac{1}{b-a}\int_a^b f(x)p(x)\,\mathrm{d}x \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b p(x)\,\mathrm{d}x \end{aligned} \tag{1.2}$$

holds, where $p : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $x = (a + b)/2$.

Various generalizations have been pointed out in many directions, for recent developments of inequalities (1.1) and (1.2) and its generalizations, see [5, 6, 7, 4, 9, 13].

In [13], Imdat Iscan gave the definition of harmonically convex functions:

Definition 1.1. [13] Let $I$ be an interval in $\mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be harmonically convex on $I$ if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.3}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

If the inequality in (1.3) is reversed, then $f$ is said to be harmonically concave.

The following result of the Hermite-Hadamard type for harmonically convex functions holds.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\,. \tag{1.4}$$

In [4], F. Chen and S. Wu proved the following Fejér inequality for harmonically convex functions.

Theorem 1.3. ([4]) *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L(a, b)$, then one has*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x &\leq \int_a^b \frac{f(x)}{x^2}p(x)\,\mathrm{d}x \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x\,, \end{aligned} \tag{1.5}$$

*where $p : [a, b] \to \mathbb{R}$ is nonnegative and integrable and satisfies*

$$p\left(\frac{ab}{x}\right) = p\left(\frac{ab}{a+b-x}\right).$$

## 2. Strongly reciprocally convex functions

In 1966 Polyak [20] introduced the notions of strongly convex and strongly quasi-convex functions. In 1976 Rockafellar [23] studied the strongly convex functions in connection with *the proximal point algorithm*. They play an important role in optimization theory and mathematical economics. Nikodem et al. have obtained some interesting properties of strongly convex functions (see [7, 12, 14]).

Definition 2.1. (See [12, 16, 22]) Let $D$ be a convex subset of $\mathbb{R}$ and let $c > 0$. A function $f : D \to \mathbb{R}$ is called strongly convex with modulus $c$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2 \tag{2.1}$$

for all $x, y \in D$ and $t \in [0, 1]$.

The usual notion of convex function correspond to the case $c = 0$. For instance, if $f$ is strongly convex, then it is bounded from below, its level sets $\{x \in I : f(x) \le \lambda\}$ are bounded for each $\lambda$ and $f$ has a unique minimum on every closed subinterval of $I$ [18, p. 268]. Any strongly convex function defined on a real interval admits a quadratic support at every interior point of its domain.

The proofs of the next two theorems can be found in [22].

THEOREM 2.2. *Let $D$ be a convex subset of $\mathbb{R}$ and let $c$ be a positive constant. A function $f : D \to \mathbb{R}$ is strongly convex with modulus $c$ if and only if the function $g(x) = f(x) - cx^2$ is convex.*

THEOREM 2.3. *The following are equivalent:*

(i) $f(tx+(1-t)y) \le tf(x)+(1-t)f(y)-t(1-t)c(x-y)^2$, *for all $x, y \in (a, b)$ and $t \in [0, 1]$.*

(ii) *For each $x_0 \in (a, b)$, there is a linear function $T$ such that $f(x) \ge f(x_0) + T(x - x_0) + c(x - x_0)^2$ for all $x, y \in (a, b)$.*

(iii) *For differentiable $f$, for each $x_0 \in (a, b)$: $f(x) \ge f(x_0)+f'(x_0)(x-x_0)+c(x - x_0)^2$, for all $x, y \in (a, b)$.*

(iv) *For twice differentiable $f$, $f''(x) \ge 2c$, for all $x, y \in (a, b)$.*

In [3] we proved the following sandwich theorem for harmonically convex functions:

THEOREM 2.4. *Let $f, g$ be real functions defined on the interval $(0, +\infty)$. The following conditions are equivalent:*

(i) *There exists a harmonically convex function $h : (0, +\infty) \to \mathbb{R}$ such that $f(x) \le h(x) \le g(x)$ for all $x \in (0, +\infty)$.*

(ii) *The following inequality holds*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le tg(y) + (1-t)g(x) \tag{2.2}$$

*for all $x, y \in (0, +\infty)$ and $t \in [0, 1]$.*

On the other hand, in [2] we introduced the notion of harmonically strongly convex function as follows:

Definition 2.5. Let $I$ be an interval in $\mathbb{R}\setminus\{0\}$ and let $c \in \mathbb{R}_+$. A function $f : I \to \mathbb{R}$ is said to be *harmonically strongly convex with modulus* $c$ on $I$, if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) - ct(1-t)(x-y)^2, \tag{2.3}$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

The symbol $\mathrm{SHC}_{(I,c)}$ will denote the class of functions that satisfy the inequality (2.3). We also establish some Hermite-Hadamard and Fejér type inequalities for the class of harmonically strongly convex functions.

Next we will explore a generalization of the concept of harmonically convex functions which we will call reciprocally strongly convex functions, it is a concept parallel to the definition presented in the definition 2.5.

Definition 2.6. Let $I$ be an interval in $\mathbb{R}\setminus\{0\}$ and let $c \in (0,\infty)$. A function $f : I \to \mathbb{R}$ is said to be *reciprocally strongly convex* with modulus $c$ on $I$, if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) - ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2, \tag{2.4}$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

The symbol $\mathrm{SRC}_{(I,c)}$ will denote the class of functions that satisfy the inequality (2.4).

Theorem 2.7. *Let $I \subset \mathbb{R}\setminus\{0\}$ be a real interval and $c \in (0,\infty)$. If $f \in SRC_{(I,c)}$, then $f$ es harmonically convex.*

*Proof.* Since $ct(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^2 \geq 0$, it is a immediate consequence of the definition. ∎

For the rest of this paper we will use $I \subset \mathbb{R}\setminus\{0\}$ to denote a real interval and $c \in (0,\infty)$.

Theorem 2.8. *Let $f : I \to \mathbb{R}$ be a function. $f \in SRC_{(I,c)}$ if and only if the function $g : I \to \mathbb{R}$, defined by $g(x) := f(x) - \frac{c}{x^2}$ es harmonically convex.*

*Proof.* Assume that $f \in \text{SRC}_{(I,c)}$, then

$$
\begin{aligned}
&g\left(\frac{xy}{tx+(1-t)y}\right)\\
&\quad= f\left(\frac{xy}{tx+(1-t)y}\right) - c\left(\frac{tx+(1-t)y}{xy}\right)^2\\
&\quad\le tf(y)+(1-t)f(x) - ct(1-t)\left(\frac{1}{y}-\frac{1}{x}\right)^2 - c\left(t\frac{1}{y}+(1-t)\frac{1}{x}\right)^2\\
&\quad= tf(y)+(1-t)f(x)\\
&\qquad - ct(1-t)\left(\frac{1}{y^2}-\frac{2}{xy}+\frac{1}{x^2}\right) c\left(\frac{t^2}{y^2}+\frac{2t(1-t)}{xy}+\frac{(1-t)^2}{x^2}\right)\\
&\quad= tf(y)+(1-t)f(x) - c\left(\frac{t}{y^2}-\frac{2t}{xy}+\frac{t}{x^2}-\frac{t^2}{y^2}+\frac{2t^2}{xy}\right.\\
&\qquad\left.-\frac{t^2}{x^2}+\frac{t^2}{y^2}+\frac{2t}{xy}-\frac{2t^2}{xy}+\frac{1}{x^2}-\frac{2t}{x^2}+\frac{t^2}{x^2}\right)\\
&\quad= tf(y)+(1-t)f(x) - c\left(\frac{t}{y^2}+\frac{1}{x^2}-\frac{t}{x^2}\right)\\
&\quad= tf(y)+(1-t)f(x) - c\left(\frac{t}{y^2}+(1-t)\frac{1}{x^2}\right)\\
&\quad= t\left(f(y)-\frac{c}{y^2}\right)+(1-t)\left(f(x)-\frac{c}{x^2}\right)\\
&\quad= tg(y)+(1-t)g(x),
\end{aligned}
$$

for all $x, y \in I$ and $t \in [0,1]$. Which proves that $g$ is harmonically convex.

Conversely, if $g$ is harmonically convex, then

$$
\begin{aligned}
f\left(\frac{xy}{tx+(1-t)y}\right) &= g\left(\frac{xy}{tx+(1-t)y}\right) + c\left(\frac{tx+(1-t)y}{xy}\right)^2\\
&\le tg(y)+(1-t)g(x) + c\left(t\frac{1}{y}+(1-t)\frac{1}{x}\right)^2\\
&= tg(y)+(1-t)g(x) + c\left(\frac{t^2}{y^2}+\frac{2t(1-t)}{xy}+\frac{(1-t)^2}{x^2}\right)
\end{aligned}
$$

$$= tg(y) + (1-t)g(x) + c\left(\frac{t(1-1+t)}{y^2} + \frac{2t(1-t)}{xy} + \frac{(1-t)(1-t)}{x^2}\right)$$

$$= tg(y) + (1-t)g(x) + c\left(\frac{t(1-1+t)}{y^2} + \frac{2t(1-t)}{xy} + \frac{(1-t)(1-t)}{x^2}\right)$$

$$= tg(y) + (1-t)g(x) + c\left(\frac{t}{y^2} - \frac{t(1-t)}{y^2} + \frac{2t(1-t)}{xy} + \frac{1-t}{x^2} - \frac{t(1-t)}{x^2}\right)$$

$$= t\left(g(y) + c\frac{1}{y^2}\right) + (1-t)\left(g(x) + c\frac{1}{x^2}\right) - ct(1-t)\left(\frac{1}{y^2} - \frac{2}{xy} + \frac{1}{x^2}\right)$$

$$= tf(y) + (1-t)f(x) - ct(1-t)\left(\frac{1}{y} - \frac{1}{x}\right)^2,$$

for all $x, y \in I$ and $t \in [0,1]$, showing that $f \in$SRC$_{(I,c)}$. ∎

EXAMPLE 2.9. (a) The constant function is harmonically convex but not reciprocally strongly convex.

(b) The function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(x) = -x^2$, is not a harmonically convex function, since $f$ is a not convex and nonincreasing function. Based on Theorem 2.7, we obtain $f \notin$ SRC$_{(I,c)}$.

(c) Since $g(x) = \log(x)$ is a harmonically convex function, the function $f(x) := \log(x) + \dfrac{c}{x^2}$ is a reciprocally strongly convex function.

LEMMA 2.10. *If $f$ is a reciprocally strongly convex function, then the function $\varphi = f + \epsilon$ is also a reciprocally strongly convex function, for any constants $\epsilon$. In fact,*

$$\varphi\left(\frac{xy}{tx+(1-t)y}\right) = f\left(\frac{xy}{tx+(1-t)y}\right) + \epsilon$$

$$\leq tf(y) + (1-t)f(x) + ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2 + \epsilon$$

$$= tf(y) + t\epsilon + (1-t)f(x) + (1-t)\epsilon + ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2$$

$$= t(f(y)+\epsilon) + (1-t)(f(x)+\epsilon) + ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2$$

$$= t\varphi(y) + (1-t)\varphi(x) + ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2.$$

Theorem 2.11. *If* $f : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *and if we consider the function* $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$, *defined by* $g(t) = f\left(\frac{1}{t}\right)$, *then* $f \in SRC_{([a,b],c)}$ *if and only if* $g$ *is strongly convex in* $\left[\frac{1}{b}, \frac{1}{a}\right]$.

*Proof.* If for all $x, y \in [a,b]$ and $t \in [0,1]$, we have

$$f\left(\frac{1}{t\frac{1}{y} + (1-t)\frac{1}{x}}\right) \leq tf(y) + (1-t)f(x) - ct(1-t)\left(\frac{1}{x} - \frac{1}{y}\right)^2;$$

this last inequality may be changed by another equivalent one:

$$g\left(tw + (1-t)u\right) \leq tg\left(w\right) + (1-t)g\left(u\right) - ct(1-t)\left(u - w\right)^2,$$

where $u, w \in \left[\frac{1}{b}, \frac{1}{a}\right]$ and $t \in [0,1]$. To complete the proof. ∎

It is easy to see that the result is also valid for intervals $(a,b) \subset \mathbb{R}\setminus\{0\}$.

Theorem 2.12. *The following are equivalent:*

(i) $f \in SRC_{((a,b),c)}$.

(ii) *For each* $x_0 \in (a,b)$, *there is a linear function* $T$ *such that*

$$f\left(\frac{1}{x}\right) \geq c(x-x_0)^2 + T(x-x_0) + f\left(\frac{1}{x_0}\right), \quad \text{for all} \quad x \in \left(\frac{1}{b}, \frac{1}{a}\right). \tag{2.5}$$

(iii) *For differentiable* $f$ *and* $x_0 \in (a,b)$,

$$f\left(\frac{1}{x}\right) \geq f\left(\frac{1}{x_0}\right) - f\left(\frac{1}{x_0}\right)\frac{x - x_0}{x_2} + c(x-x_0)^2, \tag{2.6}$$

*for all* $x, y \in (a,b)$.

(iv) *For twice differentiable* $f$,

$$\frac{1}{x^4}\left[f''\left(\frac{1}{x}\right) + 2xf'\left(\frac{1}{x}\right)\right] \geq 2c, \quad \text{for all} \quad x \in \left(\frac{1}{b}, \frac{1}{a}\right).$$

*Proof.* (i) $\Rightarrow$ (ii): Assume that $f \in \mathrm{SRC}_{((a,b),c)}$. Since all the assumptions of Theorem 2.11 are satisfied, then the function $g(x) := f\left(\frac{1}{x}\right)$ is strongly convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$. Then by Theorem 2.3, for each $x_0 \in \left(\frac{1}{b}, \frac{1}{a}\right)$, there is a linear function $T$ such that $g(x) \geq g(x_0) + T(x - x_0) + c(x - x_0)^2$, for all $x, y \in \left(\frac{1}{b}, \frac{1}{a}\right)$. This is equivalent to the inequality (2.5).

(i) $\Rightarrow$ (iii): Assume that $f \in \mathrm{SRC}_{((a,b),c)}$. By Theorem 2.11, the function $g(x) := f\left(\frac{1}{x}\right)$ is strongly convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$, then by Theorem 2.3, for each $x_0 \in \left(\frac{1}{b}, \frac{1}{a}\right)$, $g(x) \geq g(x_0) + g'(x_0)(x - x_0) + c(x - x_0)^2$, for all $x, y \in (a, b)$. This is equivalent to the inequality (2.6).

(ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i) are shown using the reciprocals of the theorem and lemma that we have used in the above part.

(i) $\Longleftrightarrow$ (iv): Suppose $f$ is twice differentiable over $(a, b)$. $f \in \mathrm{SRC}_{((a,b),c)}$ if only if the function $g(x) := f\left(\frac{1}{x}\right)$ is strongly convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$ (by the theorem 2.11). It follows from Theorem 2.3 that $g$ is a strongly convex function with modulus $c$ if only if $g''(x) \geq 2c$. Hence it is equivalent to

$$\frac{1}{x^4}\left[f''\left(\frac{1}{x}\right) + 2xf'\left(\frac{1}{x}\right)\right] \geq 2c\,, \qquad \text{for all} \quad x \in \left(\frac{1}{b}, \frac{1}{a}\right). \qquad \blacksquare$$

## 3. Main results

In this section, we derive our main results.

### 3.1. Hermite-Hadamard type inequalities

The following result is a counterpart of the Hermite-Hadamard inequality for strongly reciprocally convex functions.

Theorem 3.1. *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. If $f : I \to \mathbb{R}$ is a strongly reciprocally convex function with modulus $c$, $a, b \in I$ with $a < b$ and $f \in L[a, b]$ then*

$$f\left(\frac{2ab}{a+b}\right) + \frac{c}{12}\left(\frac{b-a}{ab}\right)^2 \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2} - \frac{c}{6}\left(\frac{b-a}{ab}\right)^2. \tag{3.1}$$

*Proof.* By Theorem 2.11 the function $g : I \to \mathbb{R}$, defined by $g(x) := f(x) - \frac{c}{x^2}$ is harmonically convex, since $f \in \mathrm{SRC}_{(I,c)}$.

Consequently, by the Hermite-Hadamard type inequality for harmonically convex functions (see [13, Theorem 1]), we have

$$g\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}\,\mathrm{d}x \le \frac{g(a)+g(b)}{2}\,,$$

$$f\left(\frac{2ab}{a+b}\right) - c\left(\frac{a+b}{2ab}\right)^2 \le \frac{ab}{b-a}\int_a^b \frac{f(x)-\frac{c}{x^2}}{x^2}\,\mathrm{d}x \le \frac{f(a)-\frac{c}{a^2}+f(b)-\frac{c}{b^2}}{2}\,.$$

This last inequality can be simplified to

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) - c\left(\frac{a+b}{2ab}\right)^2 &\le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x - \frac{abc}{3(b-a)}\left[\frac{b^3-a^3}{a^3b^3}\right] \\ &\le \frac{f(a)+f(b)}{2} - \frac{c}{2}\left(\frac{a^2+b^2}{a^2b^2}\right), \end{aligned}$$

which in turn is equivalent to the inequality

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) + \frac{c}{12}\left(\frac{b-a}{ab}\right)^2 &\le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \\ &\le \frac{f(a)+f(b)}{2} - \frac{c}{6}\left(\frac{b-a}{ab}\right)^2. \end{aligned}$$

∎

*Remark* 3.2. Letting $c \to 0^+$, in the inequalities (3.1), we obtain (1.4), which is the Hermite-Hadamard type inequalities for harmonically convex functions.

We establish some new inequalities of Hermite-Hadamard type for functions whose derivatives are strongly reciprocally convex.

We need the following lemma, which can be found in [13].

Lemma 3.3. ([13]) *Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is a differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \\ &\qquad = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t\,. \end{aligned}$$

THEOREM 3.4. *Let $f : I \subset (0, +\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is strongly reciprocally convex with modulus $c$ on $[a, b]$ for $q \geq 1$, then*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, \mathrm{d}x \right| \\ &\qquad \leq \frac{ab(b-a)}{2} \lambda_1^{1-\frac{1}{q}} \left[ \lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q - c \left( \frac{1}{b} - \frac{1}{a} \right)^2 \lambda_4 \right]^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

*where*

$$\begin{aligned} \lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right), \\ \lambda_2 &= -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right), \\ \lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right), \\ \lambda_4 &= -\frac{1}{b(b-a)} + \frac{1}{(b-a)^4} \left[ [a(a+2b) + b(b+2a)] \ln \left( \frac{(a+b)^2}{4ab} \right) \right. \\ &\qquad \left. - \frac{(a+b)^2(2a-b)}{2b} + b^2 - 3a^2 \right]. \end{aligned}$$

*Proof.* From Lemma 3.3, and letting $p := \dfrac{q}{q-1}$, we get

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, \mathrm{d}x \right| \\ &= \left| \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left( \frac{ab}{tb+(1-t)a} \right) \mathrm{d}t \right| \\ &\leq \frac{ab(b-a)}{2} \int_0^1 \left| \frac{1-2t}{(tb+(1-t)a)^2} \right| \left| f'\left( \frac{ab}{tb+(1-t)a} \right) \right| \mathrm{d}t \\ &= \frac{ab(b-a)}{2} \int_0^1 \left| \frac{1-2t}{(tb+(1-t)a)^2} \right|^{\frac{1}{p}} \left( \left| \frac{1-2t}{(tb+(1-t)a)^2} \right|^{\frac{1}{q}} \left| f'\left( \frac{ab}{tb+(1-t)a} \right) \right| \right) \mathrm{d}t \, . \end{aligned} \tag{3.3}$$

We apply Hölder's inequality to the right-hand side of (3.3) and using the hypothesis that $|f'|^q \in \mathrm{SRC}_{([a,b,c)}$, we get

$$\leq \frac{ab(b-a)}{2}\left[\int_0^1 \left(\left|\frac{1-2t}{[tb+(1-t)a]^2}\right|^{\frac{1}{p}}\right)^p \mathrm{d}t\right]^{\frac{1}{p}}$$

$$\cdot \left[\int_0^1 \left(\left|\frac{1-2t}{[tb+(1-t)a]^2}\right|^{\frac{1}{q}} \left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|\right)^q \mathrm{d}t\right]^{\frac{1}{q}}$$

$$= \frac{ab(b-a)}{2}\left[\int_0^1 \left|\frac{1-2t}{[tb+(1-t)a]^2}\right| \mathrm{d}t\right]^{1-\frac{1}{q}}$$

$$\cdot \left[\int_0^1 \left|\frac{1-2t}{[tb+(1-t)a]^2}\right| \left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q \mathrm{d}t\right]^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{2}\left[\int_0^1 \left|\frac{1-2t}{[tb+(1-t)a]^2}\right| \mathrm{d}t\right]^{1-\frac{1}{q}} \tag{3.4}$$

$$\cdot \left[\int_0^1 \frac{|1-2t|}{[tb+(1-t)a]^2}\left(t|f'(a)|^q + (1-t)|f'(b)|^q - ct(1-t)\left(\frac{1}{b}-\frac{1}{a}\right)^2\right) \mathrm{d}t\right]^{\frac{1}{q}}.$$

It can be shown that

$$\lambda_1 := \int_0^1 \frac{|1-2t|}{[tb+(1-t)a]^2}\ \mathrm{d}t = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),$$

$$\lambda_2 := \int_0^1 \frac{|1-2t|t}{[tb+(1-t)a]^2}\ \mathrm{d}t = \int_0^{\frac{1}{2}} \frac{(1-2t)t}{[tb+(1-t)a]^2}\ \mathrm{d}t - \int_{\frac{1}{2}}^1 \frac{(1-2t)t}{[tb+(1-t)a]^2}\ \mathrm{d}t$$

$$= -\frac{1}{b(b-a)} + \frac{b+3a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),$$

$$\lambda_3 := \int_0^1 \frac{|1-2t|(1-t)}{[tb+(1-t)a]^2}\ \mathrm{d}t = \lambda_1 + \lambda_2\,,$$

$$\lambda_4 := \int_0^1 \frac{t(1-t)|1-2t|}{[tb+(1-t)a]^2}\ \mathrm{d}t\,,$$

$$= -\frac{1}{b(b-a)} + \frac{1}{(b-a)^4}\left[[a(a+2b)+b(b+2a)]\ln\left(\frac{(a+b)^2}{4ab}\right) - \frac{(a+b)^2(2a-b)}{2b} + b^2 - 3a^2\right].$$

Now if we replace this values in (3.4), we get (3.2). ∎

3.2. Fejér type inequalities The following result is a counterpart of the Fejér inequality for strongly reciprocally convex functions.

Theorem 3.5. *Let* $I \subset \mathbb{R} \setminus \{0\}$ *be a real interval. If* $f : I \to \mathbb{R}$ *is a strongly reciprocally convex function with modulus* $c$*,* $a, b \in I$ *with* $a < b$ *and* $f \in L[a,b]$ *then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x + c\int_a^b \left[\frac{1}{x^2} - \left(\frac{a+b}{2ab}\right)^2\right] \frac{p(x)}{x^2}\,\mathrm{d}x \\ \leq \int_a^b \frac{f(x)}{x^2} p(x)\,\mathrm{d}x \qquad (3.5) \\ \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x - c\int_a^b \left[\frac{1}{2}\left(\frac{a^2+b^2}{a^2b^2}\right) - \frac{1}{x^2}\right]\frac{p(x)}{x^2}\,\mathrm{d}x\,, \end{aligned}$$

*where* $p : [a,b] \to [0,\infty)$ *is an integrable function and satisfies*

$$p\left(\frac{ab}{x}\right) = p\left(\frac{ab}{a+b-x}\right). \qquad (3.6)$$

*Proof.* By Theorem 2.11 the function $g : I \to \mathbb{R}$, defined by $g(x) := f(x) - \frac{c}{x^2}$ is harmonically convex, then in virtue of Theorem 1.3, we have that

$$g\left(\frac{2ab}{a+b}\right)\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x \leq \int_a^b \frac{g(x)}{x^2}p(x)\,\mathrm{d}x \leq \frac{g(a)+g(b)}{2}\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x\,.$$

The above inequality is equivalent to

$$\begin{aligned} \left[f\left(\frac{2ab}{a+b}\right) - c\left(\frac{a+b}{2ab}\right)^2\right]\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x \leq \int_a^b \frac{f(x)-\frac{c}{x^2}}{x^2}p(x)\,\mathrm{d}x \\ \leq \frac{f(a)-\frac{c}{a^2}+f(b)-\frac{c}{b^2}}{2}\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x\,. \end{aligned}$$

This last inequality can be simplified to

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x - c\left(\frac{a+b}{2ab}\right)^2\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x + c\int_a^b \frac{p(x)}{x^4}\,\mathrm{d}x \\
\le \int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \\
\le \frac{f(a)+f(b)}{2}\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x \\
- \frac{c}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x + c\int_a^b \frac{p(x)}{x^4}\,\mathrm{d}x\,,
\end{aligned}
$$

which in turn is equivalent to the inequality

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x + c\int_a^b \left[\frac{1}{x^2} - \left(\frac{a+b}{2ab}\right)^2\right]\frac{p(x)}{x^2}\,\mathrm{d}x \\
\le \int_a^b \frac{f(x)}{x^2}p(x)\,\mathrm{d}x \\
\le \frac{f(a)+f(b)}{2}\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x - c\int_a^b \left[\frac{1}{2}\left(\frac{a^2+b^2}{a^2b^2}\right) - \frac{1}{x^2}\right]\frac{p(x)}{x^2}\,\mathrm{d}x\,.
\end{aligned}
$$

∎

*Remarks* 3.6. (a) Letting $c \to 0^+$, in inequality (3.5), we obtain (1.5) which is the Fejér type inequality for harmonically convex functions.

(b) Putting $p(x) \equiv 1$ into Theorem 3.5, we obtain the inequality (3.1).

Now, we establish a new Fejér-type inequality for strongly reciprocally convex functions.

THEOREM 3.7. *Suppose $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is a strongly reciprocally convex function with modulus $c$ on $I$. If $a, b \in I$, $a < b$, and $f \in L[a, b]$, then*

$$
\begin{aligned}
f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x + \frac{c}{2ab}\int_a^b \frac{p\,(x)}{x^4}\left[2ab-(a+b)x\right]\,\mathrm{d}x \\
\le \int_a^b \frac{f(x)}{x^2}p(x)\,\mathrm{d}x \qquad (3.7)\\
\le \frac{a\,[f(a)+f(b)]}{b-a}\int_a^b (b-x)\frac{p(x)}{x^3}\,\mathrm{d}x - \frac{c}{ab}\int_a^b (b-x)(x-a)\frac{p(x)}{x^4}\,\mathrm{d}x\,,
\end{aligned}
$$

*where $p : [a, b] \to \mathbb{R}$ is a nonnegative integrable function that satisfies* (3.6).

*Proof.* According to (3.6), for $x = tb + (1-t)a$, we have

$$p\left(\frac{ab}{tb+(1-t)a}\right) = p\left(\frac{ab}{ta+(1-t)b}\right). \tag{3.8}$$

Since $f \in \mathrm{SRC}_{([a,b],c)}$, from the definition 2.6, we obtain

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(y)+f(x)}{2} - \frac{c}{4}\left(\frac{1}{x}-\frac{1}{y}\right)^2, \qquad x, y \in [a,b]. \tag{3.9}$$

Let $x = \dfrac{ab}{tb+(1-t)a}$ and $y = \dfrac{ab}{ta+(1-t)b}$ in (3.9), then

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) \leq{}& \frac{f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right)}{2} \\ &- \frac{c}{4}\left(\frac{tb+(1-t)a}{ab} - \frac{ta+(1-t)b}{ab}\right)^2. \end{aligned}$$

Thus,

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)p\left(\frac{ab}{tb+(1-t)a}\right) \leq{}& \frac{1}{2}\Bigg[f\left(\frac{ab}{ta+(1-t)b}\right)p\left(\frac{ab}{ta+(1-t)b}\right) \\ &+ f\left(\frac{ab}{tb+(1-t)a}\right)p\left(\frac{ab}{tb+(1-t)a}\right)\Bigg] \\ &- \frac{c}{4}\left(\frac{tb+(1-t)a}{ab} - \frac{ta+(1-t)b}{ab}\right)^2 p\left(\frac{ab}{tb+(1-t)a}\right). \end{aligned}$$

Integrating both sides of the above inequalities with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)&\int_0^1 p\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t \\ \leq{}& \frac{1}{2}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)p\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \\ &+ \frac{1}{2}\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)p\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t \\ &- \frac{c}{4}\int_0^1\left(\frac{tb+(1-t)a}{ab} - \frac{ta+(1-t)b}{ab}\right)^2 p\left(\frac{ab}{tb+(1-t)a}\right)\mathrm{d}t. \end{aligned}$$

By simple computation,

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b \frac{p(x)}{x^2}\,\mathrm{d}x \\ &\leq \frac{1}{2}\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}p(x)\,\mathrm{d}x + \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}p(x)\,\mathrm{d}x \\ &\quad - \frac{c}{4}\frac{2}{b-a}\int_a^b \frac{p(x)}{x^4}\left[2ab-(a+b)x\right]\mathrm{d}x\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} f\left(\frac{ab}{ta+(1-t)b}\right)p\left(\frac{ab}{ta+(1-t)b}\right) \\ &\leq \left[tf(b)+(1-t)f(a)-ct(1-t)\left(\frac{1}{a}-\frac{1}{b}\right)^2\right]p\left(\frac{ab}{ta+(1-t)b}\right). \end{aligned}$$

Again, integrating both sides of the above inequalities with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned} \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)p\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \\ &\leq \int_0^1 \left[tf(b)+(1-t)f(a)-ct(1-t)\left(\frac{1}{a}-\frac{1}{b}\right)^2\right]p\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\,. \end{aligned}$$

By simple computation,

$$\begin{aligned} \int_a^b \frac{f(x)}{x^2}p(x)\,\mathrm{d}x \\ &\leq \frac{a\left[f(a)+f(b)\right]}{b-a}\int_a^b (b-x)\frac{p(x)}{x^3}\,\mathrm{d}x - \frac{c}{ab}\int_a^b (b-x)(x-a)\frac{p(x)}{x^4}\,\mathrm{d}x\,. \end{aligned}$$

This concludes the proof. ∎

*Remarks* 3.8. (a) Letting $c \to 0^+$ in the inequalities (3.7), we obtain the left-hand side of inequality of Fejér type inequalities for harmonically convex function (see [4]).

(b) Letting $p(x) \equiv 1$ in the inequalities (3.7) we obtain inequalities of Hermite-Hadamard type (see Theorem 3.1).

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