Sobolev Spaces and Potential Spaces Associated to Hermite Polynomials Expansions

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Abstract: The aim of this paper is to study the relation existing between potential spaces and Sobolev spaces, induced by the Ornstein-Uhlenbeck differential operator and associated to Hermite polynomials expansions, where we consider the multidimensional Gaussian measure. By means of analytical methods we prove that potential spaces and Sobolev spaces are spaces with equivalent norms.

Key words: Hermite polynomials, Gaussian measure, Sobolev spaces, potential spaces, Meyer's multiplier theorem.

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1. Introduction

In the classical harmonic analysis theory the study of the differentiability and smoothness of functions, $f : \mathbb{R}^d \to \mathbb{R}$, can be described in the context of Banach spaces of functions. Thus, Sobolev spaces $W_k^p(\lambda_d)$ and potential spaces $L_s^p(\lambda_d)$ are taking into consideration, where we denote λ_d as the Lebesgue measure, $s > 0$, $k \in \mathbb{N}$ and $1 < p < \infty$. On the one hand, Sobolev spaces $W_k^p(\lambda_d)$ allows us to consider functions, such that, $f \in L^p(\lambda_d)$ and its partial derivatives $\partial^\alpha f \in L^p(\lambda_d)$, with $|\alpha| \leq k$, where the derivatives are understood in a suitable weak sense to make the space complete. On the other hand, potential spaces $L_s^p(\lambda_d)$ are defined by using fractional powers of the Laplacean $(-\triangle)^{-s/2}$, $0 < s < d$, and its variants $(I - \triangle)^{-s/2}$, $s > 0$. These fractional powers are known as Riesz potentials and Bessel potentials respectively and we have that the potential spaces $L_s^p(\lambda_d)$ are subspaces of $L^p(\lambda_d)$, such that, $f = (I - \triangle)^{-s/2}g$, with $g \in L^p(\lambda_d)$. Riesz potentials, Bessel potentials and their properties have been deeply studied and by means of these operators it has been obtained that

$$W_k^p(\lambda_d) = L_k^p(\lambda_d) \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad 1 < p < \infty,$$

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(see [15, 6]). This identity is very meaningful because the condition defining to $W_k^p(\lambda_d)$ spaces is not easy to check for any function given, nevertheless, the condition defining to $L_k^p(\lambda_d)$ can be described in terms of Bessel potentials which have integral representations.

Similar identity holds in the Hermite setting and a probabilistic proof of this fact has been given by H. Sugita in [16] and S. Watanabe in [17], where the notion of Sobolev spaces of Wiener functionals has been introduced to develop Malliavin's calculus. This question has been studied by B. Bongioanni and J. L Torrea in [1, 2], where they considered Sobolev spaces associated to Hermite functions expansions and Laguerre functions expansions. However, in [7] we introduced the fractional derivative for the gaussian measure γ_d and by means of analytical methods we obtained that

$$W_k^p(\gamma_1) = L_k^p(\gamma_1), \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad 1 < p < \infty,$$

but only in the unidimensional case.

Therefore, the purpose of this paper is to extend this identity to the multidimensional case. If $d \geq 1$, $k \in \mathbb{N}$ and $1 < p < \infty$, we shall prove that

$$W_k^p(\gamma_d) = L_k^p(\gamma_d)$$

and once more, Bessel potentials and adjoint Gaussian-Riesz transforms are key tools in the proof of this fact. In the Laguerre polynomials and Jacobi polynomials setting, some similar results have been obtained in [5] and [8], respectively. Particularly, if $1 < p < \infty$, $0 < s < 1$ and $k \geq 1$ we obtain that $W_k^p(\gamma_d) \subset L_s^p(\gamma_d)$ and the inclusion is proper.

The paper is organized as follows. Section 2 contains some basic facts notation and we obtain the result of this paper in Section 3.

2. Preliminary definitions.

Let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d \cup \{\vec{0}\}$ be a multi-index, so $\beta! = \prod_{i=1}^d \beta_i!$ and $|\beta| = \sum_{i=1}^d \beta_i$. Let us denote by $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$, and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}$. Let $\mathcal{P}$ be the subspace of polynomials functions on $\mathbb{R}^d$.

We denote the Gaussian measure $\gamma_d(dx) = \frac{e^{-|x|^2}}{\pi^{d/2}} dx$, with $x \in \mathbb{R}^d$ and the Ornstein-Uhlenbeck differential operator is defined as

$$L = \frac{1}{2}\triangle_x - \langle x, \nabla_x \rangle.$$

Let us consider the normalized Hermite polynomials of order β, in d variables, defined as

$$h_\beta(x) = \frac{1}{\left(2^{|\beta|}\beta!\right)^{1/2}} \prod_{i=1}^{d}(-1)^{\beta_i} e^{x_i^2} \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}\left(e^{-x_i^2}\right),$$

with $h_0(x) = 1$, (see [18, pages 105–106]) and it is well known that the Hermite polynomials are eigenfunctions of L; this way,

$$Lh_\beta(x) = -|\beta|\, h_\beta(x). \tag{2.1}$$

Also,

$$\partial_j h_\beta(x) = \sqrt{2\beta_j}h_{\beta-e_j}, \quad \text{for each} \quad 1 \le j \le d$$

and if α is a multi-index,

$$\partial^\alpha h_\beta(x) = \begin{cases} 2^{|\alpha|/2}\left(\displaystyle\prod_{j=1}^{d}\beta_j(\beta_j-1)\cdots(\beta_j-\alpha_j+1)\right)^{1/2} h_{\beta-\alpha}(x), & \text{if } \alpha_j \le \beta_j, \\ & \forall j = 1,\ldots,d, \\ 0, & \text{otherwise.} \end{cases}$$

Now, given a function $f \in L^1(\gamma_d)$ its β-Fourier-Hermite coefficient is defined by

$$c_\beta^f = \int_{\mathbb{R}^d} f(x)h_\beta(x)\gamma_d(dx)$$

and let C_n be the closed subspace of $L^2(\gamma_d)$ generated by the linear combinations of $\{h_\beta : |\beta| = n\}$. The orthogonality of the Hermite polynomials with respect to γ_d, lets us see that $\{C_n\}$ is an orthogonal decomposition of $L^2(\gamma_d)$

$$L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n$$

which is called the Wiener chaos decomposition, see [17]. Now, we denote J_n the orthogonal projection of $L^2(\gamma_d)$ onto C_n. If $f \in L^2(\gamma_d)$, we have that

$$J_n f = \sum_{|\beta|=n} c_\beta^f h_\beta$$

and its Hermite expansion is given by $f = \sum_{n\ge 0} J_n f$. Then, following [16] there exists a positive constant $C_{p,n}$, such that,

$$\|J_n f\|_{p,\gamma_d} \le C_{p,n}\|f\|_{p,\gamma_d} \quad \text{for} \quad 1 < p < \infty.$$

Also, if $f \in L^2(\gamma_d)$ the operator

$$Lf = \sum_{n \geq 0} -n J_n f,$$

defined on the domain $D_2(L) = \{f \in L^2(\gamma_d) : \sum_{n \geq 0} \sum_{|\beta|=n} |c_\beta^f|^2 < \infty\}$ is a self-adjoint extension of L considered on dense subspace of $L^2(\gamma_d)$. More precisely, L has a clousure wich also will denote by L.

In this context, the Ornstein-Uhlenbeck semigroup, $\{T_t\}_{t \geq 0}$, is defined as

$$T_t f(x) = \frac{1}{(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2 + |y|^2) - 2e^{-t}\langle x, y \rangle}{1 - e^{-2t}}} f(y) \gamma_d(dy), \qquad (2.2)$$

where its kernel is given by Mehler formula

$$\frac{1}{(1 - e^{-2t})^{d/2}} e^{-\frac{e^{-2t}(|x|^2 + |y|^2) - 2e^{-t}\langle x, y \rangle}{1 - e^{-2t}}} = \sum_{n \geq 0} \sum_{|\beta|=n} e^{-|\beta|t} h_\beta(x) h_\beta(y).$$

It is well known that $\{T_t\}_{t \geq 0}$ is a symmetric diffusion semigroup, with infinitesimal generator L, see [14, 16] and moreover

$$\|T_t(I - J_0 - \cdots - J_{n-1})f\|_{p, \gamma_d} \leq C_{p,n} e^{-nt} \|f\|_{p, \gamma_d}, \quad 1 < p < \infty. \qquad (2.3)$$

By means of Bochner subordination formula the Poisson-Hermite semigroup $\{P_t\}_{t \geq 0}$ is defined as

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du \qquad (2.4)$$

and similarly, $\{P_t\}_{t \geq 0}$ is a strongly continuous semigroup on $L^p(\gamma_d)$ with infinitesimal generator $(-L)^{1/2}$, (see [14]). From (2.1) we obtain that

$$T_t h_\beta(x) = e^{-t|\beta|} h_\beta(x) \quad \text{and} \quad P_t h_\beta(x) = e^{-t\sqrt{|\beta|}} h_\beta(x) \qquad (2.5)$$

Moreover,

$$T_t(J_n f) = e^{-nt} J_n f \quad \text{and} \quad P_t(J_n f) = e^{-\sqrt{n}t} J_n f. \qquad (2.6)$$

Now, if $s > 0$, similar to the classical case, the gaussian fractional integral of order s, I_s^γ, is defined by

$$I_s^\gamma := (-L)^{-s/2} \Pi_0, \quad \text{where} \quad \Pi_0 = I - J_0.$$

This (formal) definition is correct for all Hermite polynomials h_β, since by using (2.1) we have that

$$I_s^\gamma h_\beta(x) = \frac{1}{|\beta|^{s/2}} h_\beta(x), \quad \forall |\beta| > 0, \tag{2.7}$$

and define $I_s^\gamma h_0(x) = 0$, see [3, 13]. In the case that $f \in L^1(\gamma_d)$, an integral representation of I_s^γ is obtained in [13], which is given by

$$I_s^\gamma f = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} P_t(I - J_0) f dt. \tag{2.8}$$

By using (2.5), we can see that (2.8) coincides with (2.7), if $f = h_\beta$, $\forall |\beta| > 0$ and consequently, (2.8) coincides with (2.7) if f is a nonconstant polynomial or $f \in L^2(\gamma_d)$.

Again, from (2.3) and (2.4) we obtain that

$$\|P_t(I - J_0)f\|_{p,\gamma_d} \leq C_p e^{-t} \|f\|_{p,\gamma_d} \quad 1 < p < \infty, \tag{2.9}$$

since

$$e^{-t} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2/4u} du.$$

Therefore, from (2.8) and (2.9) we can conclude

$$\|I_s^\gamma f\|_{p,\gamma_d} \leq C_p \|f\|_{p,\gamma_d} \quad 1 < p < \infty. \tag{2.10}$$

Now, for $\alpha \in \mathbb{N}^d$, we consider the Riesz transform of order $|\alpha|$, associated to L, defined as, (see [10])

$$R_{|\alpha|}^\alpha := \partial^\alpha I_{|\alpha|}^\gamma \tag{2.11}$$

and if $f \in L^1(\gamma_d)$ with $c_0^f = 0$, then

$$R_{|\alpha|}^\alpha f(x)$$

$$= C_{d,\alpha} \int_{\mathbb{R}^d} \int_0^1 r^{|\alpha|-1} \left(\frac{-\log r}{1 - r^2} \right)^{\frac{|\alpha|-2}{2}} h_\alpha \left(\frac{y - rx}{\sqrt{1 - r^2}} \right) \frac{e^{-\frac{|y - rx|^2}{1 - r^2}}}{(1 - r^2)^{d/2+1}} dr f(y) dy.$$

$L^p(\gamma_d)$ estimates for $1 < p < \infty$ of the Gaussian-Riesz transform, have been showed by several authors using probabilistic and analytic methods (see for

example [4, 10, 11] among other authors). Particularly, if $f = h_\beta$, for each $i = 1, \ldots, d$ and $\beta_i \geq \alpha_i$, we have

$$R_{|\alpha|}^\alpha h_\beta(x) = \left(\frac{2^{|\alpha|}}{|\beta|^{|\alpha|}} \right)^{1/2} \left[\prod_{i=1}^d \beta_i(\beta_i - 1) \cdots (\beta_i - \alpha_i + 1) \right]^{1/2} h_{\beta-\alpha}(x)$$

and the j-th Gaussian-Riesz transform of first order, $R_1^j = \partial_j I_1^\gamma$, $j = 1, \ldots, d$, with respect to Hermite polynomials can be expressed as

$$R_1^j h_\beta = \sqrt{\frac{2\beta_j}{|\beta|}} h_{\beta-e_j}.$$

In [7], the j-th adjoint operator, $(R_1^j)^*$ of the Gaussian-Riesz transform R_1^j, has been defined as

$$\left\langle (R_1^j)^* f, g \right\rangle_{\gamma_d} = \langle f, R_1^j g \rangle_{\gamma_d}$$

and we can observe that $I_1^\gamma = (-L)^{-1/2}$ is a self-adjoint operator, then integrating by parts with respect x_j, we obtain

$$\langle f, R_1^j g \rangle_{\gamma_d} = \langle f, \partial_j I_1^\gamma g \rangle_{\gamma_d} = \langle \delta_j f, I_1^\gamma g \rangle_{\gamma_d} = \langle I_1^\gamma \delta_j f, g \rangle_{\gamma_d},$$

where $\delta_j(\cdot) = -\partial_j(\cdot) + 2x_j(\cdot)$. This way, we can express, for each $j = 1, \ldots, d$,

$$(R_1^j)^* := I_1^\gamma \delta_j = (-L)^{-1/2} \delta_j.$$

By means of the identity

$$\sqrt{2(\beta_j + 1)} h_{\beta+e_j} - 2x_j h_\beta + \sqrt{2\beta_j} h_{\beta-e_j} = 0,$$

where $h_{-e_j} = 0$, $\forall j = 1, \ldots, d$, (see [18, pages 105–106]), we have that

$$\delta_j h_\beta = \sqrt{2(\beta_j + 1)} h_{\beta+e_j} \tag{2.12}$$

and therefore,

$$(R_1^j)^* h_\beta = \sqrt{\frac{2(\beta_j + 1)}{|\beta| + 1}} h_{\beta+e_j}. \tag{2.13}$$

Also, in [7] we obtain the boundedness of j-th adjoint operator of the Gaussian-Riesz transform $(R_1^j)^*$ for $1 < p < \infty$. This follows easily from Hölder's inequality and the $L^p(\gamma_d)$ continuity of the Riesz transform. So,

$$\left\| (R_1^j)^* f \right\|_{p,\gamma_d} = \sup_{\|g\|_q \leq 1} \left| \langle f, R_1^j g \rangle_{\gamma_d} \right| \leq C_q \|f\|_{p,\gamma_d}.$$

Similarly, the j-th adjoint Gaussian-Riesz operator $(R_k^j)^*$ of higher order k, with $k \geq 1$, is defined by

$$(R_k^j)^* := I_k^\gamma \delta_j^k = (-L)^{-k/2} \delta_j^k, \quad \text{for each } j = 1, \ldots, d,$$

If $\alpha \in \mathbb{N}^d$, we can define the higher order adjoint operator of the Riesz transform by

$$(R_{|\alpha|}^\alpha)^* := I_{|\alpha|}^\gamma \delta_d^{\alpha_d} \circ \cdots \circ \delta_1^{\alpha_1},$$

thus, we get

$$(R_{|\alpha|}^\alpha)^* h_\beta(x) = \left(\frac{2^{|\alpha|}}{|\beta + \alpha|^{|\alpha|}} \right)^{1/2} \left[\prod_{i=1}^d (\beta_i + 1) \cdots (\beta_i + \alpha_i) \right]^{1/2} h_{\beta+\alpha}(x)$$

and if $1 < p < \infty$ with $f \in L^p(\gamma_d)$, we have that

$$\|(R_{|\alpha|}^\alpha)^* f\|_{p,\gamma_d} \leq C_p \|f\|_{p,\gamma_d}.$$

Now, in [7] the Gaussian fractional derivative of order $s > 0$, D_s^γ, is defined formally as

$$D_s^\gamma := (-L)^{s/2}$$

and for Hermite polynomials, from (2.1), we have that

$$D_s^\gamma h_\beta(x) = |\beta|^{s/2} h_\beta(x). \tag{2.14}$$

Particularly, from (2.14), we have that $D_s^\gamma h_0(x) = 0$ and, similar to the classical case, the Gaussian fractional derivative of a constant function is equal to zero, see [14, 12]. In the case of $0 < s < 1$, we can write

$$D_s^\gamma f = \frac{1}{c_s} \int_0^\infty t^{-s-1} (P_t f - f) dt, \tag{2.15}$$

where

$$c_s = \int_0^\infty u^{-s-1} (e^{-u} - 1) du,$$

for $f \in \mathcal{P}$ and $f \in L^2(\gamma_d)$ (see [7]). This way, (2.15) is an integral representation of $D_s^\gamma f$. By using (2.5) we get that (2.15) coincides with (2.14), if $f = h_\beta$, $\forall |\beta| > 0$ and $0 < s < 1$. Similarly, by means of the property $P_t 1 = 1$ we conclude that (2.15) coincides with (2.14), if $f = h_0$. Moreover, if $f \in \mathcal{P}$, by (2.7) and (2.14) we obtain that

$$I_s^\gamma (D_s^\gamma f) = D_s^\gamma (I_s^\gamma f) = \Pi_0 f. \tag{2.16}$$

Following S. Watanabe [17] and H. Sugita [16], we consider the Gaussian-Bessel potentials defined by

$$(I - L)^{-s/2}f = \sum_{n=0}^{\infty}(1 + n)^{-s/2}J_nf, \quad \text{for } f \in \mathcal{P}.$$

By means of the Gamma function we obtain that

$$\frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}e^{-(1+n)t}dt = (1 + n)^{-s/2},$$

thus, (2.6) lets us write

$$(I - L)^{-s/2}f = \frac{1}{\Gamma(s/2)}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}T_tf\,dt, \quad \text{for } f \in \mathcal{P} \tag{2.17}$$

and by use of the contraction property of the semigroup $\{T_t\}_{t\geq 0}$, we obtain that

$$\|(I - L)^{-s/2}f\|_{p,\gamma_d} \leq \|f\|_{p,\gamma_d}.$$

Then, the Gaussian-Bessel potential spaces of order $s \geq 0$, $L_s^p(\gamma_d)$, with $1 < p < \infty$, can be defined as the completion of the polynomials with respect to the norm

$$\|f\|_{p,s} := \left\|(I - L)^{s/2}f\right\|_{p,\gamma_d};$$

in other words, $L_s^p(\gamma_d)$ is a subspace of $L^p(\gamma_d)$ consisting of all f which can be written in the form

$$f = (I - L)^{-s/2}\psi, \quad \text{with} \quad \psi \in L^p(\gamma_d),$$

where

$$\|f\|_{p,s} = \|\psi\|_{p,\gamma_d}.$$

These potential spaces present the following inclusion properties (see [7, 17] for more details).

  i) If $p \leq q$ then $L_s^q(\gamma_d) \subseteq L_s^p(\gamma_d)$, for each $s > 0$.

  ii) If $0 < s \leq r$ then $L_r^p(\gamma_d) \subseteq L_s^p(\gamma_d)$, for each $1 < p < \infty$.

Then, in [7] the following Theorem has been obtained

THEOREM 2.1. *Let $s \geq 0$ and $1 < p < \infty$. Then $f \in L_s^p(\gamma_d)$ if and only if $D_s^\gamma f \in L^p(\gamma_d)$. Moreover,*

$$B_{p,s}\|f\|_{p,s} \leq \|D_s^\gamma f\|_{p,\gamma_d} \leq A_{p,s}\|f\|_{p,s}.$$

Particularly, we can observe that if $\{P_n\} \in \mathcal{P}$, such that, $\lim_{n \to \infty} P_n = f$ in $L_s^p(\gamma_d)$, then $\lim_n D_s^\gamma P_n$ exists in $L_s^p(\gamma_d)$ and does not depend on the choice of a sequence $\{P_n\}_n$.

Theorem 2.1 is a corollary of the following theorem, obtained by P.A. Meyer, see [9, 16]. This theorem shall be an important tool to develop our result in Section 3.

THEOREM 2.2. (Meyer's multiplier theorem) *Let T_ϕ be given by*

$$T_\phi = \sum_{n \geq 0} \phi(n) J_n,$$

where $\{\phi(n)\}_{n \geq 0}$ is a real sequence. Assume that $h(z)$ is a function, which is analytic on some neighborhood of the origin. If there are $n_0 \in \mathbb{N}$, and a positive constant s, such that $h(n^{-s}) = \phi(n) \; \forall n \geq n_0$, then T_ϕ can uniquely extend to a bounded linear operator on $L^p(\gamma_d)$, for each $1 < p < \infty$.

As an application of Meyer's multiplier theorem, in [7], the following result has been obtained.

PROPOSITION 2.1. *Given $1 < p < \infty$ and $s \geq 1$. If $f \in L_s^p(\gamma_d)$, then $f \in L_{s-1}^p(\gamma_d)$ and for each $j = 1, \ldots, d$, $\partial_j f \in L_{s-1}^p(\gamma_d)$. Moreover,*

$$\|f\|_{p,s-1} + \sum_{j=1}^d \|\partial_j f\|_{p,s-1} \leq A_{p,s,d} \|f\|_{p,s} .$$

Finally, let us consider the Gaussian Sobolev space defined as

$$W_k^p(\gamma_d) := \{f : \partial^\alpha f \in L^p(\gamma_d), \; \alpha \in \mathbb{N}^d, \; |\alpha| \leq k\},$$

where $\partial^0 f = f$, equipped with the norm

$$\|f\|_{W_k^p} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{p,\gamma_d}$$

(see [15, pages 121–122]; where $W_k^p(\lambda_d)$ spaces are considered).

Then for each $1 < p < \infty$ and $k \geq m$, we can see that $\|f\|_{W_m^p} \leq \|f\|_{W_k^p}$ and therefore, $W_k^p(\gamma_d) \subseteq W_m^p(\gamma_d)$.

Also, from the definition of $W_k^p(\gamma_d)$ spaces we can see that $f \in W_k^p(\gamma_d)$, if and only if, f and $\partial_j f \in W_{k-1}^p(\gamma_d)$ for each $j = 1, \ldots, d$. Moreover, the two norms

$$\|f\|_{W_k^p} \quad \text{and} \quad \|f\|_{W_{k-1}^p} + \sum_{j=1}^d \|\partial_j f\|_{W_{k-1}^p}$$

are equivalent.

Remark. In [7], as has been mentioned previously, we proved that if $k \geq 1$, $1 < p < \infty$ and $f \in L_k^p(\gamma_1)$, then there exist a positive constant, $B_{p,k}$, such that,

$$\|f\|_{p,k} \leq B_{p,k} \left\| \frac{d^k}{dx^k} f \right\|_{p,\gamma_1}$$

for the unidimensional case. Nevertheless, in order to motivate the results to be developed in Section 3, we recall how this inequality was proved.

In fact, if $f \in \mathcal{P}$ we define the operator T_k as

$$T_k f = \sum_{n \geq 0} \frac{(n+k)^k}{2^k(n+k)\cdots(n+1)} c_n^f h_n.$$

Then using Meyer's Multipliers Theorem with the function

$$h(z) = \frac{(1+kz)^k}{2^k(1+kz)\cdots(1+z)},$$

we obtain that

$$\|T_k f\|_{p,\gamma_d} \leq C_p \|f\|_{p,\gamma_d}.$$

Now, we introduce the operator U_k as

$$U_k h_n = \left(\frac{n+k}{2} \right)^{k/2} \left((n+k)\cdots(n+1) \right)^{-1/2} h_{n+k}$$

and if $f \in \mathcal{P}$ we get that $U_k f = \sum_{n \geq 0} c_n^f U_k(h_n)$.

Denoting $(R_k)^* = (-L)^{-k/2}\delta^k$ as the unidimensional adjoint Gaussian Riesz transform of order k, with $k \geq 1$, we can see that

$$(R_k)^* h_n(x) = 2^{k/2} \frac{[(n+1)\cdots(n+k)]^{1/2}}{(n+k)^{k/2}} h_{n+k}.$$

Therefore, for each $k \geq 1$ and $n \geq 1$, we have

$$U_k(h_n) = [(R_k)^* \circ T_k](h_n).$$

For this reason, if $f \in \mathcal{P}$, by means of the $L^p(\gamma_d)$-continuity of the operators $(R^k)^*$ and T_k we obtain that

$$\|U_k f\|_{p,\gamma_d} \leq C_{p,k} \|f\|_{p,\gamma_d}. \tag{2.18}$$

But by definition $(U_k \circ R^k)(h_n) = h_n$ and if $f \in \mathcal{P}$ with $c_0^f = 0$, we get

$$D_k^\gamma f = \left(U_k \circ R_k \circ D_k^\gamma\right)f = \left(U_k \circ \frac{d^k}{dx^k} \circ I_k^\gamma \circ D_k^\gamma\right)f = U_k\left(\frac{d^k}{dx^k}f\right).$$

Therefore, by using the Theorem 2.1 and (2.18) we have that

$$\|f\|_{p,k} \leq B_{p,k}\left\|\frac{d^k}{dx^k}f\right\|_{p,\gamma_1},$$

and we can use the density of the polynomials in $L^p(\gamma_1)$ in the general case.

Now, let us consider the multidimensional case with $d > 1$ and we would like to repeat a similar argument. In this case, we could define the operators

$$T_\alpha(h_\beta) = \frac{|\beta + \alpha|^{|\alpha|}}{2^{|\alpha|}\prod\limits_{i=1}^{d}(|\beta| + 1)\cdots(|\beta| + \alpha_i)}h_\beta$$

and

$$U_\alpha(h_\beta) = \left(\frac{|\beta + \alpha|}{2}\right)^{|\alpha|/2}\left[\prod_{i=1}^{d}(\beta_i + 1)\cdots(\beta_i + \alpha_i)\right]^{-1/2}h_{\beta+\alpha}$$

for $\alpha \in \mathbb{N}^d$. Thus, if $f \in \mathcal{P}$ we introduce

$$T_\alpha f = \sum_{n \geq 0} T_\alpha(J_n f), \quad \text{and} \quad U_\alpha f = \sum_{n \geq 0} U_\alpha(J_n f)$$

and by using Meyer's Multipliers Theorem with

$$h(z) = \frac{2^{-|\alpha|}(1 + |\alpha|z)}{\prod\limits_{i=1}^{d}(1 + \alpha_i z)\cdots(1 + z)} \quad \text{and} \quad \phi(|\beta|) = \frac{2^{-|\alpha|}\left(1 + \frac{|\alpha|}{|\beta|}\right)}{\prod\limits_{i=1}^{d}\left(1 + \frac{1}{|\beta|}\right)\cdots\left(1 + \frac{\alpha_i}{|\beta|}\right)},$$

we obtain that T_α has a $L^p(\gamma_d)$-continuous extension. However, we can see that

$$[(R_{|\alpha|}^\alpha)^* \circ T_\alpha](h_\beta) = \left(\frac{|\beta + \alpha|^{|\alpha|}}{2^{|\alpha|}}\right)^{1/2}\left[\frac{\prod\limits_{i=1}^{d}(\beta_i + 1)\cdots(\beta_i + \alpha_i)}{\prod\limits_{i=1}^{d}(|\beta| + 1)\cdots(|\beta| + \alpha_i)}\right]^{1/2}h_{\beta+\alpha}$$

and therefore
$$U_\alpha f \neq [(R_{|\alpha|}^\alpha)^* \circ T_\alpha] f,$$

if $f \in \mathcal{P}$. In consequence, this reasoning fails in the multidimensional case. This way, to prove that
$$\|f\|_{p,k} \simeq \|f\|_{W_k^p},$$

we need to developed a different argument.

Consequently, we are able to present the results of this paper in the following section.

3. The results

By using (2.11) and (2.16) we can write
$$\partial^\alpha f = R_{|\alpha|}^\alpha D_{|\alpha|}^\gamma f,$$

if f is a nonconstant polynomial. Then, by means of the $L^p(\gamma_d)$-continuity of the Gaussian-Riesz transform and Theorem 2.1, we get
$$\|\partial^\alpha f\|_{p,\gamma_d} \leq C_{p,|\alpha|} \|f\|_{p,|\alpha|} \tag{3.1}$$

and therefore, the density of the polynomials in $L_{|\alpha|}^p(\gamma_d)$, with $|\alpha| \leq k$, the fact that $\|f\|_{p,|\alpha|} \leq \|f\|_{p,k}$ and (3.1), allows us to conclude that
$$\|f\|_{W_k^p} \leq C_{p.k} \|f\|_{p,k}. \tag{3.2}$$

Consequently,
$$L_k^p(\gamma_d) \subseteq W_k^p(\gamma_d) \quad \text{for each} \quad k \in \mathbb{N}. \tag{3.3}$$

Now, we shall prove the converse inequality in (3.2). First, we establish the following Lemma, where the relation between Riesz potentials and Bessel potentials are obtained in the gaussian context; see [3] for similar results and compare with [15, pages 133–134].

LEMMA 3.1. *Let $s \geq 0$ and $1 < p < \infty$.*

i) *Suppose $f \in \mathcal{P}$, then there exists a constant $C_p > 0$, such that,*
$$\left\|\left(D_s^\gamma \circ (I - L)^{-s/2}\right) f\right\|_{p,\gamma_d} \leq C_p \|f\|_{p,\gamma_d}.$$

ii) *Suppose $f \in \mathcal{P}$, then we have*
$$\left\|\left((I - L)^{s/2} \circ I_s^\gamma\right) f\right\|_{p,\gamma_d} \leq A_p \|f\|_{p,\gamma_d}.$$

iii) *There exists a pair of operators T_1 and T_2, which are bounded operators on $L^p(\gamma_d)$, so that*

$$(I - L)^{s/2} = T_1 + D_s^\gamma \circ T_2.$$

Proof. Items i) and ii) have been obtained in [3]. However, we present it with details for the sake of completeness.

i) If $f \in \mathcal{P}$ then $f = \sum_{n \geq 0} J_n f$. Particularly, we observe that

$$\left(D_s^\gamma \circ (I - L)^{-s/2}\right) J_0 f = c_0^f \left(D_s^\gamma \circ (I - L)^{-s/2}\right) h_0 = 0$$

and we can express

$$\left(D_s^\gamma \circ (I - L)^{-s/2}\right) f = \sum_{n \geq 0} \frac{n^{s/2}}{(1 + n)^{s/2}} J_n f.$$

Therefore, the result follows from Theorem 2.2 considering the function

$$h(z) = (z + 1)^{-s/2}.$$

ii) Again, we consider $f \in \mathcal{P}$, such that $f \in (C_0)^\perp$, then

$$\left((I - L)^{s/2} \circ I_s^\gamma\right) f = \sum_{n > 0} \frac{(1 + n)^{s/2}}{n^{s/2}} J_n f$$

and similarly, by means of Meyer's multiplier theorem with the function

$$h(z) = (z + 1)^{s/2},$$

we get that

$$\left\|\left((I - L)^{s/2} \circ I_s^\gamma\right) f\right\|_{p,\gamma_d} \leq A_p \|f\|_{p,\gamma_d}.$$

Particularly, if f is a constant function, then $f = c_0^f h_0$ and since by definition $I_s^\gamma h_0 = 0$, we have that

$$\left((I - L)^{s/2} \circ I_s^\gamma\right) f = c_0^f \left((I - L)^{s/2} \circ I_s^\gamma\right) h_0 = 0.$$

iii) We choose $T_1 = I$ and

$$T_2 = \left((I - L)^{s/2} \circ I_s^\gamma\right) - I_s^\gamma.$$

Then if $f \in \mathcal{P}$, such that, $f \in (C_0)^\perp$, we obtain

$$\|T_2 f\|_{p,\gamma_d} \leq \left\|\left((I - L)^{s/2} \circ I_s^\gamma\right) f\right\|_{p,\gamma_d} + \|I_s^\gamma f\|_{p,\gamma_d}$$

and iii) follows from ii) and (2.10).

Particularly, if f is a constant function, we have that $I_s^\gamma f = 0$ and therefore, $T_2 f = 0$. ∎

Remark. It should be emphasized, that polynomials are dense in $L^p(\gamma_d)$, for $1 < p < \infty$. This way, the Lemma 3.1 states that the operators defined by

$$D_s^\gamma \circ (I - L)^{-s/2} \quad \text{and} \quad (I - L)^{s/2} \circ I_s^\gamma,$$

are bounded operators on $L^p(\gamma_d)$, on every $1 < p < \infty$.

Now, on the one hand, by using (2.12) and (2.13) we observe that

$$(R_1^j)^*(\partial_j h_\beta) = (R_1^j)^*\big(\sqrt{2\beta_j}\,h_{\beta-e_j}\big) = \frac{2\beta_j}{|\beta|^{1/2}}h_\beta,$$

for each $j = 1, ..., d$ and $|\beta| \neq 0$. Thus,

$$\sum_{j=1}^d (R_1^j)^*(\partial_j h_\beta) = 2|\beta|^{1/2}h_\beta$$

and if $g \in \mathcal{P}$, we obtain that

$$\sum_{j=1}^d (R_1^j)^*(\partial_j J_n g) = 2n^{1/2}J_n g.$$

In consequence,

$$D_1^\gamma g = \frac{1}{2}\sum_{j=1}^d (R_1^j)^*(\partial_j g). \tag{3.4}$$

Particularly, if g is a constant function we can see that $D_1^\gamma g = 0$ and since $\partial_j g = 0$, then (3.4) is also true.

On other hand, if $s \geq 1$, we claim that

$$(I - L)^{-(s-1)/2}(\partial_j g) = \big(T_0 \circ \partial_j(I - L)^{-(s-1)/2}\big)g, \tag{3.5}$$

where we set $T_0 := (I - L)^{(s-1)/2} \circ I_{s-1}^\gamma$.

In fact, suppose $g \in \mathcal{P}$, such that, $g \in (C_0)^\perp$. Then $g = \sum_{n>0} J_n g$ and

$$\partial_j(I - L)^{-(s-1)/2}g = \sum_{n>0}\left(\frac{1}{n+1}\right)^{(s-1)/2}\sum_{|\beta|=n}\sqrt{2\beta_j}\,c_\beta^g h_{\beta-e_j}.$$

Also,

$$(I - L)^{-(s-1)/2}\partial_j g = \sum_{n>0} \sum_{|\beta|=n} \frac{\sqrt{2\beta_j}c_\beta^g}{|\beta|^{(s-1)/2}} h_{\beta-e_j}$$

$$= \sum_{n>0} \left(\frac{1+n}{n}\right)^{(s-1)/2} \left(\frac{1}{1+n}\right)^{(s-1)/2} \sum_{|\beta|=n} \sqrt{2\beta_j}c_\beta^g h_{\beta-e_j}$$

$$= ((I - L)^{(s-1)/2} \circ I_{s-1}^\gamma \circ \partial_j (I - L)^{-(s-1)/2})g,$$

which proves (3.5). Particularly, if g is a constant function we have that

$$(I - L)^{-(s-1)/2}(\partial_j g) = 0 \quad \text{and} \quad \partial_j (I - L)^{-(s-1)/2}g = c_0^g \partial_j h_0 = 0$$

and therefore, (3.5) is also true.

Then, we are able to prove the following proposition (see [15, pages 136–138]).

PROPOSITION 3.1. *Given* $1 < p < \infty$ *and* $s \geq 1$. *Suppose that* $f \in L_{s-1}^p(\gamma_d)$ *and* $\partial_j f \in L_{s-1}^p(\gamma_d)$ *for each* $j = 1, \ldots, d$. *Then,* $f \in L_s^p(\gamma_d)$ *and also,*

$$\|f\|_{p,s} \leq B_{p,s}\left(\|f\|_{p,s-1} + \sum_{j=1}^d \|\partial_j f\|_{p,s-1}\right).$$

Proof. Let $f \in L_{s-1}^p(\gamma_d) \cap \mathcal{P}$. Then, there exists $\psi \in \mathcal{P}$, such that $f = (I - L)^{-(s-1)/2}\psi$ and therefore,

$$f = \sum_{n\geq 0} \left(\frac{1}{n+1}\right)^{(s-1)/2} J_n \psi$$

and

$$\partial_j f = \sum_{n\geq 0} \left(\frac{1}{n+1}\right)^{(s-1)/2} \sum_{|\beta|=n} \sqrt{2\beta_j}c_\beta^\psi h_{\beta-e_j}.$$

Since $\psi \in L_1^p(\gamma_d) \cap \mathcal{P}$, we can write $\psi = (I - L)^{-1/2}h$ and according to the Lemma 3.1, part iii), where we consider $s = 1$, $T_1 = I$ and

$$T_2 = ((I - L)^{1/2} \circ I_1^\gamma) - I_1^\gamma,$$

we obtain that

$$h = (I - L)^{1/2}\psi = T_1\psi + (D_1^\gamma \circ T_2)\psi.$$

Therefore,

$$\|h\|_{p,\gamma_d} \leq \|T_1\psi\|_{p,\gamma_d} + \left\|(D_1^\gamma \circ T_2)\psi\right\|_{p,\gamma_d}. \tag{3.6}$$

Now, we can see that

$$\|h\|_{p,\gamma_d} = \left\|(I - L)^{1/2}\psi\right\|_{p,\gamma_d} = \left\|(I - L)^{1/2}(I - L)^{(s-1)/2}f\right\|_{p,\gamma_d} = \|f\|_{p,s}$$

and

$$\|T_1\psi\|_{p,\gamma_d} = \|\psi\|_{p,\gamma_d} = \left\|(I - L)^{(s-1)/2}f\right\|_{p,\gamma_d} = \|f\|_{p,s-1}.$$

On the other hand, by means of (3.4) we have that

$$(D_1^\gamma \circ T_2)\psi = (T_2 \circ D_1^\gamma)\psi = \frac{1}{2}T_2\left[\sum_{j=1}^{d}(R_1^j)^*(\partial_j\psi)\right]$$

and therefore,

$$
\begin{aligned}
\|(D_1^\gamma \circ T_2)\psi\|_{p,\gamma_d} &= \frac{1}{2}\left\|T_2\left[\sum_{j=1}^{d}(R_1^j)^*(\partial_j\psi)\right]\right\|_{p,\gamma_d} \\
&\leq A_p\left\|\sum_{j=1}^{d}(R_1^j)^*(\partial_j\psi)\right\|_{p,\gamma_d} \\
&\leq C_p\sum_{j=1}^{d}\|\partial_j\psi\|_{p,\gamma_d}.
\end{aligned}
$$

Since $f \in L_{s-1}^p(\gamma_d) \cap \mathcal{P}$ and $\partial_j f \in L_{s-1}^p(\gamma_d) \cap \mathcal{P}$, for each $j = 1, \ldots, d$, according to (3.5) with $g = \psi$ we have that

$$T_0(\partial_j f) = (I - L)^{-(s-1)/2}\partial_j\psi$$

and consequently,

$$
\begin{aligned}
\|\partial_j\psi\|_{p,\gamma_d} &= \left\|(T_0 \circ (I - L)^{(s-1)/2})(\partial_j f)\right\|_{p,\gamma_d} \\
&\leq A_p\left\|(I - L)^{(s-1)/2}(\partial_j f)\right\|_{p,\gamma_d} \\
&= C_p\|\partial_j f\|_{p,s-1},
\end{aligned}
$$

because $T_0 \circ (I - L)^{(s-1)/2} = (I - L)^{(s-1)/2} \circ T_0$. Then, from (3.6) we obtain that

$$\|f\|_{p,s} \leq B_{p,s}\left(\|f\|_{p,s-1} + \sum_{j=1}^{d} \|\partial_j f\|_{p,s-1} \right).$$

In the general case, the density of the polynomials in $L_s^p(\gamma_d)$ is used. ∎

Remark. In the proof of the Proposition 3.1, if f is a constant function, we can see that $f = \psi = h$. Then, trivially we can write

$$h = (I - L)^{1/2}\psi = T_1\psi + (D_1^\gamma \circ T_2)\psi$$

because $(I - L)^{1/2}\psi = \psi$, $T_1\psi = \psi$ and $T_2\psi = 0$. This way, we obtain that

$$\|f\|_{p,s-1} = \|f\|_{p,s}.$$

The Proposition 2.1 and the Proposition 3.1 let us obtain the following corollary.

COROLLARY 3.1. *Suppose $1 < p < \infty$ and $s \geq 1$. Then, $f \in L_s^p(\gamma_d)$ if and only if $f \in L_{s-1}^p(\gamma_d)$ and for each $j = 1, \ldots, d$, $\partial_j f \in L_{s-1}^p(\gamma_d)$. Moreover, the two norms, $\|f\|_{p,s}$ and $\|f\|_{p,s-1} + \sum_{j=1}^{d} \|\partial_j f\|_{p,s-1}$, are equivalent.*

Also, by means of the Proposition 3.1 we get the following result.

COROLLARY 3.2. *Given $1 < p < \infty$ and $k \geq 1$. Suppose $f \in W_k^p(\gamma_d)$, then there exists a positive constant $A_{p,k}$ such that*

$$\|f\|_{p,k} \leq A_{p.k}\left(\|f\|_{p,\gamma_d} + \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{p,\gamma_d} \right).$$

Proof. By using the Proposition 3.1, with $s = 1$, we have

$$\|f\|_{p,1} \leq B_{p,1}\left(\|f\|_{p,\gamma_d} + \sum_{j=1}^{d} \|\partial_j f\|_{p,\gamma_d} \right).$$

If $s = 2$ we can see that

$$\|f\|_{p,2} \leq B_{p,2}\left(\|f\|_{p,1} + \sum_{i=1}^{d} \|\partial_i f\|_{p,1} \right)$$

and for each $i = 1, ..., d$,

$$\|\partial_i f\|_{p,1} \leq A_{p,1}\left(\|\partial_i f\|_{p,\gamma_d} + \sum_{j=1}^{d} \|\partial_{ij}^2 f\|_{p,\gamma_d} \right).$$

Then by means of the estimates above, we get

$$\|f\|_{p,2} \leq C_{p,2}\left(\|f\|_{p,\gamma_d} + \sum_{j=1}^{d} \|\partial_j f\|_{p,\gamma_d} + \sum_{i=1}^{d}\sum_{j=1}^{d} \|\partial_{ij}^2 f\|_{p,\gamma_d} \right),$$

and the result of this corollary is obtained by induction. ∎

Therefore, we can conclude that

$$W_k^p(\gamma_d) \subseteq L_k^p(\gamma_d) \quad \text{for each} \quad k \in \mathbb{N} \tag{3.7}$$

and in consequence, by using (3.3) and (3.7) we obtain the following theorem which it is the principal result of this paper.

THEOREM 3.1. *Suppose k is a positive integer and $1 < p < \infty$. Then*

$$L_k^p(\gamma_d) = W_k^p(\gamma_d).$$

Particularly, as a final comment we observe that

Remark. If $0 < s < 1$, $1 < p < \infty$ and $k \geq 1$, then $W_k^p(\gamma_d) \subset L_s^p(\gamma_d)$ and the inclusion is proper.

In fact, if $0 < s < 1$ and $k \geq 1$, the inclusion properties of the potential spaces allows us to write

$$W_k^p(\gamma_d) = L_k^p(\gamma_d) \subset L_s^p(\gamma_d).$$

Now, let us consider $d = 1$, $k = 1$, $p = 2$, $0 < s < 1$ and the function f defined as

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

First, one checks that $f \in L^2(\gamma_1)$ but $f' \notin L^2(\gamma_1)$ and therefore $f \notin W_1^2(\gamma_1)$.

Now, for each $s \in (0, 1)$ we define the family of functions

$$g_s(x) = \begin{cases} \sqrt{2^s x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and from (2.2) we get that the semigroup, $T_t g_s$, can be written as

$$T_t g_s(x) = \frac{\sqrt{2^s}}{\sqrt{\pi(1 - e^{-2t})}} \int_0^\infty e^{-\frac{|y - e^{-t}x|^2}{1 - e^{-2t}}} \sqrt{y}\, dy.$$

By using the change of variable $u = \frac{y - e^{-t}x}{\sqrt{1 - e^{-2t}}}$ in the above identity, we have that

$$T_t g_s(x) = \sqrt{\frac{2^s}{\pi}} \int_{-\frac{xe^{-t}}{(1 - e^{-2t})^{1/2}}}^\infty \left(\sqrt{1 - e^{-2t}}\, u + e^{-t}x\right)^{1/2} e^{-u^2}\, du$$

$$\geq e^{-t/2}\sqrt{x}\sqrt{\frac{2^s}{\pi}} \int_0^\infty e^{-u^2}\, du,$$

so, $T_t g_s(x) \geq 2^{s/2} e^{-t/2}\sqrt{x}$, for all $t > 0$ and $x \geq 0$. Therefore,

$$(I - L)^{-s/2} g_s(x) = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{\frac{s}{2} - 1} e^{-t} T_t g_s(x)\, dt$$

$$\geq 2^{s/2}\sqrt{x}\left(\frac{1}{\Gamma(s/2)} \int_0^\infty t^{\frac{s}{2} - 1} e^{-2t}\, dt\right)$$

and then, by use of the change of variable $t = u/2$, we obtain

$$(I - L)^{-s/2} g_s(x) \geq f(x) \quad \text{for each} \quad x \geq 0. \tag{3.8}$$

From (2.17) if $h_1 \geq h_2$, we have that

$$(I - L)^{-s/2} h_1 \geq (I - L)^{-s/2} h_2, \tag{3.9}$$

because T_t is linear and $T_t h \geq 0$, if $h \geq 0$. Also, if $h_1 \geq h_2$, we can see that

$$(I - L)^{s/2} h_1 \geq (I - L)^{s/2} h_2.$$

In fact, suppose that $h_1 \geq h_2$ but $(I - L)^{s/2} h_1 < (I - L)^{s/2} h_2$, then by using (3.9) we obtain a contradiction. Thus, by applying the operator $(I - L)^{s/2}$ in (3.8), we get

$$g_s(x) \geq (I - L)^{s/2} f(x)$$

and consequently,

$$\|f\|_{2,s} = \left\|(I - L)^{s/2} f\right\|_{2,\gamma_1} \leq \|g_s\|_{2,\gamma_1} = 2^{s/2}\|f\|_{2,\gamma_1} \leq \sqrt{2}\|f\|_{2,\gamma_1},$$

since $s \in (0, 1)$. Therefore, we can conclude that $f \in L_s^2(\gamma_1)$.

This way, we have obtained a function f, such that, $f \notin W_1^2(\gamma_1)$ but $f \in L_s^2(\gamma_1)$, if $0 < s < 1$.

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