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Exposed Polynomials of $\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)$

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Abstract: We show that every extreme polynomials of $\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)$ is exposed.

Key words: The Krein-Milman Theorem, extreme polynomials, exposed polynomials, the plane with a hexagonal norm.

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1. Introduction

According to the Krein-Milman Theorem, every nonempty convex set in a Banach space is fully described by the set of its extreme points. Let $n \in \mathbb{N}$. We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. We recall that if $x \in B_E$ is said to be an *extreme point* of B_E if $y, z \in B_E$ and $x = \lambda y + (1-\lambda)z$ for some $0 < \lambda < 1$ implies that $x = y = z$. $x \in B_E$ is called an *exposed point* of B_E if there is an $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $\text{ext}B_E$ and $\exp B_E$ the sets of extreme and exposed points of B_E, respectively. We denote by $\mathcal{L}({}^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \ldots, x_n)|$. A n-linear form T is symmetric if $T(x_1, \ldots, x_n) = T\big(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\big)$ for every permutation σ on $\{1, 2, \ldots, n\}$. We denote by $\mathcal{L}_s({}^nE)$ the Banach space of all continuous symmetric n-linear forms on E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a unique $T \in \mathcal{L}_s({}^nE)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. In this case it is convenient to write $T = \check{P}$. We denote by $\mathcal{P}({}^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. Note that the spaces $\mathcal{L}({}^nE)$,

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$\mathcal{L}_s(^nE)$, $\mathcal{P}(^nE)$ are very different from a geometric point of view. In particular, for integral multilinear forms and integral polynomials one has ([2], [9], [42])

$$\mathrm{ext}B_{\mathcal{L}_I(^nE)} = \{\phi_1\phi_2\cdots\phi_n : \phi_i \in \mathrm{ext}B_{E^*}\},$$
$$\mathrm{ext}B_{\mathcal{P}_I(^nE)} = \{\pm\phi^n : \phi \in E^*,\ \|\phi\| = 1\},$$

where $\mathcal{L}_I(^nE)$ and $\mathcal{P}_I(^nE)$ are the spaces of integral n-linear forms and integral n-homogeneous polynomials on E, respectively. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [10].

Let us say about the stories of the classification problems of $\mathrm{ext}B_X$ and $\mathrm{exp}B_X$ if $X = \mathcal{P}(^nE)$. Choi *et al.* ([4], [5]) initiated the classification problems and classified $\mathrm{ext}B_X$ if $X = \mathcal{P}\big(^2l_p^2\big)$ for $p = 1, 2$, where $l_p^2 = \mathbb{R}^2$ with the l_p-norm. B. Grecu [14] classified $\mathrm{ext}B_X$ if $X = \mathcal{P}\big(^2l_p^2\big)$ for $1 < p < 2$ or $2 < p < \infty$. Kim [18] classified $\mathrm{exp}B_X$ if $X = \mathcal{P}\big(^2l_p^2\big)$ for $1 \le p \le \infty$. Kim *et al.* [34] showed that every extreme 2-homogeneous polynomials on a real separable Hilbert space is also exposed. Kim ([20], [26]) characterized $\mathrm{ext}B_X$ and $\mathrm{exp}B_X$ for $X = \mathcal{P}\big(^2d_*(1,w)^2\big)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm

$$\|(x,y)\|_{d_*} = \max\Big\{|x|,\ |y|,\ \tfrac{|x|+|y|}{1+w}\ :\ 0 < w < 1\Big\}.$$

He showed [26] that $\mathrm{ext}B_{\mathcal{P}(^2d_*(1,w)^2)} \ne \mathrm{exp}B_{\mathcal{P}(^2d_*(1,w)^2)}$. In [31], Kim classified $\mathrm{ext}B_X$ and using the classification of $\mathrm{ext}B_X$, Kim computed the polarization and unconditional constants of the space X if $X = \mathcal{P}\big(^2\mathbb{R}^2_{h(\frac{1}{2})}\big)$, where $\mathbb{R}^2_{h(w)}$ denotes the space $\mathbb{R}^2$ endowed with the hexagonal norm

$$\|(x,y)\|_{h(w)} := \max\{|y|,\ |x| + (1-w)|y|\}.$$

We refer to ([1]–[9], [11]–[43]) and references therein for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

We will denote by $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1)$ and $P(x,y) = ax^2 + by^2 + cxy$ a symmetric bilinear form and a 2-homogeneous polynomial on a real Banach space of dimension 2, respectively. Recently, Kim [31] classified the extreme points of the unit ball of $\mathcal{P}\big(^2\mathbb{R}^2_{h(\frac{1}{2})}\big)$ as follows:

$$\begin{aligned}\mathrm{ext}B_{\mathcal{P}\left(^2\mathbb{R}^2_{h(\frac{1}{2})}\right)} = \Big\{ &\pm y^2,\ \pm\big(x^2 + \tfrac{1}{4}y^2 \pm xy\big),\ \pm\big(x^2 + \tfrac{3}{4}y^2\big),\\ &\pm\big[x^2 + \big(\tfrac{c^2}{4} - 1\big)y^2 \pm cxy\big],\\ &\pm\big[cx^2 + \big(\tfrac{c+4\sqrt{1-c}}{4} - 1\big)y^2 \pm \big(c + 2\sqrt{1-c}\big)xy\big]\ (0 \le c \le 1)\Big\}.\end{aligned}$$

In this paper, we show that that every extreme polynomials of $\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)$ is exposed.

2. Results

THEOREM 2.1. ([31]) *Let* $P(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)$ *with* $a \geq 0$, $c \geq 0$ *and* $a^2 + b^2 + c^2 \neq 0$. *Then:*

Case $1: \ c < a$.

If $a \leq 4b$, *then*

$$\begin{aligned}\|P\| &= \max\Big\{a,\, b,\, \big|\tfrac{1}{4}a + b\big| + \tfrac{1}{2}c,\, \tfrac{4ab-c^2}{4a},\, \tfrac{4ab-c^2}{2c+a+4b},\, \tfrac{4ab-c^2}{|2c-a-4b|}\Big\} \\ &= \max\Big\{a,\, b,\, \big|\tfrac{1}{4}a + b\big| + \tfrac{1}{2}c\Big\}.\end{aligned}$$

If $a > 4b$, *then* $\|P\| = \max\Big\{a,\, |b|,\, \big|\tfrac{1}{4}a + b\big| + \tfrac{1}{2}c,\, \tfrac{|c^2-4ab|}{4a}\Big\}$.

Case $2: \ c \geq a$.

If $a \leq 4b$, *then* $\|P\| = \max\Big\{a,\, b,\, \big|\tfrac{1}{4}a + b\big| + \tfrac{1}{2}c,\, \tfrac{|c^2-4ab|}{2c+a+4b}\Big\}$.

If $a > 4b$, *then* $\|P\| = \max\Big\{a,\, |b|,\, \big|\tfrac{1}{4}a + b\big| + \tfrac{1}{2}c,\, \tfrac{c^2-4ab}{2c-a-4b}\Big\}$.

THEOREM 2.2. ([31])

$$\begin{aligned}\mathrm{ext}B_{\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)} = \Big\{ &\pm y^2,\ \pm\big(x^2 + \tfrac{1}{4}y^2 \pm xy\big),\ \pm\big(x^2 + \tfrac{3}{4}y^2\big), \\ &\pm\big[x^2 + \big(\tfrac{c^2}{4} - 1\big)y^2 \pm cxy\big], \\ &\pm\big[cx^2 + \big(\tfrac{c+4\sqrt{1-c}}{4} - 1\big)y^2 \pm \big(c + 2\sqrt{1-c}\big)xy\big]\ (0 \leq c \leq 1)\Big\}.\end{aligned}$$

THEOREM 2.3. *Let* $f \in \mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)^*$ *with* $\alpha = f(x^2)$, $\beta = f(y^2)$, $\gamma = f(xy)$. *Then*

$$\begin{aligned}\|f\| = \sup\Big\{&|\beta|,\ \big|\alpha + \tfrac{1}{4}\beta\big| + |\gamma|,\ \big|\alpha + \tfrac{3}{4}\beta\big|,\ \big|\alpha + \big(\tfrac{c^2}{4} - 1\big)\beta\big| + c|\gamma|, \\ &\big|c\alpha + \big(\tfrac{c+4\sqrt{1-c}}{4} - 1\big)\beta\big| + (c + 2\sqrt{1-c})|\gamma|\ (0 \leq c \leq 1)\Big\}.\end{aligned}$$

Proof. It follows from Theorem 2.2 and the fact that

$$\|f\| = \sup\Big\{|f(P)|\ :\ P \in \mathrm{ext}B_{\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)}\Big\}.$$

∎

Note that if $\|f\| = 1$, then $|\alpha| \leq 1$, $|\beta| \leq 1$, $|\gamma| \leq \frac{1}{2}$.

We are in a position to show the main result of this paper.

THEOREM 2.4.

$$\exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)} = \operatorname{ext} B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}.$$

Proof. Let $(0 \leq c \leq 1)$

$$\begin{aligned}
P_1(x,y) &= y^2\,,\\
P_2^+(x,y) &= x^2 + \tfrac{1}{4}y^2 + xy\,,\\
P_2^-(x,y) &= x^2 + \tfrac{1}{4}y^2 - xy\,,\\
P_3(x,y) &= x^2 + \tfrac{3}{4}y^2\,,\\
P_{4,c}^+(x,y) &= x^2 + \left(\tfrac{c^2}{4} - 1\right)y^2 + cxy\,,\\
P_{4,c}^-(x,y) &= x^2 + \left(\tfrac{c^2}{4} - 1\right)y^2 - cxy\,,\\
P_{5,c}^+(x,y) &= cx^2 + \left(\tfrac{c+4\sqrt{1-c}}{4} - 1\right)y^2 + (c + 2\sqrt{1-c})xy\,,\\
P_{5,c}^-(x,y) &= cx^2 + \left(\tfrac{c+4\sqrt{1-c}}{4} - 1\right)y^2 - (c + 2\sqrt{1-c})xy\,.
\end{aligned}$$

Claim 1: $P_1 = y^2 \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$.

Let $f \in \mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)^*$ be such that

$$\alpha = \frac{1}{5}\,, \qquad \beta = 1\,, \qquad \gamma = 0\,.$$

Indeed,

$$f(P_1) = 1\,, \quad |f(P_2^{\pm})| = \frac{9}{20}\,, \quad |f(P_3)| = \frac{19}{20}\,. \tag{*}$$

Note that for all $0 \leq c \leq 1$,

$$|f(P_{4,c}^{\pm})| = \frac{4}{5} - \frac{c^2}{4} \leq \frac{4}{5}\,, \tag{**}$$

$$|f(P_{5,c}^{\pm})| = |\sqrt{1-c} + \frac{9c}{20} - 1| \leq \frac{11}{20}\,. \tag{***}$$

Hence, by Theorem 2.3, $1 = \|f\|$. We will show that f exposes P_1. Let $Q(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)$ such that $1 = \|Q\| = f(Q)$. We will show that $Q = P_1$. Since $\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)$ is a finite dimensional Banach space with dimension 3, by the Krein-Milman Theorem, $B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$ is the closed convex hull of $\mathrm{ext}B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$. Then,

$$\begin{aligned}Q(x,y) &= uP_1(x,y) + v^+P_2^+(x,y) + v^-P_2^-(x,y) + tP_3(x,y)\\ &\quad + \sum_{n=1}^{\infty}\lambda_n^+P^+_{4,c_n^+}(x,y) + \sum_{n=1}^{\infty}\lambda_n^-P^-_{4,c_n^-}(x,y)\\ &\quad + \sum_{m=1}^{\infty}\delta_m^+P^+_{5,a_m^+}(x,y) + \sum_{m=1}^{\infty}\delta_m^-P^-_{5,a_m^-}(x,y)\,,\end{aligned}$$

for some $u, v^{\pm}, t, \lambda_n^{\pm}, \delta_m^{\pm}, \in \mathbb{R}$ $(n, m \in \mathbb{N})$ with $0 \le c_n^{\pm}, a_m^{\pm} \le 1$ and

$$|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty}|\lambda_n^+| + \sum_{n=1}^{\infty}|\lambda_n^-| + \sum_{m=1}^{\infty}|\delta_m^+| + \sum_{m=1}^{\infty}|\delta_m^-| = 1\,.$$

We will show that $v^{\pm} = t = \lambda_n^{\pm} = \delta_m^{\pm} = 0$ for every $n, m \in \mathbb{N}$.

Subclaim: $v^{\pm} = t = 0$.

Assume that $v^+ \neq 0$. It follows that

$$\begin{aligned}1 = f(Q) &= uf(P_1) + v^+f(P_2^+) + v^-f(P_2^-) + tf(P_3) + \sum_{n=1}^{\infty}\lambda_n^+f(P^+_{4,c_n})\\ &\quad + \sum_{n=1}^{\infty}\lambda_n^-f(P^-_{4,c_n}) + \sum_{m=1}^{\infty}\delta_m^+f(P^+_{5,a_m}) + \sum_{m=1}^{\infty}\delta_m^-f(P^-_{5,a_m})\\ &\le |u| + |v^+||f(P_2^+)| + |v^-||f(P_2^-)| + |t||f(P_3)| + \sum_{n=1}^{\infty}|\lambda_n^+||f(P^+_{4,c_n})|\\ &\quad + \sum_{n=1}^{\infty}|\lambda_n^-||f(P^-_{4,c_n})| + \sum_{m=1}^{\infty}|\delta_m^+||f(P^+_{5,a_m})| + \sum_{m=1}^{\infty}|\delta_m^-||f(P^-_{5,a_m})|\\ &\le |u| + \frac{9}{20}|v^+| + \frac{9}{20}|v^-| + \frac{19}{20}|t| + \frac{4}{5}\sum_{n=1}^{\infty}|\lambda_n^+|\\ &\quad + \frac{4}{5}\sum_{n=1}^{\infty}|\lambda_n^-| + \frac{11}{20}\sum_{m=1}^{\infty}|\delta_m^+| + \frac{11}{20}\sum_{m=1}^{\infty}|\delta_m^-| \qquad \text{(by (*), (**), (***))}\end{aligned}$$

$$< |u| + |v^+| + \frac{9}{20}|v^-| + \frac{19}{20}|t| + \frac{4}{5}\sum_{n=1}^{\infty}|\lambda_n^+|$$
$$+ \frac{4}{5}\sum_{n=1}^{\infty}|\lambda_n^-| + \frac{11}{20}\sum_{m=1}^{\infty}|\delta_m^+| + \frac{11}{20}\sum_{m=1}^{\infty}|\delta_m^-|$$
$$\leq |u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty}|\lambda_n^+| + \sum_{n=1}^{\infty}|\lambda_n^-| + \sum_{m=1}^{\infty}|\delta_m^+| + \sum_{m=1}^{\infty}|\delta_m^-| \;=\; 1\,,$$

which is impossible. Therefore, $v^+ = 0$. Using a similar argument as above, we have $v^- = t = 0$.

Subclaim: $\lambda_n^{\pm} = \delta_m^{\pm} = 0$ for every $n, m \in \mathbb{N}$.

Assume that $\lambda_{n_0}^+ \neq 0$ for some $n_0 \in \mathbb{N}$. It follows that

$$1 = f(Q) = uf(P_1) + \lambda_{n_0}^+ f(P_{4,c_{n_0}}^+) + \sum_{n\in\mathbb{N}, n\neq n_0} \lambda_n^+ f(P_{4,c_n^+}^+)$$
$$+ \sum_{n=1}^{\infty}\lambda_n^- f(P_{4,c_n^-}^-) + \sum_{m=1}^{\infty}\delta_m^+ f(P_{5,a_m^+}^+) + \sum_{m=1}^{\infty}\delta_m^- f(P_{5,a_m^-}^-)$$
$$\leq |u| + |\lambda_{n_0}^+||f(P_{4,c_{n_0}^+}^+)| + \sum_{n\in\mathbb{N}, n\neq n_0}|\lambda_n^+||f(P_{4,c_n^+}^+)| + \sum_{n=1}^{\infty}|\lambda_n^-||f(P_{4,c_n^-}^-)|$$
$$+ \sum_{m=1}^{\infty}|\delta_m^+||f(P_{5,a_m^+}^+)| + \sum_{m=1}^{\infty}|\delta_m^-||f(P_{5,a_m^-}^-)|$$
$$< |u| + |\lambda_{n_0}^+| + \frac{4}{5}\sum_{n\in\mathbb{N}, n\neq n_0}|\lambda_n^+| + \frac{4}{5}\sum_{n=1}^{\infty}|\lambda_n^-| + \frac{11}{20}\sum_{m=1}^{\infty}|\delta_m^+| + \frac{11}{20}\sum_{m=1}^{\infty}|\delta_m^-|$$
$$\leq |u| + \sum_{n=1}^{\infty}|\lambda_n^+| + \sum_{n=1}^{\infty}|\lambda_n^-| + \sum_{m=1}^{\infty}|\delta_m^+| + \sum_{m=1}^{\infty}|\delta_m^-| \;=\; 1\,,$$

which is impossible. Therefore, $\lambda_n^+ = 0$ for every $n \in \mathbb{N}$. Using a similar argument as above, we have $\lambda_n^- = \delta_m^{\pm} = 0$ for every $n, m \in \mathbb{N}$. Therefore, $Q(x,y) = uP_1(x,y)$. Hence $u = 1$, so $Q = P_1$. Therefore, f exposes P_1.

Claim 2: $P_{5,0} = 2xy \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$.

Let $f \in \mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)^*$ be such that

$$\alpha = \beta = 0\,, \qquad \gamma = \frac{1}{2}\,.$$

We will show that f exposes $P_{5,0}$. Indeed, $f(P_{5,0})=1$, $f(P_1)=0$, $f(P_2^{\pm})=\pm\frac{1}{2}$, $f(P_3)=0$,

$$-\frac{1}{2}\le f(P_{4,c}^{\pm})=\pm\frac{c}{2}\le\frac{1}{2}\qquad(0\le c\le 1)\,.$$

Note that, for $0<c\le 1$,

$$-1<f(P_{5,c}^{\pm})=\pm\frac{c+2\sqrt{1-c}}{2}<1\,. \tag{†}$$

Hence, by Theorem 2.3, $1=\|f\|$. Let

$$\begin{aligned}Q(x,y)&=uP_1(x,y)+v^+P_2^+(x,y)+v^-P_2^-(x,y)+tP_3(x,y)\\&\quad+\sum_{n=1}^{\infty}\lambda_n^+P_{4,c_n^+}^+(x,y)+\sum_{n=1}^{\infty}\lambda_n^-P_{4,c_n^-}^-(x,y)\\&\quad+\sum_{m=1}^{\infty}\delta_m^+P_{5,a_m^+}^+(x,y)+\sum_{m=1}^{\infty}\delta_m^-P_{5,a_m^-}^-(x,y)\,,\end{aligned}$$

for some $u,v^{\pm},t,\lambda_n^{\pm},\delta_m^{\pm},\in\mathbb{R}$ $(n,m\in\mathbb{N})$ with $0\le c_n^{\pm},a_m^{\pm}\le 1$ and

$$|u|+|v^+|+|v^-|+|t|+\sum_{n=1}^{\infty}|\lambda_n^+|+\sum_{n=1}^{\infty}|\lambda_n^-|+\sum_{m=1}^{\infty}|\delta_m^+|+\sum_{m=1}^{\infty}|\delta_m^-|\;=1\,.$$

We will show that $v^{\pm}=t=\lambda_n^{\pm}=\delta_m^{\pm}=0$ for every $n,m\in\mathbb{N}$.

Subclaim: $v^+=0$.

Assume that $v^+\ne 0$. It follows that

$$\begin{aligned}1=f(Q)&=v^+f(P_2^+)+v^-f(P_2^-)+\sum_{n=1}^{\infty}\lambda_n^+f(P_{4,c_n^+}^+)\\&\quad+\sum_{n=1}^{\infty}\lambda_n^-f(P_{4,c_n^-}^-)+\sum_{m=1}^{\infty}\delta_m^+f(P_{5,a_m^+}^+)+\sum_{m=1}^{\infty}\delta_m^-f(P_{5,a_m^-}^-)\\&<|v^+|+\frac{1}{2}|v^-|+\sum_{n=1}^{\infty}|\lambda_n^+||f(P_{4,c_n^+}^+)|+\sum_{n=1}^{\infty}|\lambda_n^-||f(P_{4,c_n^-}^-)|\\&\quad+\sum_{m=1}^{\infty}|\delta_m^+||f(P_{5,a_m^+}^+)|+\sum_{m=1}^{\infty}|\delta_m^-||f(P_{5,a_m^-}^-)|\\&\le|v^+|+|v^-|+\sum_{n=1}^{\infty}|\lambda_n^+|+\sum_{n=1}^{\infty}|\lambda_n^-|+\sum_{m=1}^{\infty}|\delta_m^+|+\sum_{m=1}^{\infty}|\delta_m^-|\;\le\;1\,,\end{aligned}$$

which is impossible. Therefore, $v^+ = 0$. Using a similar argument as Claim 1, we have $v^- = \lambda_n^{\pm} = 0$ for every $n \in \mathbb{N}$. Hence,

$$Q(x,y) = uP_1(x,y) + tP_3(x,y) + \sum_{m=1}^{\infty} \delta_m^+ P_{5,a_m^+}^+(x,y) + \sum_{m=1}^{\infty} \delta_m^- P_{5,a_m^-}^-(x,y)\,.$$

It follows that

$$\begin{aligned} 1 = f(Q) &= \sum_{m=1}^{\infty} \delta_m^+ f(P_{5,a_m^+}^+) + \sum_{m=1}^{\infty} \delta_m^- f(P_{5,a_m^-}^-) \\ &\le \sum_{m=1}^{\infty} |\delta_m^+||f(P_{5,a_m^+}^+)| + \sum_{m=1}^{\infty} |\delta_m^-||f(P_{5,a_m^-}^-)| \\ &\le \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| \ \le\ 1\,, \end{aligned}$$

which shows that

$$f(P_{5,a_m^+}^+) = f(P_{5,a_m^-}^-) = \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1\,, \quad u = t = 0 \quad \text{for all } m \in \mathbb{N}\,.$$

By (†), $P_{5,a_m^{\pm}}^{\pm} = P_{5,0}$ for every $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \delta_m^+ + \sum_{m=1}^{\infty} \delta_m^- = 1$. Therefore, $Q = P_{5,0}$. Hence, f exposes $P_{5,0}$.

Claim 3: $P_2^+ = x^2 + \frac{1}{4}y^2 + xy \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$.

Let $f \in \mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)^*$ be such that

$$\alpha = \frac{1}{2} = \beta\,, \qquad \gamma = \frac{3}{8}\,.$$

We will show that f exposes P_2. Indeed, $f(P_2^+) = 1$, $f(P_2^-) = \frac{1}{4}$, $f(P_1) = \frac{1}{2}$, $f(P_3^{\pm}) = \frac{7}{8}$. By some calculation, we have

$$|f(P_{4,c}^{\pm})| \le \frac{1}{2}\,, \qquad |f(P_{5,c}^{\pm})| \le \frac{57}{64} \qquad \text{for } 0 \le c \le 1\,.$$

Hence, by Theorem 2.3, $1 = \|f\|$. By similar arguments as Claims 1 and 2, f exposes P_2^+. Obviously, $P_2^- \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$.

Claim 4: $P_{4,0}^+ = x^2 - y^2 \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$.

Let $f \in \mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)^*$ be such that

$$\alpha = \frac{1}{2} = -\beta\,, \qquad \gamma = 0\,.$$

We will show that f exposes $P_{4,0}$. Indeed,

$$f(P^+_{4,0}) = 1\,, \quad |f(P_1)| = \frac{1}{2}\,, \quad |f(P^\pm_2)| = \frac{3}{8}\,, \quad |f(P_3)| = \frac{1}{8}\,.$$

Note that

$$|f(P^\pm_{4,c})| = 1 - \frac{c^2}{8} < 1 \qquad \text{for } 0 < c \le 1\,.$$

Note that, for $0 \le c \le 1$,

$$|f(P^\pm_{5,c})| = \frac{3c + 4 - 4\sqrt{1-c}}{8} \le \frac{7}{8}\,.$$

Hence, by Theorem 2.3, $1 = \|f\|$. By similar arguments as Claims 1 and 2, f exposes $P^+_{4,0}$.

Claim 5: $P_3 = x^2 + \frac{3}{4}y^2 \in \exp B_{\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)}$.

Let $f \in \mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)^*$ be such that

$$\alpha = \frac{5}{8}\,, \qquad \beta = \frac{1}{2}\,, \qquad \gamma = 0\,.$$

We will show that f exposes P_3. Indeed,

$$f(P_3) = 1\,, \quad |f(P_1)| = \frac{1}{2}\,, \quad |f(P^\pm_2)| = \frac{3}{4}\,.$$

Note that

$$|f(P^\pm_{4,c})| \le \frac{1}{4}\,, \quad |f(P^\pm_{5,c})| \le \frac{1}{3} \qquad \text{for } 0 \le c \le 1\,.$$

Hence, by Theorem 2.3, $1 = \|f\|$. By similar arguments as Claims 1 and 2, f exposes P_3.

Claim 6: $P^+_{5,1} = x^2 - \frac{3}{4}y^2 + xy \in \exp B_{\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)}$.

Let $f \in \mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)^*$ be such that

$$\alpha = \frac{11}{16}\,, \qquad \beta = -\frac{1}{4}\,, \qquad \gamma = \frac{1}{8}.$$

We will show that f exposes $P_{5,1}^+$. Indeed,

$$f(P_{5,1}^+) = 1\,, \quad |f(P_1)| = \frac{1}{4}\,, \quad |f(P_2^\pm)| \le \frac{3}{4}\,, \quad |f(P_3)| = \frac{1}{2}\,.$$

Note that

$$\frac{3}{4} \le f(P_{4,c}^\pm) < 1\,, \quad -\frac{1}{4} \le f(P_{5,c}^\pm) < 1 \qquad \text{for } 0 \le c < 1\,.$$

Hence, by Theorem 2.3, $1 = \|f\|$. By similar arguments as Claims 1 and 2, f exposes $P_{5,1}^+$. Obviously, $P_{5,1}^- \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$.

Claim 7: $P_{4,c}^+ = x^2 + (\frac{c^2}{4} - 1)y^2 + cxy \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$ for $0 < c < 1$.

Let $f \in \mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)^*$ be such that

$$\alpha = \frac{3}{4} - \frac{c^2}{16}\,, \qquad \beta = -\frac{1}{4}\,, \qquad \gamma = \frac{c}{8}\,.$$

Indeed,

$$\begin{gathered} f(P_{4,c}^+) = 1\,, \qquad \frac{3}{4} \le f(P_{4,c}^-) = 1 - \frac{c^2}{4} < 1\,, \qquad |f(P_1)| = \frac{1}{4}\,, \\ \frac{1}{2} \le f(P_2^\pm) \le \frac{3}{4}\,, \qquad \frac{1}{2} \le f(P_3) < \frac{9}{16}\,. \end{gathered} \tag{*}$$

Note that for every $t \in [0,1]$ with $t \neq c$,

$$f(P_{4,t}^+) = -\frac{1}{16}t^2 + \frac{c}{8}t + \left(1 - \frac{c^2}{16}\right)$$

and

$$f(P_{4,t}^-) = -\frac{1}{16}t^2 - \frac{c}{8}t + \left(1 - \frac{c^2}{16}\right).$$

Hence, we have, for every $t \in [0,1]$ with $t \neq c$,

$$1 < \min\left\{1 - \frac{c^2}{16}, 1 - \frac{(1-c)^2}{16}\right\} \le f(P_{4,t}^+) < 1 \tag{**}$$

and

$$-1 < 1 - \frac{(1+c)^2}{16} \le f(P_{4,t}^-) \le 1 - \frac{c^2}{16} < 1\,.$$

Note that, for every $t \in [0,1]$,

$$f(P^+_{5,t}) = \left(\frac{-c^2+2c+11}{16}\right)t + \left(\frac{c-1}{4}\right)\sqrt{1-t} + \frac{1}{4}$$

and

$$f(P^-_{5,t}) = \left(\frac{-c^2-2c+11}{16}\right)t + \left(\frac{c+1}{4}\right)\sqrt{1-t} + \frac{1}{4}\,.$$

Hence, we have that, for every $t \in [0,1]$,

$$-1 < \frac{c}{4} \le f(P^+_{5,t}) \le \frac{-c^2+2c+15}{16} < 1 \qquad (***)$$

and

$$-1 < \frac{c+2}{4} \le f(P^-_{5,t}) \le \frac{-c^2-2c+15}{16} < 1\,.$$

Hence, by Theorem 2.3, $1 = \|f\|$. We will show that f exposes $P^+_{4,c}$. Let $Q(x,y) = ax^2+by^2+cxy \in \mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)$ such that $1 = \|Q\| = f(Q)$. We will show that $Q = P^+_{4,c}$. By the Krein-Milman Theorem,

$$\begin{aligned} Q(x,y) &= uP_1(x,y) + v^+P^+_2(x,y) + v^-P^-_2(x,y) + tP_3(x,y) \\ &\quad + \sum_{n=1}^{\infty} \lambda^+_n P^+_{4,c^+_n}(x,y) + \sum_{n=1}^{\infty} \lambda^-_n P^-_{4,c^-_n}(x,y) \\ &\quad + \sum_{m=1}^{\infty} \delta^+_m P^+_{5,a^+_m}(x,y) + \sum_{m=1}^{\infty} \delta^-_m P^-_{5,a^-_m}(x,y)\,, \end{aligned}$$

for some $u, v^\pm, t, \lambda^\pm_n, \delta^\pm_m, \in \mathbb{R}$ $(n, m \in \mathbb{N})$ with $0 \le c^\pm_n, a^\pm_m \le 1$ and

$$|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda^+_n| + \sum_{n=1}^{\infty} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m| = 1\,.$$

We will show that $u = v^\pm = t = \lambda^-_n = \delta^\pm_m = 0$ for every $n, m \in \mathbb{N}$. Assume that $\delta^+_{m_0} \neq 0$ for some $m_0 \in \mathbb{N}$. It follows that

$$\begin{aligned} 1 = f(Q) &= uf(P_1) + v^+ f(P_2^+) + v^- f(P_2^-) + tf(P_3) + \sum_{n=1}^{\infty} \lambda_n^+ f(P^+_{4,c_n^+}) \\ &+ \sum_{n=1}^{\infty} \lambda_n^- f(P^-_{4,c_n^-}) + \sum_{m=1}^{\infty} \delta_m^+ f(P^+_{5,a_m^+}) + \sum_{m=1}^{\infty} \delta_m^- f(P^-_{5,a_m^-}) \\ &< \frac{1}{4}|u| + \frac{3}{4}|v^+| + \frac{3}{4}|v^-| + \frac{9}{16}|t| + \sum_{n=1}^{\infty} |\lambda_n^+| \\ &+ \sum_{n=1}^{\infty} |\lambda_n^-| + |\delta^+_{m_0}| + \sum_{m\neq m_0} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| \quad \text{(by (*), (**), (***))} \ \leq 1\,, \end{aligned}$$

which is impossible. Therefore, $\delta_m^+ = 0$ for every $m \in \mathbb{N}$. Using a similar argument as above, we have $u = v^{\pm} = t = \lambda_n^- = 0$. Therefore,

$$Q(x,y) = \sum_{n=1}^{\infty} \lambda_n^+ P^+_{4,c_n^+}(x,y)\,.$$

We will show that if $c^+_{n_0} \neq c$ for some $n_0 \in \mathbb{N}$, then $\lambda^+_{n_0} = 0$. Assume that $\lambda^+_{n_0} \neq 0$. It follows that

$$\begin{aligned} 1 = f(Q) &= \lambda^+_{n_0} f(P^+_{4,c^+_{n_0}}) + \sum_{n\neq n_0} \lambda_n^+ f(P^+_{4,c_n^+}) \\ &< |\lambda^+_{n_0}| + \sum_{n\neq n_0} |\lambda_n^+| \ = \ 1\,, \end{aligned}$$

which is impossible. Therefore, $\lambda_n^+ = 0$ for every $n \in \mathbb{N}$. Therefore,

$$Q(x,y) = \Bigg(\sum_{c_n^+ = c} \lambda_n^+\Bigg) P^+_{4,c}(x,y) = P^+_{4,c}(x,y)\,.$$

Therefore, f exposes $P^+_{4,c}$. Obviously, $P^-_{4,c} \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$ for $0 < c \leq 1$.

Claim 8: $P^+_{5,c} = cx^2 + \left(\frac{c+4\sqrt{1-c}}{4} - 1\right)y^2 + (c+2\sqrt{1-c})xy \in \exp B_{\mathcal{P}\left({}^2\mathbb{R}^2_{h(\frac{1}{2})}\right)}$ for $0 < c < 1$.

Let $f \in \mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)^*$ be such that

$$\alpha = \frac{1}{2}\left(1 - \frac{c + 4\sqrt{1-c}}{4}\right), \qquad \beta = -\frac{c}{2}, \qquad \gamma = \frac{c + 2\sqrt{1-c}}{4}.$$

Note that

$$0 \le \alpha < \frac{3}{8}, \qquad -\frac{1}{2} < \beta \le 0, \qquad \frac{1}{4} < \gamma \le \frac{1}{2}.$$

We will show that f exposes $P^+_{5,c}$. Indeed,

$$\begin{gathered} f(P^+_{5,c}) = 1, \qquad |f(P_1)| < \frac{1}{2}, \qquad 0 < f(P^+_2) < \frac{1}{2}, \\ -1 < f(P^-_2) < -\frac{1}{8}, \qquad -\frac{1}{8} \le f(P_3) < 0. \end{gathered} \tag{*}$$

Note that for every $t \in [0,1]$,

$$f(P^+_{4,t}) = -\frac{c}{8}t^2 + \left(\frac{c + 2\sqrt{1-c}}{4}\right)t + \frac{1}{2} + \frac{3c}{8} - \frac{\sqrt{1-c}}{2}$$

and

$$f(P^-_{4,t}) = -\frac{c}{8}t^2 - \left(\frac{c + 2\sqrt{1-c}}{4}\right)t + \frac{1}{2} + \frac{3c}{8} - \frac{\sqrt{1-c}}{2}.$$

Hence, we have for every $t \in [0,1]$,

$$\begin{gathered} -1 < \frac{1}{2} + \frac{3c}{8} - \frac{\sqrt{1-c}}{2} \le f(P^+_{4,t}) \le \frac{c+1}{2} < 1, \\ -1 < \frac{1}{2} - \sqrt{1-c} \le f(P^-_{4,t}) \le \frac{1}{2} + \frac{3c}{8} - \frac{\sqrt{1-c}}{2} < 1. \end{gathered} \tag{**}$$

Note that for every $t \in [0,1]$ with $t \ne c$,

$$f(P^+_{5,t}) = \frac{1}{2}t + \sqrt{1-c}\,\sqrt{1-t} + \frac{c}{2}$$

and

$$f(P^-_{5,t}) = \left(\frac{1 - c - \sqrt{1-c}}{2}\right)t - (c + \sqrt{1-c})\,\sqrt{1-t} + \frac{c}{2}.$$

Hence, we have for every $t \in [0,1]$ with $t \ne c$,

$$\begin{gathered} -1 < \min\left\{\frac{c}{2} + \sqrt{1-c}, \frac{c+1}{2}\right\} \le f(P^+_{5,t}) < 1, \\ -1 < -\left(\frac{c}{2} + \sqrt{1-c}\right) \le f(P^-_{5,t}) \le \frac{1}{2} - \sqrt{1-c} < 1. \end{gathered} \tag{***}$$

Hence, by Theorem 2.3, $1 = \|f\|$. Let $Q(x,y) = ax^2 + by^2 + cxy$ in $\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)$ such that $1 = \|Q\| = f(Q)$. By the Krein-Milman Theorem,

$$\begin{aligned}Q(x,y) &= uP_1(x,y) + v^+P_2^+(x,y) + v^-P_2^-(x,y) + tP_3(x,y)\\ &\quad + \sum_{n=1}^{\infty} \lambda_n^+ P^+_{4,c_n^+}(x,y) + \sum_{n=1}^{\infty} \lambda_n^- P^-_{4,c_n^-}(x,y)\\ &\quad + \sum_{m=1}^{\infty} \delta_m^+ P^+_{5,a_m^+}(x,y) + \sum_{m=1}^{\infty} \delta_m^- P^-_{5,a_m^-}(x,y)\,,\end{aligned}$$

for some $u, v^\pm, t, \lambda_n^\pm, \delta_m^\pm, \in \mathbb{R}$ $(n, m \in \mathbb{N})$ with $0 \le c_n^\pm, a_m^\pm \le 1$ and

$$|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda_n^+| + \sum_{n=1}^{\infty} |\lambda_n^-| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1\,.$$

We will show that $u = v^\pm = t = \lambda_n^\pm = \delta_m^- = 0$ for every $n, m \in \mathbb{N}$. Assume that $\lambda_{n_0} \neq 0$ for some $n_0 \in \mathbb{N}$. It follows that

$$\begin{aligned}1 = f(Q) &= uf(P_1) + v^+f(P_2^+) + v^-f(P_2^-) + tf(P_3) + \sum_{n=1}^{\infty} \lambda_n^+ f(P^+_{4,c_n^+})\\ &\quad + \sum_{n=1}^{\infty} \lambda_n^- f(P^-_{4,c_n^-}) + \sum_{m=1}^{\infty} \delta_m^+ f(P^+_{5,a_m^+}) + \sum_{m=1}^{\infty} \delta_m^- f(P^-_{5,a_m^-})\\ &< \frac{1}{2}|u| + \frac{1}{2}|v^+| + \frac{1}{2}|v^-| + \frac{1}{2}|t| + |\lambda_{n_0}^+| + \sum_{n\neq n_0} |\lambda_n^+| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-|\\ &\le 1 \quad \text{(by (*), (**), (***))}\,,\end{aligned}$$

which is impossible. Therefore, $\lambda_n^+ = 0$ for every $n \in \mathbb{N}$. Using a similar argument as above, we have $u = v^\pm = t = \lambda_n^- = \delta_m^- = 0$ for every $n, m \in \mathbb{N}$. Therefore,

$$Q(x,y) = \sum_{m=1}^{\infty} \delta_m^+ P^+_{5,a_m^+}(x,y)\,.$$

We will show that if $a_{m_0}^+ \neq c$ for some $m_0 \in \mathbb{N}$, then $\delta_{m_0}^+ = 0$. Assume that

$\delta^+_{m_0} \neq 0$. It follows that

$$\begin{aligned} 1 = f(Q) &= \delta^+_{m_0} f(P^+_{5,a^+_{m_0}}) + \sum_{m \neq m_0} \delta^+_m f(P^+_{5,a_m}) \\ &< |\delta^+_{m_0}| + \sum_{m \neq m_0} |\delta^+_m| \;=\; 1 \end{aligned}$$

which is impossible. Therefore, $\delta^+_{m_0} = 0$. Therefore,

$$Q(x,y) = \Bigg(\sum_{a_m = a} \delta^+_m\Bigg) P^+_{5,c}(x,y) = P^+_{5,c}(x,y)\,.$$

Therefore, f exposes $P^+_{5,c}$. Obviously, $P^-_{5,c} \in \exp B_{\mathcal{P}\big({}^2\mathbb{R}^2_{h(\frac{1}{2})}\big)}$ for $0 < c < 1$. Therefore, we complete the proof. ∎

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