EXTRACTA MATHEMATICAE doi:10.17398/2605-5686.34.1.1
Vol. **34**, Num. 1 (2019), 1 – 17 Available online February 3, 2019

Ideal operators and relative Godun sets

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Received August 13, 2017 Presented by David Yost
Accepted July 2, 2018

Abstract: In this paper we study ideals in Banach spaces through ideal operators. We provide characterisation of recently introduced notion of *almost isometric* ideal which is a version of *Principle of Local Reflexivity* for a subspace of a Banach space. Studying ideals through ideal operators give us better insight in to the properties of these subspaces *vis-a-vis* properties of the space itself. We provide a few applications of our characterisation theorem.

Key words: Ideals, almost isometric ideals, strict ideals, maximal ideal operator, Godun sets, VN-subspaces.

AMS *Subject Class.* (2010): 46E30, 46B20.

1. Introduction and preliminaries

Principle of local reflexivity (henceforth PLR) states that finite dimensional subspaces of X^{**}, the bidual of a Banach space X, are almost isometric to finite dimensional subspaces of X. PLR also provides control over actions of a fixed finite dimensional subspace $F \subseteq X^*$ on a finite dimensional subspace $E \subseteq X^{**}$ and its almost isometric copy in X. It is immediately realised that PLR is a consequence of finite representability of X^{**} in X and X^* is norming for X^{**}, which in turn is a consequence of existence of norm one projection in X^{***} with range X^* and kernel $X^\perp$. Since finite representability is a notion which can be defined for arbitrary pair of Banach spaces, the notion of ideals were introduced and studied extensively. A closed subspace Y of a Banach space X is said to be an ideal in X if there exists a norm one projection $P : X^* \longrightarrow X^*$ with $\ker(P) = Y^\perp$. The following characterisation of ideals, though it was widely known, was stated explicitly in [12].

*Professor Dutta passed away while the paper was being finalized.

ISSN: 0213-8743 (print), 2605-5686 (online)

THEOREM 1.1. *Let Y be a subspace of a Banach space X. Then Y is an ideal in X if and only if there exists a Hahn–Banach extension operator $\phi : Y^* \longrightarrow X^*$ such that for every $\epsilon > 0$ and every finite dimensional subspaces $E \subset X$ and $F \subset Y^*$ there exists $T : E \longrightarrow Y$ such that*

(a) *$Te = e$ for all $e \in Y \cap E$,*

(b) *$||Te|| \leq (1 + \epsilon)||e||$ for all $e \in E$,*

(c) *$\phi f^*(e) = f^*(Te)$ for all $e \in E$, $f^* \in F$.*

Clearly the notion of ideal imitates PLR. However it lacks two important aspects of X in X^{**} situation; namely, we do not get almost isometry from the finite dimensional subspace $E \subseteq X$ to Y and Range(P) is not norming for X, where by norming we mean 1-norming. So following two possible strengthening of notion of ideals were considered.

DEFINITION 1.2. [9] A subspace Y of a Banach space X is said to be a strict ideal in X if Y is an ideal in X and $Range(P)$ is norming for X where $P : X^* \longrightarrow X^*$ is a norm one projection with $\ker(P) = Y^\perp$.

DEFINITION 1.3. [2] A subspace Y of a Banach space X is said to be an almost isometric ideal (henceforth *ai*-ideal) in X if for every $\epsilon > 0$ and every finite-dimensional subspace $E \subset X$ there exists $T : E \longrightarrow Y$ which satisfies condition (a) in Theorem 1.1 and $(1 - \epsilon)||e|| \leq ||Te|| \leq (1 + \epsilon)||e||$ for all $e \in E$.

On the other hand there is a notion of u-ideal, which is a generalisation of M-ideal and is also a strengthening of PLR.

DEFINITION 1.4. [9] A subspace Y of a Banach space X is said to be a u-ideal in X if there exists a norm one projection $P : X^* \to X^*$ with $\ker(P) = Y^\perp$ and $\|I_{X^*} - 2P\| = 1$, where I_{X^*} denotes the identity operator on X^*.

It is straightforward to see that strict ideals are *ai*-ideals and *ai*-ideals are of course ideals. The inclusions are strict as shown by examples in [2].

If we view above notion of ideals and its subsequent strengthening as generalisations of X in X^{**} situation, many isometric properties of X are carried to Y and much of the studies in this area are devoted to that.

However PLR can be viewed simply as: identity operator on X^{**} is an extension of identity operator on X and X^* is norming for X. On the same vein, the property of Y being an ideal in X may be viewed as there exists $T : X \to Y^{**}$ such that $\|T\| \leq 1$ and $T|_Y = I_Y$. To see this consider the following.

Suppose $Y \subseteq X$ and $T : X \to Y^{**}$ is such that $T|_Y = I_Y$ and $\|T\| \leq 1$. We will refer T as an ideal operator. Given T we define $P : X^* \to X^*$ as $P(x^*) = T^*|_{Y^*}(x^*|_Y)$ and $\phi : Y^* \to X^*$ by $\phi(y^*) = T^*|_{Y^*}(y^*)$. Then P is a norm one projection on X^* with $\ker(P) = Y^\perp$ and ϕ is a Hahn–Banach extension operator. In the sequel, we will refer these P and ϕ as ideal projection and Hahn–Banach extension operator corresponding to T. We also note that this definition is reversible, in the sense that given an ideal projection P, we may define T as above. Same is true for ϕ.

With this view point, in Section 2 of this paper we provide characterisations of ideal, strict ideal and ai-ideal in terms of the properties of ideal operator T. In the case of ideal and strict ideal (see Proposition 2.1) the results are essentially known.

Proposition 2.1 has some immediate corollaries. It is straightforward to see that if Y is 1-complemented in X then Y is an ideal in X (if $P : X \to X$ is a projection with $\|P\| = 1$ then P^* is an ideal projection). For a space Y which is 1-complemented in its bidual Y^{**} we show Y is an ideal in any superspace X if and only if Y is 1-complemented in X. In particular $L_1(\mu)$ or any reflexive space is an ideal in a superspace if and only if it is 1-complemented. It also follows that any ideal in $L_1(\mu)$ is 1-complemented.

Coming to ai-ideals we show that reflexive spaces with a smooth norm cannot have any proper ai-ideal. Situation becomes more interesting for the space $C[0,1]$, the space of all real-valued continuous functions on [0,1] equipped with the supremum norm. It is known that $C[0,1]$ is universal for the class of separable Banach spaces. We show that any ai-ideal of $C[0,1]$ inherits the universality property from $C[0,1]$. Also, any separable ai-ideal in $L_1(\mu)$, μ non-atomic, is isometric to $L_1[0,1]$.

Section 3 of this paper is mainly devoted to the study of properties of ideal operator T. There is a need to exercise some caution while dealing with this operator. It may well be the case that for some ideal operator, Y has 'nice' property in X but there are other ideal operators for which such properties fail. For example, let Y be a u-ideal in Y^{**}. Then Y is always a strict ideal in Y^{**} under canonical projection π determined by $Y^{***} = Y^* \oplus Y^\perp$. But canonical projection may not satisfy u-ideal condition, namely, $\|I - 2\pi\| = 1$. So we introduce the notion of a maximal ideal operator (vis-a-vis, maximal ideal projection) and discuss properties of maximal ideal operator. However, there are situations where there is only one possible ideal operator. Here we introduce relative Godun set $G(Y,X)$ of X with respect to Y. We recall that $Y \subseteq X$ is said to have *unique ideal property* (henceforth UIP) in X if there is only one possible norm one projection P on X^* with $\ker(P) = Y^\perp$. Similarly $Y \subseteq X$ is said to have *unique extension property* (henceforth UEP) in X if there is only one possible $T : X \to Y^{**}$ such that $\|T\| \leq 1$ and $T|_Y = I_Y$. From the above relation of P and T it is clear that Y has UIP in X if and only if it has UEP in X (and in this case ϕ is also unique).

If Y has UEP in Y^{**} then we just say Y has UEP.

UEP also provides a sufficient condition for an ai-ideal in a dual space to be a local dual for predual. See [6] and references there in for recent results on local duals. In particular a pertinent question in this area is if a separable Banach space with non separable dual always has a separable local dual.

We now provide some brief preliminaries needed throughout this paper.

To show that finite representability of X in a subspace Y with condition (a) of Theorem 1.1 is enough to characterise ideals through a global property we use following two lemmas from [12].

LEMMA 1.5. *Let E be a finite dimensional Banach space and $T : E \longrightarrow Y^{**}$ be a linear map for any Banach space Y. Then there exists a net (T_α), $T_\alpha : E \longrightarrow Y$ such that*

(a) $||T_\alpha|| \longrightarrow ||T||$,
(b) $T_\alpha e \longrightarrow Te$ *for all* $e \in T^{-1}(Y)$,
(c) $T_\alpha^* y^* \longrightarrow T^* y^*$ *for all* $y^* \in Y^*$.

Next result from [12] shows that if we are given with T and (T_α) satisfying the conditions of Lemma 1.5, then we can modify (T_α) so that it satisfies the conditions of the following lemma.

LEMMA 1.6. *Let E be a finite dimensional Banach space and $T : E \longrightarrow Y^{**}$ be a linear map for any Banach space Y. Let $F \subseteq Y^*$ be a finite dimensional Banach space, then there exists a net (T_α), $T_\alpha : E \longrightarrow Y$ such that*

(a) $||T_\alpha|| \longrightarrow ||T||$,
(b) $T_\alpha e = Te$ *for all* $e \in T^{-1}(Y)$,
(c) $T_\alpha^* y^* = T^* y^*$ *for all* $y^* \in F$,
(d) $T_\alpha^* y^* \longrightarrow T^* y^*$ *for all* $y^* \in Y^*$.

In Section 3 we will refer to VN-subspaces (very non-constraint subspaces) and their characterisation done in [3]. We recall definitions of VN-subspaces of a Banach space and nicely smooth spaces from [3] and [7] respectively.

DEFINITION 1.7. Let Y be a subspace of a Banach space X.

(a) The ortho-complement $O(Y, X)$ of Y in X is defined as

$$O(Y, X) = \{x \in X : \|x - y\| \geq \|y\| \text{ for all } y \in Y\}.$$

We denote $O(X, X^{**})$ by $O(X)$.

(b) Y is said to be a VN-subspace of X if $O(Y, X) = \{0\}$.

(c) X is said to be nicely smooth if it is a VN-subspace of its bidual.

In this article, we consider only Banach spaces over the real field $\mathbb{R}$ and all subspaces we consider are assumed to be closed.

2. GENERALISATION OF PLR

We start with the following proposition which is essentially known. However it provides a way to look ideals and strict ideals through some global property. Certain known properties of ideals and strict ideals follow trivially if we take this global view point.

PROPOSITION 2.1. *Let X be a Banach space and Y be a subspace of X. Then*

(a) *Y is an ideal in X if and only if there exists $T : X \longrightarrow Y^{**}$ such that $||T|| \leq 1$ and $T|_Y = I_Y$.*

(b) *Y is a strict ideal in X if and only if there exists an isometry $T : X \longrightarrow Y^{**}$ such that $T|_Y = I_Y$.*

We now note some immediate corollaries.

COROLLARY 2.2. *Let Y be a subspace of a Banach space X and Y be 1-complemented in its bidual. Then Y is an ideal in X if and only if Y is 1-complemented in X.*

Proof. If Y is 1-complemented in X then trivially Y is an ideal in X.

For the converse consider the map $T : X \longrightarrow Y^{**}$ from Proposition 2.1 (a). Let $P : Y^{**} \longrightarrow Y^{**}$ be a norm one projection with $Range(P) = Y$. If we take $Q = PT : X \longrightarrow X$ then $||Q|| = ||PT|| \leq 1$, $Q^2 = Q$ and $Range(Q) = Y$. ∎

COROLLARY 2.3. (a) *If Y is isometric to any dual space then $Y \subseteq X$ is an ideal if and only if Y is 1-complemented in X. In particular no reflexive space can simultaneously be a VN-subspace and an ideal in any superspace.*

(b) *$L_1(\mu) \subseteq X$ is an ideal in X if and only if $L_1(\mu)$ is 1-complemented in X.*

(c) *Y is an ideal in $L_1(\mu)$ if and only if Y is 1-complemented in $L_1(\mu)$.*

(d) *No infinite dimensional reflexive space can be an ideal in a space with Dunford Pettis property (see [4, Definition 1.10]).*

Proof. Proofs of (a) and (b) follow from Corollary 2.2.

(c) Let Y be an ideal in $L_1(\mu)$. Then Y^* is isometric to a 1-complemented subspace of $L_\infty(\mu)$. Thus Y is isometric to $L_1(\nu)$ for some positive measure ν. Hence Y is 1-complemented in its bidual. Then, by Corollary 2.2, Y is 1-complemented in $L_1(\mu)$.

(d) If Y is an infinite dimensional reflexive space and $Y \subseteq X$ is an ideal then by Corollary 2.2 Y is 1-complemented in X. However complemented subspaces of a space with Dunford Pettis property have Dunford Pettis property and a reflexive space with Dunford Pettis property is finite dimensional. ∎

The notion of *ai*-ideals is strictly in between the notions of ideals and strict ideals. In the next theorem we characterise *ai*-ideals in terms of the operator T defined in Theorem 2.1.

DEFINITION 2.4. Let X and Z be Banach spaces. For $\epsilon > 0$, an operator $T : X \to Z$ is said to be an ϵ-isometry if $\|T\|\,\|T^{-1}\| \leq 1 + \epsilon$.

THEOREM 2.5. *Let Y be a subspace of X. Then Y is an ai-ideal in X if and only if following condition is satisfied.*

Given a finite dimensional subspace E of X and $\epsilon > 0$, there exists a bounded linear map $T^E : X \to Y^{**}$ such that $T^E|_Y = I_Y$ and T^E is an ϵ-isometry on $E \cap (T^E)^{-1}(Y)$.

Proof. Let E be a finite dimensional subspace of X. Without loss of generality, (by possibly adding an element from Y) we assume $E \cap Y \neq \{0\}$. Now consider a net (E_α) of finite dimensional subspaces of X such that $E_\alpha \supseteq E$ and the ϵ_{E_α}-isometry $\widetilde{T_{E_\alpha}^{\epsilon_{E_\alpha}}} : E_\alpha \to Y$ with $\widetilde{T_{E_\alpha}^{\epsilon_{E_\alpha}}}$ is identity on $E \cap Y$. Let T^E be a weak* limit point of this net in the sense that for all $x \in X$ and $x^* \in X^*$, $x^*(T^E x) = \lim_\alpha x^*(\widetilde{T_{E_\alpha}^{\epsilon_{E_\alpha}}} x)$. Let $T_\alpha = \widetilde{T_{E_\alpha}^{\epsilon_{E_\alpha}}}|_E$ and $T_E = T^E|_E$.

It is straightforward to verify that

(a) $||T_\alpha|| \longrightarrow ||T_E||$,

(b) $T_\alpha y \longrightarrow T_E y$ for all $y \in T_E^{-1}(Y)$,

(c) $T_\alpha^* y^* \longrightarrow T_E^* y^*$ for all $y^* \in Y^*$.

Thus T_α and the operator T_E satisfy conditions of Lemma 1.5.

Now applying a perturbation argument as in Lemma 1.6, given any $\epsilon > 0$ we can find S_α satisfying conditions of Lemma 1.6 and $||S_\alpha - T_\alpha|| < \epsilon$ for large $\dim(E_\alpha)$. Hence S_α is $(\epsilon_{E_\alpha} + \epsilon)$-isometry and $S_\alpha \longrightarrow T_E$ in the weak* topology. Thus T_E is an ϵ-isometry on $E \cap T^{-1}(Y)$.

Conversely, let E be a finite dimensional subspace of X and there exists an operator T^E satisfying the condition of the theorem. Consider $T^E|_E : E \to Y^{**}$ and apply PLR to get the desired operator satisfying the definition of ai-ideal. ∎

Remark 2.6. Following example of ai-ideal in c_0 was considered in [2].

Let $Y = \{(a_n) \in c_0 : a_1 = 0\}$. Then $T : c_0 \longrightarrow Y^{**}$ in this case is given by $T(a) = (0, a_2, a_3, \ldots)$. Hence T can not be extended as an isometry beyond $T^{-1}(Y)$.

Corollary 2.7. *Let $Y \subseteq X$ be an ai-ideal and Y be reflexive. If Y has UEP in X, then Y is isometric to X. In particular, in the following cases Y is isometric to X.*

(a) *The norm of X is smooth on Y.*

(b) *Y is a u-ideal in X* [2, Theorem 2.3].

Proof. Given $x \in X$, consider $E = \text{span}\{x\}$ and map $T^E : X \longrightarrow Y^{**}$ as in Theorem 2.5. Since Y is reflexive and $T^E|_Y = I|_Y$ we get T^E is onto. Thus $(T^E)^{-1}(Y) = X$. Since Y has UEP in X, there exists a unique T such that $T^E = T$ for all E. It follows that the ideal operator T is one-one as well. Hence T is an isometry. ∎

Let μ be a non-atomic σ-finite measure. It is proved in [2] that any copy of ℓ_1 in $L_1(\mu)$ can not be an ai-ideal. However $L_1(\mu)$ contains a 1-complemented copy of ℓ_1. Hence a copy of ℓ_1 can be an ideal in $L_1(\mu)$. It follows from Corollary 2.7 that any copy of ℓ_p in $L_p(\mu)$ can not be an ai-ideal for $1 < p < \infty$.

We will now prove that *ai*-ideals of $C[0,1]$ are universal for separable Banach spaces.

PROPOSITION 2.8. *Let Y be an ai-ideal in $C[0,1]$. Then Y is universal for separable Banach spaces.*

Proof. Let Y be an *ai*-ideal in $C[0,1]$. Then Y is an L_1-predual space and it follows from [2, Proposition 3.8] that Y inherits Daugavet property from $C[0,1]$. Also from [13, Theorem 2.6] it follows that Y contains a copy of ℓ_1. Thus Y is an L_1-predual with non separable dual and hence Y is also universal for separable Banach spaces (see [10, Theorem 2.3]). ∎

We now show that for a non-atomic measure μ, any separable *ai*-ideal in $L_1(\mu)$ is isometric to $L_1[0,1]$.

PROPOSITION 2.9. *Let $Y \subseteq L_1(\mu)$ be a separable ai-ideal where μ is a non-atomic probability measure. Then Y is isometric to $L_1[0,1]$.* ∎

Proof. Let $Y \subseteq L_1(\mu)$ be a separable *ai*-ideal. Then by [2, Proposition 3.8], it follows that Y inherits Daugavet property from $L_1(\mu)$ and thus Y^* is non separable. Since Y is an ideal in $L_1(\mu)$ we have Y^* is isometric to a 1-complemented subspace of $L_\infty(\mu)$ and thus Y is isometric to $L_1(\nu)$ space for some measure ν. But Y has Daugavet property so ν can not have atoms. Now since ν is non atomic and Y is separable we can conclude that Y is isometric to $L_1[0,1]$. ∎

The property of being an *ai*-ideal is inherited from bidual.

PROPOSITION 2.10. *Suppose $Y \subseteq X$ and $Y^{\perp\perp}$ is an ai-ideal in X^{**}. Then Y is an ai-ideal in X^{**} and hence in particular in X.*

Proof. We note that the property of being *ai*-ideal is transitive. Since Y is always an *ai*-ideal in $Y^{\perp\perp}$ and $Y^{\perp\perp}$ is an *ai*-ideal in X^{**}, Y is *ai*-ideal in X^{**} as well. ∎

We end this section stating a result which connects *ai*-ideals in a dual space to local dual of preduals.

DEFINITION 2.11. [6] A closed subspace Z of X^* is said to be a local dual space of a Banach space X if for every $\epsilon > 0$ and every pair of finite dimensional subspaces F of X^* and G of X, there exists an ϵ-isometry $L : F \to Z$ satisfying the following conditions.

(a) $L(f)|_G = f|_G$ for all $f \in F$,

(b) $L(f) = f$ for $f \in F \cap Z$.

The following result is simple to observe from above definition.

PROPOSITION 2.12. *Let X be a Banach space and Z be a subspace of X^*. If Z is a local dual of X, then Z is an ai-ideal in X^*.*

Remark 2.13. Let Y be an ai-ideal in a Banach space X. Since X is an ai-ideal in X^{**}, Y is an ai-ideal in X^{**}. But Y cannot be a local dual space of X^* as it is not norming for X^*. So the converse of Proposition 2.12 need not be true.

THEOREM 2.14. *Let X be a Banach space. Let Z be an ai-ideal in X^* with UEP and Z be norming for X. Then Z is a local dual space of X.*

Proof. Let F and G be finite dimensional subspaces of X^* and X respectively. Also, let $\epsilon > 0$. Now let $\widehat{G} = span\{\hat{g}|_Z : g \in G\}$, where $\hat{g}$ is the canonical image of g in X^{**}. Then, by [2, Theorem 1.4], there exists a Hahn–Banach extension operator $\varphi : Z^* \to X^{**}$ and an ϵ-isometry $L : F \to Z$ such that $Lf = f$ for all $f \in F \cap Z$ and $\varphi(\hat{g}|_Z)(f) = (\hat{g}|_Z)(Lf) = (Lf)(g)$ for all $g \in G$ and $f \in F$. Now to prove Z is a local dual space of X, it is enough to prove that $L(f)(g) = f(g)$ for all $f \in F$ and $g \in G$. Now let $g \in G$. Since Z is norming for X, $\hat{g}$ is a Hahn–Banach extension of $\hat{g}|_Z$. Further, by UEP, $\hat{g}$ is the only Hahn–Banach extension of $\hat{g}|_Z$. Therefore $(Lf)(g) = \varphi(\hat{g}|_Z)(f) = f(g)$ for all $f \in F$ and $g \in G$. Hence Z is a local dual space of X. ∎

3. PROPERTIES OF IDEAL OPERATORS

In this section we explore conditions for the ideal operator T to be unique or one-one. Any nicely smooth space has UEP (see [3, 7]). However any Banach space X is a strict ideal in X^{**}. So for an ideal Y in X, to get uniqueness we mostly have to consider Y to be a strict ideal in X. As mentioned in the introduction, while considering UEP one needs to exercise some caution here. For an ideal Y in X we will first make sense of a maximal ideal projection through the use of Godun set of X with respect to Y.

We formulate the following lemma for which the equivalence of first three parts is established in [9, Lemma 2.2] and the proof for the fourth part goes verbatim as in the proof of (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3) in [9, Proposition 2.3].

Lemma 3.1. *Let Y be an ideal in X and T be the corresponding ideal operator. For $\lambda, a \in \mathbb{R}$, the following assertions are equivalent.*

(a) $\|I - \lambda P\| \leq a$.

(b) *For any $\epsilon > 0$, $x \in X$ and convex subset A of Y such that Tx is in the weak* closure of A there exists $y \in A$ such that $\|x - \lambda y\| < a\|x\| + \epsilon$.*

(c) *For any $x \in X$ there exists a net $(y_\alpha) \subseteq Y$ such that (y_α) converges to Tx in the weak* topology and $\limsup_\alpha \|x - \lambda y_\alpha\| \leq a\|x\|$.*

Moreover, if Y is a strict ideal in X and T is the corresponding strict ideal operator then above assertions are also equivalent to the following.

(d) *For $\epsilon > 0$ and any sequence (y_n) in B_Y with (y_n) converges in the weak* topology to Tx for some $x \in B_X$, there exist n and $u \in \mathrm{co}\{\mathrm{y_k}\}_{\mathrm{k=1}}^{\mathrm{n}}$, $t \in \mathrm{co}\{\mathrm{y_k}\}_{\mathrm{k=n+1}}^{\infty}$ such that $\|t - \lambda u\| < a + \epsilon$.*

Let π be the canonical projection of X^{***} onto X^*. The Godun set $G(X)$ is defined to be $G(X) = \{\lambda : \ \|I - \lambda\pi\| = 1\}$ (see [9]).

For ideal $Y \subseteq X$ and ideal projection P we define Godun set of X with respect to Y and P as $G_P(Y, X) = \{\lambda : \ \|I - \lambda P\| = 1\}$. Then it follows that $0 \in G_P(Y, X)$ and $G_P(Y, X)$ is a closed convex subset of $[0, 2]$ and thus itself an interval.

Our next result is an analogue of [9, Lemma 2.5] and has interesting consequences.

Lemma 3.2. *Let Y be a strict ideal in X and P be the corresponding strict ideal projection.*

(a) *If Z is a subspace of X such that $Y \subseteq Z \subseteq X$ then there exists an ideal projection Q on Z^* with $\ker(Q) = Y^\perp$ such that $G_P(Y, X) \subseteq G_Q(Y, Z)$.*

(b) *If Z is a closed subspace of Y then Y/Z is an ideal in X/Z and there exists an ideal projection $\tilde{P}$ on $(X/Z)^*$ such that $G_P(Y, X) \subseteq G_{\tilde{P}}(Y/Z, X/Z)$.*

Proof. (a) Let T be the corresponding strict ideal operator from X to Y^{**}. Consider $T_Z = T|_Z : Z \to Y^{**}$. We define $Q : Z^* \to Z^*$ as $Q(z^*) = (T_Z)^*(z^*|_Y)$. It is straightforward to check that $\ker(Q) = Y^\perp \subseteq Z^*$. Thus Q is an ideal projection on Z^*. The proof now follows from equivalence of (d) and (a) in Lemma 3.1.

(b) We again let T be the corresponding strict ideal operator from X to Y^{**}. Let $q : X \to X/Z$ be the quotient map. We define $\tilde{T} : X/Z \to (Y/Z)^{**}$ by

$\tilde{T}(\overline{x}) = q^{**}(Tx)$, where $\overline{x}$ is the equivalence class containing x. Let $\tilde{P}$ be the ideal projection corresponding to $\tilde{T}$. Again a straightforward application of equivalence of (d) and (a) in Lemma 3.1 gives the desired conclusion. ∎

For an ideal Y in X, we define Godun set of X with respect to Y as $G(Y,X) = \cup\{G_P(Y,X) : P \text{ is an ideal projection}\}$. We now verify that $G(Y,X)$ is $G_P(Y,X)$ for some ideal projection P. In the sequel we will refer such projection as maximal ideal projection and the corresponding T as maximal ideal operator.

THEOREM 3.3. *Let Y be an ideal in X. Then there exists an ideal projection P such that $G(Y,X) = G_P(Y,X)$.*

Proof. Suppose for all ideal projection P, $G_P(Y,X) = \{0\}$. Then $G(Y,X) = \{0\}$ and we choose any P as maximal ideal projection.

Suppose there exists an ideal projection P and $\lambda \in G_P(Y,X)$ with $\lambda \neq 0$. Then we claim that $[0,1] \subseteq G_P(Y,X)$. To see this, suppose on the contrary that $G_P(Y,X) \subseteq [0,\gamma]$ for some $0 < \gamma < 1$. We choose $\mu \in (0,\gamma)$. It is straightforward to verify that $\gamma+\mu-\gamma\mu \in G_P(Y,X)$ as well. Thus $\gamma+\mu-\gamma\mu \leq \gamma$. Hence $\mu(1-\gamma) \leq 0$ which is a contradiction.

Now let us consider $\lambda = \sup\{\mu : \mu \in G(Y,X)\}$.

If $\lambda \neq 0$, then by above argument either $\lambda = 1$ or $1 < \lambda \leq 2$. In the case $\lambda = 1$, there exists an ideal projection P such that $G(Y,X) = G_P(Y,X) = [0,1]$.

If $\lambda > 1$, then choose a sequence (λ_n) in $G(Y,X)$ such that $\lambda_n > 1$ and λ_n converges to λ. Let P_n be an ideal projection corresponding to Y with $\|I - \lambda_n P_n\| = 1$ for all n. Since $B(X^*)$ is isometric to the dual of projective tensor product of X^* and X, there exists a bounded linear map $P : X^* \to X^*$ and a subsequence (denoted again by (P_n)) of (P_n) such that for every $x^* \in X^*$, $P_n(x^*)$ converges to $P(x^*)$ in the weak* topology. Since for every $x^* \in X^*$ and $n \in \mathbb{N}$, $P_n(x^*)$ is a Hahn-Banach extension of $x^*|_Y$, we can see that $P(x^*)$ is also a Hahn-Banach extension of $x^*|_Y$. Thus $\ker(P) = Y^\perp$. For any $x^* \in X^*$, since $x^* - P(x^*) \in Y^\perp = \ker(P)$, we can see that $P(P(x^*)) = P(x^*)$. Hence P is an ideal projection corresponding to Y. Since, for every $x^* \in X^*$, $x^* - \lambda_n P_n(x^*)$ converges to $x^* - \lambda P(x^*)$ in the weak* topology, we can see that $\|(I - \lambda P)(x^*)\| \leq \liminf_n \|(I - \lambda_n P_n)(x^*)\| \leq \|x^*\|$ for every $x^* \in X^*$. Thus $\|I - \lambda P\| = 1$. Hence $G(Y,X) = G_P(Y,X) = [0,\lambda]$. ∎

Remark 3.4. We note that $G(Y,X) = \{0\}$ is possible. If $Y = \ell_1$ and

$X = Y^{**}$, then following the same argument used in [9, Proposition 2.6] it follows that $G(Y, X) = \{0\}$.

We now show that if Y is nicely smooth and Y embeds in a superspace X as a strict ideal then strict ideal operator T is unique.

PROPOSITION 3.5. *Let Y be a nicely smooth Banach space and Y be a strict ideal in a superspace X. Then the strict ideal operator is unique.*

Proof. Let T_1 and T_2 be two strict ideal operator. Then for any $x \in X$, $\|T_1x - y\| = \|T_2x - y\|$ for all $y \in Y$. Hence $T_1x - T_2x \in O(Y)$ (see [3]). But since Y is nicely smooth, $O(Y) = \{0\}$ and hence $T_1x = T_2x$. ∎

We will now give a sufficient condition for a strict ideal Y in X to be a VN-subspace of X. We will first provide an analogue of [9, Proposition 2.7].

PROPOSITION 3.6. *Let Y be a strict ideal in X and P a strict ideal projection for Y in X^*. If $1 < \lambda \leq 2$ and $\|I - \lambda P\| = a < \lambda$ then for any proper subespace $M \subseteq X^*$, M norming for Y, we have M is weak* dense in X^*.*

Proof. Let $r_Y(M)$ be the greatest constant r such that $\sup_{x^* \in S_M} |x^*(y)| \geq r\|y\|$ for all $y \in Y$.

We will first show that for any weak* closed subspace $M \subseteq X^*$, $r_Y(M) \leq \lambda^{-1}a$. Without loss of generality let $M = \ker(x)$ for some $x \in S_X$.

Consider the isometry $T : X \longrightarrow Y^{**}$ corresponding to P. Then by Lemma 3.1 there exists a net $\{y_\alpha\} \subseteq Y$ such that $y_\alpha \longrightarrow Tx$ in the weak* topology and $\limsup \|x - \lambda y_\alpha\| \leq a\|x\|$.

Now since T is an isometry we have $\|y_\alpha\| \longrightarrow 1$. For any $x^* \in S_M$, $\lambda|x^*(y_\alpha)| = |x^*(x - \lambda y_\alpha)| \leq \|x - \lambda y_\alpha\|$.

Since $\sup_{x^* \in S_M} |x^*(y_\alpha)| \geq r_Y(M)\|y_\alpha\|$ and $\|y_\alpha\| \longrightarrow 1$ it follows that $r_Y(M) \leq \lambda^{-1}a$.

If there exists $1 < \lambda \leq 2$ and $\|I - \lambda P\| = a < \lambda$ then it follows that for any weak* closed proper subspace M of X^* which is norming for Y we have $r_Y(M) < 1$. This contradicts M is norming for Y and hence we have the result. ∎

THEOREM 3.7. *Let $Y \subseteq X$ be a strict ideal such that $\|I - \lambda P\| < \lambda$ for some $1 < \lambda \leq 2$ where P is a strict ideal projection. Then Y is a VN-subspace of X. In particular a strict u-ideal is always a VN-subspace.*

Proof. Let $Y \subseteq X$ be a strict ideal such that $\|I - \lambda P\| < \lambda$ for some $1 < \lambda \leq 2$ where P is a strict ideal projection. Then it follows from Proposition 3.6 that any norming subspace for Y separates points in X. Hence Y is a VN-subspace of X (see [3]). ∎

COROLLARY 3.8. *Let X be a Banach space. Then the following assertions are equivalent.*

(a) *X^* is separable.*

(b) *There exists a renorming of X such that X is nicely smooth, that is X^* has no proper norming subspace.*

(c) *There exists a renorming of X such that every subspace and quotient of X in the new norm are nicely smooth.*

Proof. (a) $\Rightarrow$ (b) From [9, Theorem 2.9] it follows that given $1 < \lambda < 2$ there exists a renorming of X such that $\lambda \in G(X)$. Conclusion follows from Theorem 3.7.
(b) $\Rightarrow$ (c) Follows from Lemma 3.2 and Theorem 3.7.
(c) $\Rightarrow$ (b) $\Rightarrow$ (a) Is trivial. ∎

COROLLARY 3.9. *Let Y be a strict ideal in X with strict ideal projection P and $O(Y, X) \neq \{0\}$. Then either $G_P(Y, X) = \{0\}$ or $G_P(Y, X) = [0, 1]$ and the later happens only if P is bicontractive.*

Proof. Let $0 \neq x \in O(Y, X)$ and $M = \ker(x)$. Then M is norming for Y. Now if we assume that $\|I - \lambda P\| < \lambda$ then $r_Y(M) \leq \|I - \lambda P\|\lambda^{-1} < 1$. But M is norming for Y so it follows that $\|I - \lambda P\| \geq \lambda$ and thus $G_P(Y, X) \subseteq [0, 1]$.

If $1 \notin G_P(Y, X)$ that is $\|I - P\| > 1$ then $G_P(Y, X) = \{0\}$. Hence the conclusion follows. ∎

We now provide a sufficient condition for $x \in X$ to be in $O(Y, X)$.

PROPOSITION 3.10. *Let $Y \subseteq X$ be an ideal and T be the corresponding ideal operator. If $Tx = 0$, then $x \in O(Y, X)$. Consequently, if Y is also a VN-subspace of X, then any ideal operator T is one-one.*

Proof. Let $Tx = 0$. Then $Px^*(x) = 0$ for all $x^* \in X^*$ where P is the ideal projection corresponding to T. Thus $Range(P) \subseteq \ker(x)$. But $Range(P)$ is norming for Y, hence $\ker(x)$ is norming for Y and $x \in O(Y, X)$. ∎

We now present an extension of [9, Theorem 7.4]. Towards this, for an ideal Y in X with associated ideal operator T, we define

$$Ba(Y,X) = \{x \in X : \text{there exists } \{y_n\} \subseteq Y \text{ such that } y_n \longrightarrow Tx \text{ in the weak}^* \text{ topology}\}$$

and

$$k_u^Y(x) = \inf\Big\{a : Tx = \sum y_n \text{ in weak}^* \text{ topology and for any } n, \|\textstyle\sum_{k=1}^n \theta_k y_k\| \leq a, \theta_k = \pm 1\Big\}.$$

It follows from [11] that if Y does not contain a copy of ℓ_1 then $Ba(Y,X) = X$. As considered in [9], we will say pair (Y,X) has property u if $k_u^Y(x) < \infty$ for all $x \in X$. In this case by closed graph theorem there exists a constant C such that $k_u^Y(x) \leq C||Tx||$ for all $x \in Ba(Y,X)$. We denote least constant C by $k_u^Y(X)$.

We will need the following lemma.

LEMMA 3.11. *Let Y be a strict ideal in X such that Y does not contain a copy of ℓ_1 and T be a strict ideal operator. Then Y is a u-ideal in X if and only if $k_u^Y(X) = 1$.*

Proof. If Y is a u-ideal in X then the result follows from [9, Lemma 3.4]. Conversely, by following the similar arguments as in [9, Lemma 5.3] that in this case $\|I - 2P\| \leq k_u^Y(X)$ where P is the ideal projection corresponding to the ideal operator T. Thus $\|I - 2P\| = 1$ and Y is a strict u-ideal in X. ∎

THEOREM 3.12. *Let Y be a Banach space not containing ℓ_1. Then the following assertions are equivalent.*

(a) *Y is a u-ideal in Y^{**}.*

(b) *Whenever Y is a strict ideal in X, Y is a strict u-ideal in X.*

(c) *Whenever Y is a strict ideal in X, $k_u^Y(X) < 2$.*

Proof. (a) ⇒ (b) Since Y is a strict ideal in X, the ideal operator T is an extension of identity operator on Y and we have the results by [9, Proposition 3.6].
(b) ⇒ (c) Follows from Lemma 3.11.
(c) ⇒ (a) Follows from [9, Theorem 7.4] by taking $X = Y^{**}$. ∎

We will now give examples where the ideal operator T is unique/ one-one.

We first see how does the ideal operator corresponding to an M-ideal in $C(K)$ behave. It is well-known that M-ideals in $C(K)$ are precisely of the form $J_D = \{f \in C(K) : f|_D = 0\}$ for some closed subset D of K.

PROPOSITION 3.13. *Let D be a closed subset of a compact Hausdorff space K. Then the following are equivalent.*

(a) *J_D is a strict ideal in $C(K)$.*

(b) *J_D is a VN-subspace of $C(K)$.*

(c) $\overline{K \setminus D} = K$.

Proof. (a) $\iff$ (b) Observe that J_D^* is norming if and only if $\overline{K \setminus D} = K$. (b) $\iff$ (c) Follows from standard arguments. ∎

EXAMPLE 3.14. Since the ideal projection corresponding to an M-ideal is unique, the ideal operator T corresponding to J_D is unique. In addition, if $\overline{K \setminus D} = K$, then, by Proposition 3.13, the unique ideal operator T corresponding to J_D is an isometry.

We next note that the ideal operators corresponding to $C(K, X)$ and $C(K, X^*)$ are isometries.

For a compact Hausdorff space K and for any Banach space X, let $WC(K, X)$ denote the space of X-valued functions on K that are continuous when X has the weak topology, equipped with the supremum norm. Also, $W^*C(K, X^*)$ denotes the space of X^*-valued functions on K that are continuous when X^* has the weak* topology, equipped with the supremum norm.

PROPOSITION 3.15. *Let X be a Banach space. Then $C(K, X)$ is a strict ideal in $WC(K, X)$. Moreover, $C(K, X^*)$ is a strict ideal in $W^*C(K, X^*)$.*

Proof. The former conclusion follows from the fact that there exists an isometry from $WC(K, X)$ to $C(K, X)^{**}$ whose restriction to $C(K, X)$ is the canonical embedding.

To prove the later conclusion recall that $C(K, X^*) = K(X, C(K))$, the space of compact operators from X to $C(K)$ and $W^*C(K, X^*) = L(X, C(K))$, the space of bounded linear operators from X to $C(K)$. It follows from [8, Lemma 2] that if Y is a Banach space having metric approximation property (in short MAP), then there exists an isometry from $L(X, Y)$ to $K(X, Y)^{**}$

whose restriction to $K(X,Y)$ is the canonical embedding. Since $C(K)$ has MAP, it follows that $C(K,X^*)$ is a strict ideal in $W^*C(K,X^*)$. ∎

We know that $C(K,X) \subseteq Ba(K,X) \subseteq C(K,X)^{**}$, where $Ba(K,X)$ denotes the class of Baire-1 functions from K to X. Since $C(K,X)$ is a strict ideal in $C(K,X)^{**}$, it follows that $C(K,X)$ is also a strict ideal in $Ba(K,X)$. So the corresponding ideal operator is an isometry.

If X has MAP, then, by [8, Lemma 2], $K(X)$ is a strict ideal in $L(X)$. Now it follows from [5] that a reflexive space with compact approximation property has MAP. Hence if X is a reflexive space with compact approximation property such that either weak* denting points of B_{X^*} separates points of X^{**} or denting points of B_X separates points of X^*, then $K(X)$ is a strict ideal in $L(X)$ and is also a VN-subspace of $L(X)$ (see [3]). So the ideal operator T is an isometry.

Acknowledgements

The second author is grateful to the NBHM, India for its financial support (No.2/40(2)/2014/R&D-II/6252). He also likes to thank Prof. Pradipta Bandyopadhyay for the discussion regarding local dual spaces.

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