Ricci solitons on para-Kähler manifolds

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Abstract: The main purpose of the paper is to study the nature of Ricci soliton on para-Kähler manifolds satisfying some certain curvature conditions. In particular, if we consider certain pseudosymmetric and parallel symmetric tensor on para-Kähler manifolds we prove that V is solenoidal if and only if it is shrinking or steady or expanding depending upon the sign of scalar curvature for dimension $M > 4$, where (g, V, λ) be a Ricci soliton in a paraholomorphic projectively, pseudosymmetric para-Kähler manifolds. Moreover, we obtain some results related to the quasi-conformal curvature tensor on such manifolds.

Key words: Para-Kähler manifold, Ricci soliton, pseudosymmetry, paraholomorphic projective curvature, quasi-conformal curvature.

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1. Introduction

Ricci flow is an excellent tool in simplifying the structure of the manifolds. It is defined for Riemannian manifolds of any dimension. It is a process that deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the irregularity in the metric which is given by

$$\frac{\partial g(t)}{\partial t} = -2 \operatorname{Ric}(g(t)),$$

where g is the Riemannian metric dependent on time t and $\operatorname{Ric}(g(t))$ is the Ricci tensor. We consider $\phi_t : M \to M$, $t \in \mathbb{R}$, be a family of diffeomorphisms and $(\phi_t : t \in \mathbb{R})$ is a one parameter family of abelian group called flow. It generates a vector field X_q given by

$$X_q f = \frac{df(\phi_t(q))}{dt}, \, f \in C^\infty(M).$$

If Y is a vector field then $\mathfrak{L}_X Y = \lim_{t \to 0} \frac{\phi^\bullet{}_t Y - Y}{t}$ is known as Lie-derivative of Y with respect to X. Ricci solitons move under the Ricci flow under

$\phi_t : M \rightarrow M$ of the initial metric, that is, they are stationary points of the Ricci flow in the space of metric. If g_0 is a metric on the co-domain, then $g(t) = \phi^{\bullet}_t g_0$ is the pullback of g_0, is a metric on the domain. Thus if g_0 is a soliton of the Ricci flow on the co-domain, subjected to the condition $\mathfrak{L}_V g_0 + 2 \operatorname{Ric} g_0 + 2\lambda g_0 = 0$ on the co-domain then $g(t)$ is the soliton of Ricci flow on the domain subjected to the condition $\mathfrak{L}_V g + 2 \operatorname{Ric} g + 2\lambda g = 0$ on the domain by [13] under suitable conditions. So g_0 and $g(t)$ are metrics which satisfy Ricci flow. Thus the following equation

$$\mathfrak{L}_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

is called Ricci soliton. It is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ respectively. Therefore, Ricci solitons are generalization of Einstein manifolds and they are also known as quasi-Einstein manifolds by theoretical physicists.

Para-Kähler manifolds are examples of symplectic, locally product and semi-Riemannian manifolds. Some authors studied on paracomplex geometry [3]. Besides, many authors considered the notion of "hyperbolic" instead of "para". It was first used by Prvanovic [8]. In this paper, the references [4, 5] have been our motivation in studying the para-Kähler manifolds, where the structure tensor P is an almost complex and metric g is positive definite.

This paper is organized as follows: In Section 2, we give the basic concept of para-Kähler manifolds and some certain curvature tensor. In Section 3, we introduce certain pseudosymmetric conditions on such manifolds. In Section 4, we discuss the parallel symmetric second order tensor field. In Section 5, we consider quasi-conformally flat para-Kähler manifolds. Finally in Section 6, we study the parallel quasi-conformal para-Kähler manifolds.

2. Para-Kähler manifolds

By a para-Kählerian manifold we mean a triple (M, P, g), where M is a connected differentiable manifold of dimension $n = 2m$, P is a $(1,1)$-tensor field and g is a pseudo-Riemannian metric on M satisfying the conditions

$$P^2 = I, \qquad g(PX, PY) = -g(X, Y), \qquad \nabla P = 0, \tag{2.1}$$

for any $X, Y \in \aleph(M)$, where $\aleph(M)$ is the Lie algebra of vector fields on M, ∇ is the Levi-Civita connection of g and I is the identity tensor field.

Let (M, P, g) be a para-Kählerian manifold. The Riemann-Christoffel curvature tensor R, the Ricci curvature S and the scalar curvature r are defined

by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W),$$
$$S(X,Y) = \text{Tr}\{Z \to R(Z,X)Y\},$$
$$r = \text{Tr}_g S.$$

Let Q be the Ricci operator given by $S(X,Y) = g(QX,Y)$, for these tensor fields, the following identities are satisfied

$$R(PX,PY) = -R(X,Y), \qquad R(PX,Y) = -R(X,PY),$$
$$S(PX,Y) = -S(PY,X), \qquad S(PX,PY) = -S(X,Y). \tag{2.2}$$

$$\text{Tr}\{Z \to R(X,Y)PZ\} = -2S(X,PY),$$
$$\text{Tr}\{Z \to R(PZ,X)Y\} = S(X,PY),$$
$$QY = -\sum_i \varepsilon_i R(e_i,Y)e_i.$$

For any $(0,2)$-type tensor field Φ on M and $X,Y \in \aleph(M)$, we define the endomorphism $X \wedge_\Phi Y$ of $\aleph(M)$ by

$$(X \wedge_\Phi Y)Z = \Phi(Y,Z)X - \Phi(X,Z)Y, \qquad Z \in \aleph(M).$$

The paraholomorphic projective curvature tensor $\widetilde{P}$ of (M,P,g) is defined as follows ([7, 8, 9]):

$$\widetilde{P}(X,Y) = R(X,Y) - \frac{1}{n+2}\{X \wedge_S Y - (PX)\wedge_S(PY) + 2g(QPX,Y)P\}. \tag{2.3}$$

We recall that

$$\widetilde{P}(X,Y) = -\widetilde{P}(Y,X), \qquad \text{Tr}\{Z \to \widetilde{P}(Z,X)Y\} = 0,$$
$$\sum_i \varepsilon_i \widetilde{P}(X,e_i,e_i,W) = \frac{1}{n+2}\{nS(X,W) - rg(X,W)\}, \tag{2.4}$$

where $\{e_1,e_2,\ldots,e_n\}$ is an orthonormal frame and ε_i is the indicator of e_i, $\varepsilon_i = g(e_i,e_i) = \pm 1$. In [8], Prvanovic defined the following $(0,4)$-type tensor field: given $X,Y,Z,V \in \aleph(M)$,

$$R_0(X,Y,Z,V) = \frac{1}{4}\Big\{g(X,Z)g(Y,V) - g(X,V)g(Y,Z)$$
$$- g(X,PZ)g(Y,PV) + g(X,PV)g(Y,PZ) \tag{2.5}$$
$$- 2g(X,PY)g(Z,PV)\Big\}.$$

For any $q \in M$, a subspace $U \subset T_q M$ is called non-degenerate if g restricted to U is non-degenerate. If $\{u, v\}$ is a basis of a plane $\sigma \subset T_q M$, then σ is non-degenerate if and only if $g(u, u)g(v, v) - [g(u, v)]^2 \neq 0$. Thus the sectional curvature of $\sigma = \text{span}\{u, v\}$ is

$$k(\sigma) = \frac{R(u, v, u, v)}{g(u, u)g(v, v) - [g(u, v)]^2}.$$

From (2.1) it follows that X and PX are orthogonal for any $X \in \aleph(TM)$. By a P-plane we mean a plane which is invariant by P. For any $q \in M$, a vector $u \in T_q M$ is isotropic if $g(u, u) = 0$. If $u \in T_q M$ is not isotropic, then the sectional curvature $K(u)$ of the P-plane $\text{span}\{u, Pu\}$ is called the P-sectional curvature defined by u. When $K(u)$ is constant, then (M, P, g) is called of constant P-sectional curvature, or a para-Kähler space form.

The notion of a quasi-conformal curvature tensor C is given by Yano and Sawaki [14] and is defined by

$$\begin{aligned}
C(X, Y)Z = {} & \alpha R(X, Y)Z + \beta \big[S(Y, Z)X - S(X, Z)Y \\
& + g(Y, Z)QX - g(X, Z)QY \big] \\
& - \frac{r}{n}\left(\frac{\alpha}{n-1} + 2\beta \right) \{ g(Y, Z)X - g(X, Z)Y \},
\end{aligned} \tag{2.6}$$

where α and β are constants. If $\alpha = 1$ and $\beta = -\frac{1}{n-2}$, then (2.6) reduces to conformal curvature tensor [6]. A manifold (M^n, g) $(n > 3)$ is said to be quasi-conformally flat manifold if $C = 0$. In [1], it is known that a quasi-conformally flat manifold is either conformally flat under $\alpha \neq 0$ or Einstein manifold under the conditions $\alpha = 0$ and $\beta \neq 0$. The authors give no restrictions for $\alpha = 0$ and $\beta = 0$. However, we consider the condition $\alpha \neq 0$, or $\beta \neq 0$ in this study.

In view of the equations (2.2), (2.4) and (2.6), we have

$$\sum_i \varepsilon_i \, g(C\,(Pe_i, PY)e_i, W) = -\frac{\alpha}{2}g(PY, W) + \beta[2S(PY, PW) - \tau g(PY, W)]$$

$$- \frac{r}{n}\left\{ \frac{\alpha}{n-1} + 2\beta \right\} g(PY, PW), \tag{2.7}$$

which implies

$$\sum_i \varepsilon_i \, C\,(Pe_i, PY)e_i = -\frac{\alpha}{2}PY + \beta[2QPY - \tau \, PY]$$

$$+ \frac{r}{n}\left\{ \frac{\alpha}{n-1} + 2\beta \right\} Y, \tag{2.8}$$

where τ is the special scalar curvature, which is defined as the trace of PQ. It is remarked that $\sum_i \varepsilon_i g(Pe_i, e_i) = 0$.

For any $(0, k)$-type tensor $(k \geq 1)$ field T on a pseudo-Riemannian manifold (M, g), we define a $(0, k + 2)$-tensor field $R \cdot T$ by the following condition

$$(R \cdot T)(U, V, X_1, \ldots, X_k) = -\sum_{s=1}^{k} T(X_1, \ldots, R(U, V)X_s, \ldots, X_k). \quad (2.9)$$

A pseudo-Riemannian manifold (M, g) is called semisymmetric if $R \cdot R = 0$; Ricci-symmetric if $R \cdot S = 0$ ([2, 4, 11, 12]).

We also define a $(0, k + 2)$-tensor $(k \geq 1)$ field $Q(g, T)$ as follows

$$Q(g, T)(U, V, X_1, \ldots, X_k) = -\sum_{s=1}^{k} T(X_1, \ldots, (U \wedge V)X_s, \ldots, X_k). \quad (2.10)$$

A pseudo-Riemannian manifold (M, g) is Ricci-pseudosemisymmetric [4] if there exists a function $L_S : M \to \mathbb{R}$ such that

$$R \cdot S = L_S Q(g, S). \quad (2.11)$$

It is clear that every Ricci-semisymmetric manifold is Ricci-pseudosymmetric. In general, the converse is not true [4]. The Riemannian curvature $(1, 3)$-tensor field associated to the Levi-Civita connection ∇ of g is given by $R = [\nabla, \nabla] - \nabla$. Then

$$\begin{aligned} R(X, Y, Z, V) &= -R(Y, X, Z, V) \\ &= -R(X, Y, V, Z) = R(PX, PY, Z, V), \\ \sum_{\sigma} R(X, Y, Z, V) &= 0, \end{aligned} \quad (2.12)$$

where σ represents the sum over all cyclic permutations.

3. Ricci-pseudosymmetric and pseudosymmetric

In this section, we study the Ricci-pseudosymmetric and pseudodymmetric condition on para-Kähler manifolds and deduce some results.

THEOREM 3.1. *Every Ricci-pseudosymmetric para-Kählerian manifold is Ricci-semisymmetric.*

Proof. Let the manifold (M, P, g) be para-Kähler satisfying the condition

$$(R \cdot S)(X, Y, U, V) = L_S Q(g, S)(X, Y, U, V). \qquad (3.1)$$

In view of (2.2) and (2.9), we have

$$(R \cdot S)(PX, PY, U, V) = -(R, S)(X, Y, U, V). \qquad (3.2)$$

Using (3.2) in (3.1), it follows that

$$L_S Q(g, S)(X, Y, U, V) = -L_S Q(g, S)(PX, PY, U, V). \qquad (3.3)$$

Since L_S is non-zero at a certain point $q \in M$, from (3.3), we get

$$Q(g, S)(X, Y, U, V) = -Q(g, S)(PX, PY, U, V).$$

In view of (2.10), we have

$$S(X, V)g(Y, U) - S(Y, V)g(X, U) + S(X, U)g(Y, V) - S(Y, U)g(X, V)$$
$$= -S(V, PX)g(U, PY) + S(V, PY)g(U, PX) \qquad (3.4)$$
$$- S(U, PX)g(V, PY) + S(U, PY)g(V, PX).$$

Then taking contraction of (3.4) with respect to Y and U, and using (2.2), we obtain

$$S(X, V) = \frac{r}{n} g(X, V). \qquad (3.5)$$

Which implies (M, P, g) is an Einstein manifold with $R \cdot S = 0$. Thus it completes the proof. ∎

COROLLARY 3.2. *Let (g, V, λ) be a Ricci soliton in a Ricci-pseudosymmetric para-Kählerian manifold (M, P, g). Then V is solenoidal iff it is shrinking or steady or expanding depending on the sign of the scalar curvature.*

Proof. In view of (1.1) and (3.5), we get

$$(\mathfrak{L}_V g)(X, V) + 2\frac{r}{n} g(X, V) + 2\lambda g(X, V) = 0. \qquad (3.6)$$

Taking $X = V = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \leq i \leq n)$, we have

$$(\mathfrak{L}_V g)(e_i, e_i) + 2\frac{r}{n} g(e_i, e_i) + 2\lambda g(e_i, e_i) = 0, \qquad (3.7)$$

which implies

$$\text{div } V + r + \lambda n = 0. \tag{3.8}$$

If V is solenoidal, then div $V = 0$. Thus (3.8) reduces to $\lambda = -(\frac{r}{n})$. Therefore, we obtain the desired result. ∎

A pseudo-Riemannian manifold (M, g) is said to be pseudosymmetric [4] if there exists a function $L_R : M \to \mathbb{R}$ such that

$$R \cdot R = L_R Q(g, R). \tag{3.9}$$

It is well-known that every semisymmetric manifold is also pseudosymmetrc but converse is not true, in general [4].

THEOREM 3.3. *Let* (M, P, g) *be a pseudosymmetric para-Kähler manifold. Then*

  (i) (M, P, g) *is Ricci flat, for dim* $M = 4$.

  (ii) (M, P, g) *is semisymmetric, for dim* $M > 4$.

Proof. Let (M, P, g) be para-Kähler manifold satisfying the condition (3.9). Then using the analogy with Theorem 3.1, we get

$$(n-4)R(X, U, V, W) - 2\,S(V, PW)g(U, PX) + S(W, PU)g(V, PX) \tag{3.10}$$
$$-S(V, PU)g(W, PX) + S(U, W)g(U, V) - S(U, V)g(X, W) = 0.$$

Putting PU instead of U in (3.10) and contacting with respect to U and X we have $S = 0$, for $n = 4$.

On the other hand, for $n > 4$, taking contraction (3.10) with respect to U and V we have

$$S(X, W) = \frac{r}{n}g(X, W), \tag{3.11}$$

which implies $R \cdot S = 0$. In view of (3.10), we get $R \cdot R = 0$. So Theorem 3.3 is proved. ∎

COROLLARY 3.4. *Let* (g, V, λ) *be a Ricci soliton in a pseudosymmetric para-Kähler manifold* (M, P, g) $(n = 4)$. *Then* V *is solenoidal iff it is always steady.*

COROLLARY 3.5. *Let* (g, V, λ) *be a Ricci soliton in a pseudosymmetric para-Kählerian manifold* (M, P, g) $(n > 4)$. *Then* V *is solenoidal iff it is shrinking or steady or expanding depending on the sign of the scalar curvature.*

A para-Kähler manifold with paraholomorphic projective curvature tensor satisfies the condition

$$R \cdot \widetilde{P} = L_P Q(g, \widetilde{P}),$$

where L_P is a function on M. Such type of manifold is called paraholomorphic projective-pseudosymmetric. If $R \cdot \widetilde{P} = 0$, then it is said to be paraholomorphic projective-semisymmetric manifold (see [7]).

THEOREM 3.6.   Let (M, P, g) be a paraholomorphic projective-pseudosymmetric para-Kähler manifold. Then we have

(i) (M, P, g) is Ricci flat, for dim $M = 4$.

(ii) (M, P, g) is semisymmetric, for dim $M > 4$.

Proof. It is well know that, if $R \cdot \widetilde{P} = 0$ at a certain point q of the manifold M, then $R \cdot R = 0$ at this point (see [7]). Now, we suppose that $R \cdot \widetilde{P} \neq 0$ and let the contraction of tensor $\widetilde{P}$ is $\tilde{W}$, defined by

$$\tilde{W}(X, W) = \sum_i \varepsilon_i \widetilde{P}(X, e_i, e_i, W). \tag{3.12}$$

From (2.4), we have

$$\tilde{W}(X, W) = \frac{1}{n+2}\{nS(X, W) - rg(X, W)\}. \tag{3.13}$$

Thus (M, A, g) is paraholomorphic projective-pseudosymmetric. So we have

$$(R \cdot \widetilde{P})(X, Y, Z, U, V, W) = L_P Q(g, \widetilde{P})(X, Y, Z, U, V, W). \tag{3.14}$$

Taking contraction of (3.14) with respect to U and V we get

$$(R \cdot \tilde{W})(X, Y, Z, W) = L_P Q(g, \widetilde{P})(X, Y, Z, W).$$

Also, from (2.9) and (3.13), we obtain

$$(R \cdot S)(X, Y, Z, W) = L_P Q(g, S)(X, Y, Z, W).$$

According to Theorem 3.1, we get $R \cdot S = 0$. Since L_P does not vanish at point q, we have $Q(g, S) = 0$ by the help of above equation. From (2.4) and (3.14), we conclude that $R \cdot R = L_P Q(g, R)$, i.e., (M, P, g) is pseudosymmetric. Therefore, the proof is completed. ∎

COROLLARY 3.7. *Let (g, V, λ) be a Ricci soliton in a paraholomorphic projectively-pseudo symmetric para-Kähler manifold (M, P, g) $(n = 4)$. Then V is solenoidal iff it is always steady.*

COROLLARY 3.8. *Let (g, V, λ) be a Ricci soliton in a paraholomorphic projectively pseudo symmetric para-Kähler manifold (M, P, g) $(n > 4)$. Then V is solenoidal iff it is shrinking or steady or expanding depending on the sign of the scalar curvature.*

4. PARALLEL SYMMETRIC SECOND ORDER COVARIANT TENSOR

In this section, we study the second order parallel tensor on a para-Kähler manifold. Thus we give the following results.

THEOREM 4.1. *A second order parallel tensor in a para-Kähler space form is a linear combination of para-Kähler metric and para-Kähler 2-form.*

Proof. Let $\hbar$ be a $(0, 2)$-tensor which is parallel in view of ∇, that is, $\nabla \hbar = 0$. Then from Ricci identity [10], we have

$$\hbar(R(X, Y)Z, V) + \hbar(Z, R(X, Y)V) = 0. \tag{4.1}$$

Using (2.5) in (4.1) and replacing $X = V = e_i$, $1 \le i \le n$, on simplification, we get

$$\left\{ \begin{array}{c} \hbar(Y, Z) - g(Y, Z)(trH) + \hbar(PY, PZ) - g(Y, PZ)(\text{tr}.HP) \\ + 2\hbar(PY, PZ) + (n - 1)\hbar(Y, Z) + 3\hbar(Z, P^2 Y) \end{array} \right\} = 0, \tag{4.2}$$

where H is a $(1, 1)$ tensor and $\text{tr}.H = \sum_{i=1}^{n} \hbar(e_i, e_i)$. Using the notion of symmetrization and anti-symmetrization. Then from (4.2), we obtain

$$(n + 3)\hbar(Y, Z) + 3\hbar(PY, PZ) = g(Y, Z)(\text{tr}.H) \tag{4.3}$$

and

$$(n + 3)\hbar(Y, Z) + 3\hbar(PY, PZ) = g(Y, PZ)(\text{tr}.HP). \tag{4.4}$$

Replacing Y and Z by PY and PZ in (4.3) and (4.4) respectively, using (2.1), we have

$$\hbar_s(Y, Z) = -\frac{1}{(n + 6)}(\text{tr}.H)g(Y, Z) \tag{4.5}$$

and

$$\hbar_a(Y, Z) = -\frac{1}{n}(\text{tr}.HP)g(PY, Z). \tag{4.6}$$

In view of (4.5) and (4.6), we get

$$\hbar(Y,Z) = \{\, \vartheta\,(\mathrm{tr}.H)\,g(Y,Z) + \omega\,(\mathrm{tr}.HP)\,\psi \},\qquad(4.7)$$

where $\vartheta = -\frac{1}{(n+6)}$, $\omega = -\frac{1}{n}$ and $\psi = g(PY,Z)$. Thus it completes the proof. ▮

COROLLARY 4.2. *A locally Ricci symmetric para-Kähler space form is an Einstein manifold.*

Proof. According to our hypothesis, if we put $H = S$, in (4.7) then we have $\mathrm{tr}.H = r$ and $\mathrm{tr}.HP = 0$. Then follows from (4.7), we obtain

$$S(Y,Z) = \vartheta\,r\,g(Y,Z).\qquad(4.8)$$

Which completes the proof. ▮

COROLLARY 4.3. *Let (g,V,λ) be a Ricci soliton in a para-Kähler space form. Then V is solenoidal iff it is shrinking or steady or expanding depending on the sign of the scalar curvature.*

Proof. In view of (1.1) and (4.8), we get

$$(\mathfrak{L}_V g)(Y,Z) + 2\vartheta r g(Y,Z) + 2\lambda g(Y,Z) = 0.\qquad(4.9)$$

Taking $Y = Z = e_i$ in (4.9), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i $(1 \leq i \leq n)$, we have

$$(\mathfrak{L}_V g)(e_i,e_i) + 2\vartheta\,r\,g(e_i,e_i) + 2\lambda g(e_i,e_i) = 0,\qquad(4.10)$$

which implies

$$\mathrm{div}\,V + \vartheta\,r\,n + \lambda n = 0.\qquad(4.11)$$

Here, if V is solenoidal, then we get $\mathrm{div}\,V = 0$. Thus (4.11) reduces to $\lambda = -\vartheta\,r$. So this completes the proof. ▮

THEOREM 4.4. *A Ricci semi-symmetric para-Kähler space form is an Einstein manifold.*

Proof. Let the para-Kähler space form holds the condition $R \cdot S = 0$. Then we have

$$(R(X,Y) \cdot S)(V,U) = 0,\qquad(4.12)$$

which reduces to

$$S(R(X,Y)V,U) + S(V, R(X,Y)U) = 0. \tag{4.13}$$

Using (2.5) in (4.13) and putting $Y = V = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and again taking summation over i, $1 \leq i \leq n$, we get

$$S(X,U) = \frac{r}{(n+2)}g(X,U). \tag{4.14}$$

Thus this proves the theorem. ∎

COROLLARY 4.5. *Let (g, V, λ) be a Ricci soliton in a Ricci semi-symmetric para-Kähler space form. Then V is solenoidal iff it is shrinking or steady or expanding depending on the sign of the scalar curvature.*

Proof. From (1.1) and (4.14), we get

$$(\mathfrak{L}_V g)(X,U) + 2\frac{r}{(n+2)}g(X,U) + \lambda g(X,U) = 0. \tag{4.15}$$

Then putting $X = U = e_i$ in (4.15) and taking summation over i $(1 \leq i \leq n)$, we have

$$(\mathfrak{L}_V g)(e_i, e_i) + 2\frac{r}{(n+2)}g(e_i, e_i) + 2\lambda g(e_i, e_i) = 0, \tag{4.16}$$

which implies

$$\text{div } V + \frac{r}{(n+2)}n + \lambda n = 0. \tag{4.17}$$

In view of (4.17), if V is solenoidal then we have div $V = 0$. Thus (4.17) reduces to $\lambda = -\frac{r}{(n+2)}$. So the proof is clear. ∎

COROLLARY 4.6. *If the $(0,2)$-type tensor field $L_V + 2S$ is parallel, where V is a vector field on a para-Kähler space form. Then (g, V) admits a Ricci soliton if PV is solenoidal. Moreover, it is shrinking or steady or expanding depending on the sign of the scalar curvature.*

5. QUASI-CONFORMALLY FLAT PARA-KÄHLER MANIFOLDS

In this section, we consider the notion of quasi-conformally flat para-Kähler space form, i.e., $C(X,Y)Z = 0$. Thus we can state the following result.

THEOREM 5.1. *Let (g, V, λ) be a Ricci soliton in a quasi-conformally flat para-Kähler space form. Then V is solenoidal iff it is always steady.*

Proof. In view of (2.1), (2.2) and (2.7), we have

$$\frac{\alpha}{2} g(PY, W) + \beta \left[2S(Y, W) + \tau g(PY, W) \right]$$
$$- \frac{r}{n} \left\{ \frac{\alpha}{n-1} + 2\beta \right\} g(Y, W) = 0. \tag{5.1}$$

Contracting (5.1) over the pair of argument Y and W, we obtain

$$r \left\{ \frac{\alpha}{(n-1)} \right\} = 0, \tag{5.2}$$

which implies

$$r = 0, \qquad \alpha \neq 0. \tag{5.3}$$

In view of (5.1) and (5.3), we get

$$S(Y, W) = - \left\{ \frac{\tau}{2} + \frac{\alpha}{4\beta} \right\} g(PY, W). \tag{5.4}$$

Then using (5.3) and (5.4) in (2.6), we obtain

$$R(X, Y)Z = \frac{\beta}{\alpha} \left[\left(\frac{\tau}{2} + \frac{\alpha}{\beta} \right) \left(g(PY, Z)X - g(PX, P)Y \right. \right.$$
$$\left. \left. + g(Y, Z)PX - g(X, Z)PY \right) \right]. \tag{5.5}$$

Again from (1.1) and (5.4), it yields

$$(\mathfrak{L}_V g)(Y, W) - 2 \left\{ \frac{\tau}{2} + \frac{\alpha}{4\beta} \right\} g(PY, W) + 2\lambda g(Y, W) = 0. \tag{5.6}$$

Taking contraction with respect to Y and W for i $(1 \leq i \leq n)$, we have

$$(\mathfrak{L}_V g)(e_i, e_i) - 2 \left\{ \frac{\tau}{2} + \frac{\alpha}{4\beta} \right\} g(Pe_i, e_i) + 2\lambda g(e_i, e_i) = 0, \tag{5.7}$$

which reduces to

$$\operatorname{div} V + \lambda n = 0. \tag{5.8}$$

Suppose that V is solenoidal. Then (5.8), takes the form $\lambda = 0$. Thus the proof is obvious. ∎

COROLLARY 5.2. *In a quasi-conformally flat para-Kähler space, the Ricci and the curvature tensors have the shapes (5.4) and (5.5), respectively.*

6. Parallel quasiconformal curvature tensor

This section deals with the parallelity condition ($\nabla C = 0$) of quasi-conformal curvature tensor on para-Kähler space form. It is essential to state the following result.

Theorem 6.1. *A para-Kähler space form is quasi-conformally symmetric iff it is locally symmetric.*

Proof. In view of the condition $\nabla C = 0$, (2.7) reduces to

$$-\frac{\alpha}{2}g(PY,W) + \beta\big[2S(Y,W) - \tau g(PY,W)\big]$$
$$+ \frac{r}{n}\left\{\frac{\alpha}{n-1} + 2\beta\right\}g(Y,W) = 0. \tag{6.1}$$

Taking covariant derivation of (6.1) along the vector field Z, we get

$$\beta\big[2(\nabla_Z S)(Y,W) - d\tau(Z)g(PY,W)\big]$$
$$+ \frac{dr(Z)}{n}\left\{\frac{\alpha}{n-1} + 2\beta\right\}g(Y,W) = 0. \tag{6.2}$$

Taking contraction in (6.2) with respect to Y and W, multiplying by ε_i, we obtain

$$dr(Z)\left\{\frac{\alpha}{n-1} + 4\beta\right\} = 0. \tag{6.3}$$

Thus (6.3) implies that $dr(Z) = 0$. Using this equation in (6.2), we have

$$(\nabla_Z S)(Y,W) = \frac{1}{2}d\tau(Z)g(PY,W). \tag{6.4}$$

Replacing Y by AY in (6.4) and using (2.1), we get

$$(\nabla_Z S)(PY,W) = \frac{1}{2}d\tau(Z)g(Y,W). \tag{6.5}$$

Again taking contraction in (6.2) with respect to Y and W, multiplying by ε_i, we obtain

$$(\nabla_Z S)(PY,W) = 0. \tag{6.6}$$

Taking covariant derivation of (2.6) and using (6.6), we get

$$(\nabla_Z C)(X,Y)W = \alpha(\nabla_Z R)(X,Y)W, \qquad \alpha \neq 0. \tag{6.7}$$

Thus it completes the proof. ∎

On the other-hand, if we consider the condition $R \cdot C = 0$, then we have $R \cdot Q = 0$ by using (2.7). This implies that $R \cdot S = 0$. Taking into account of $R \cdot C = 0$ and $R \cdot S = 0$, we get $R \cdot R = 0$. Thus we can state the following result.

COROLLARY 6.2. *A para-Kähler space form is quasi-conformally semisymmetric iff it is semisymmetric.*

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