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Sharp Upper Estimates for the First Eigenvalue of a Jacobi Type Operator

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Abstract: Our purpose in this article is to obtain sharp upper estimates for the first positive eigenvalue of a Jacobi type operator, which is a suitable extension of the linearized operators of the higher order mean curvatures of a closed hypersurface immersed either in the Euclidean space or in the Euclidean sphere.

Key words: Euclidean space, Euclidean sphere, closed hypersurfaces, r-th mean curvatures, Jacobi operator, Reilly type inequalities.

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1. Introduction

In the last decades has been increasing the study of the first positive eigenvalue of certain elliptic operators defined on Riemannian manifolds. This study was initiated in 1977 when Reilly [13] established some inequalities estimates for the first positive eigenvalue λ_1 of the Laplacian operator Δ of a closed hypersurface M^n immersed in the Euclidean space $\mathbb{R}^{n+1}$. For instance, he obtained the following sharp estimate

$$\lambda_1 \left(\int_M H_r \, \mathrm{d}M \right)^2 \leq n \, \mathrm{vol}(M) \int_M H_{r+1}^2 \, \mathrm{d}M \,,$$

for every $0 \leq r \leq n-1$, where H_r stands for the r-th mean curvature of M^n, and the equality holds precisely when M^n is a round sphere of $\mathbb{R}^{n+1}$.

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Several authors presented generalizations and extensions of the previous Reilly's inequality to some other ambient spaces (we refer, for instance, the works [1], [6], [7], [8], [9], [10], [11] and [16]). Also in this setting, we note that Alías and Malacarne [3] extended techniques due to Takahashi [15] and Veeravalli [16] in order to derive sharp upper bounds for the first positive eigenvalue of the linearized operator L_r of the r-th mean curvature H_r of a closed hypersurface immersed either in the Euclidean space $\mathbb{R}^{n+1}$ or in the Euclidean sphere $\mathbb{S}^{n+1}$.

Our aim in this work is study the first positive eigenvalue $\lambda_1^{\mathcal{L}_{r,s}}$ of the Jacobi type (or simply, Jacobi) operator $\mathcal{L}_{r,s}$, which is defined as follows: fixed integer numbers r, s such that $0 \leq r \leq s \leq n-1$, $\mathcal{L}_{r,s} : C^\infty(M) \to C^\infty(M)$ is given by

$$\mathcal{L}_{r,s}(f) = \sum_{j=r}^{s} (j+1) a_j L_j(f)\,, \tag{1.1}$$

where L_j are the linearized operators of the j-th mean curvatures H_j, a_j are nonnegative real numbers (with at least one nonzero) for all $j \in \{r, \ldots, s\}$ and f is a smooth function on the hypersurface M^n which is supposed immersed either in $\mathbb{R}^{n+1}$ or in $\mathbb{S}^{n+1}$.

We point out that the authors in [17] established the notion of (r, s)-stability concerning closed hypersurfaces with higher order mean curvatures linearly related in a space form. In this setting, they obtained a suitable characterization of the (r, s)-stability through of the analysis of the first positive eigenvalue $\lambda_1^{\mathcal{L}_{r,s}}$ of the Jacobi operator $\mathcal{L}_{r,s}$, which is associated to the corresponding variational problem (cf. [17, Theorem 5.3]). Our purpose in this work, is exactly obtain sharp upper estimates for $\lambda_1^{\mathcal{L}_{r,s}}$. Consequently, the results that we will present along this paper are naturally attached with the study of (r, s)-stable closed hypersurfaces in a space form.

This manuscript is organized in the following way: in Section 2, we recall some basic facts concerning r-th mean curvatures H_r and their corresponding linearized operators L_r. Afterwards, in Section 3 we obtain a version of the classical result of Takahashi [15] (cf. Proposition 1) for the Jacobi operator $\mathcal{L}_{r,s}$ defined in (1.1) and we apply it to obtain a Reilly type inequality for $\lambda_1^{\mathcal{L}_{r,s}}$ (cf. Lemma 3). Next, in Section 4 we apply our previous Reilly type inequality in order to prove sharp upper bound for $\lambda_1^{\mathcal{L}_{r,s}}$ (cf. Theorem 1, Theorem 2, Theorem 3 and Corollary1). Finally, in Section5 we consider the case when the ambient space is $\mathbb{S}^{n+1}$ (cf. Theorem 4).

2. Preliminaries

Given a connected and orientable hypersurface $x : M^n \to \overline{M}^{n+1}(c)$ into a Riemannian space form of constant sectional curvature c, one can choose a globally defined unit normal vector field N on M^n. Let A denote the shape operator with respect to N, so that, at each $p \in M^n$, A restricts to a self-adjoint linear map $A_p : T_pM \to T_pM$.

Associated to the shape operator A of M^n one has n algebraic invariants, namely, the elementary symmetric functions S_r of the principal curvatures $\kappa_1, \ldots, \kappa_n$ of A, given by

$$S_r = \sigma_r(\kappa_1, \ldots, \kappa_n) = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r},$$

where, for $1 \leq r \leq n$, $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

The *r-th mean curvature* H_r of M^n is then defined by

$$\binom{n}{r} H_r = S_r.$$

For $0 \leq r \leq n$, let

$$P_r : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

be the r-th *Newton transformation* of M^n, defined inductively by putting $P_0 = I$ (the identity of $\mathfrak{X}(M)$) and, for $1 \leq r \leq n$,

$$P_r = \binom{n}{r} H_r I - AP_{r-1}.$$

A standard fact concerning the Newton transformations is that, $1 \leq r \leq n$,

$$\operatorname{tr}(P_r) = b_r H_r \qquad \text{and} \qquad \operatorname{tr}(AP_r) = b_r H_{r+1}, \tag{2.1}$$

where $b_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$ (see, for instance, [4] and [12]).

On the other hand, the divergence of P_r is defined by

$$\operatorname{div} P_r = \operatorname{tr}(\nabla P_r) = \sum_{i=1}^{n} (\nabla_{e_i} P_r) e_i,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M^n.

Associated to each P_r, one has the second order linear differential operator $L_r : C^\infty(M) \to C^\infty(M)$, given by

$$L_r(f) = \mathrm{tr}(P_r \,\mathrm{Hess} f)\,, \qquad 0 \le r \le n\,. \tag{2.2}$$

Note that, when $r = 0$, the operator L_r reduces to the Laplacian operator of M^n and, since $\overline{M}^{n+1}(c)$ has constant sectional curvature, then L_r is a divergence (cf. [14]), more precisely

$$L_r(f) = \mathrm{div}(P_r \nabla f)\,, \qquad 0 \le r \le n\,,$$

for $f \in C^\infty(M)$.

The following result gives sufficient conditions to the ellipticity of the operators L_r (cf. [4, Proposition 3.2]).

Lemma 1. *Let $\overline{M}^{n+1}(c)$ be the Euclidian space $\mathbb{R}^{n+1}$ (when $c = 0$) or an open hemisphere of the an Euclidian sphere $\mathbb{S}^{n+1}$ (when $c > 0$), and $x : M^n \to \overline{M}^{n+1}(c)$ be a closed hypersurface. If $H_{r+1} > 0$ then*

(a) *each operator L_j is elliptic,*

(b) *each j-th mean curvature H_j is positive,*

for all $j \in \{1, \ldots, r\}$.

When $\overline{M}^{n+1}(c)$ is the Euclidian space, [3, Corollary 3] also gives the following another sufficient criteria of ellipticity.

Lemma 2. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a closed hypersurface with positive Ricci curvature (hence, necessarily embedded). Then*

(a) *each operator L_j is elliptic,*

(b) *each j-th mean curvature H_j is positive,*

for all $j \in \{1, \ldots, r\}$.

3. A Reilly-type inequality in the Euclidean space

Given a closed hypersurface $x : M^n \to \mathbb{R}^{n+1}$, its center of gravity $\mathbf{c}$ is defined by

$$\mathbf{c} = \frac{1}{\mathrm{vol}(M)} \int_M x \, \mathrm{d}M \in \mathbb{R}^{n+1}, \tag{3.1}$$

where $\mathrm{vol}(M)$ denotes the n-dimensional volume of M^n. In this setting, let us consider on M^n the support functions $l_a = \langle x - \mathbf{c}, a\rangle$ and $f_a = \langle N, a\rangle$ with respect to a fixed nonzero vector $a \in \mathbb{R}^{n+1}$. It is not difficult to verify that the gradient of function l_a is given by $\nabla l_a = a^\top$, where $a^\top = a - f_a N \in \mathfrak{X}(M)$. Thus, for $X \in \mathfrak{X}(M)$ we have that

$$\nabla_X \nabla l_a = f_a AX\,. \tag{3.2}$$

From (2.1) and (3.2), for each $j \in \{r, \ldots, s\}$, we get

$$L_j(l_a) = b_j H_{j+1} f_a\,. \tag{3.3}$$

Consequently, considering the Jacobi operator $\mathcal{L}_{r,s}$ defined in (1.1), from (3.3) we obtain

$$\mathcal{L}_{r,s}(l_a) = \left(\sum_{j=r}^{s}(j+1)a_j b_j H_{j+1}\right) f_a\,. \tag{3.4}$$

Thus, denoting by $\{e_1, \ldots, e_{n+1}\}$ the canonical orthonormal basis of $\mathbb{R}^{n+1}$, from (3.4) we can write

$$\mathcal{L}_{r,s}(x - \mathbf{c}) = \left(\sum_{j=r}^{s}(j+1)a_j b_j H_{j+1}\right) N. \tag{3.5}$$

Now, we are in position to present a version of a classical result due to Takahashi [15].

Proposition 1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface and* $\mathbf{c}$ *its center of gravity. If $\mathcal{L}_{r,s}$ is the Jacobi operator defined in* (1.1)*, then*

$$\mathcal{L}_{r,s}(x - \mathbf{c}) + \lambda(x - \mathbf{c}) = 0\,, \tag{3.6}$$

for some real number $\lambda \neq 0$ if, and only if, $x(M)$ is a round sphere of $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$.

Proof. Suppose that (3.6) is true for some $\lambda \neq 0$. From expression (3.5) we have

$$\left(\sum_{j=r}^{s}(j+1)a_j b_j H_{j+1}\right) N + \lambda(x - \mathbf{c}) = 0\,. \tag{3.7}$$

Taking the covariant derivative in (3.7) we obtain

$$X\left(\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1}\right)N-\left(\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1}\right)AX+\lambda X=0\,, \tag{3.8}$$

for all $X\in\mathfrak{X}(M)$. Consequently, taking into account that $\lambda\neq 0$, from (3.8) we conclude that $\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1}$ is a nonzero constant.

Thus, returning to (3.8), we obtain

$$A=\left(\sum_{j=r}^{s}(j+1)a_jb_jH_{j+1}\right)^{-1}\cdot\lambda I,$$

an, hence, $x(M)$ is a totally umbilical hypersurface of $\mathbb{R}^{n+1}$. Therefore, unless of translations and homotheties, $x(M)$ is a round sphere of $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$.

Reciprocally, for a the round sphere of $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$ and of radius $\rho>0$, let us consider $N=-\frac{1}{\rho}(x-\mathbf{c})$, and thus its j-th mean curvature is $H_{j+1}=\frac{1}{\rho^{j+1}}$. Then, from (3.5) we have that (3.6) is satisfied for

$$\lambda=\sum_{j=r}^{s}\frac{(j+1)a_jb_j}{\rho^{j+2}}\neq 0\,,$$

since at least on of a_j are supposed be nonzero. ∎

Remark 1. We note that the first positive eigenvalue of the operator $\mathcal{L}_{r,s}$ on a round sphere $\mathbb{S}^n(\rho)\subset\mathbb{R}^{n+1}$ of radius $\rho>0$ is given by

$$\lambda_1^{\mathcal{L}_{r,s}}=\sum_{j=r}^{s}(j+1)a_jb_jH_{j+2}\,.$$

Indeed, since $\mathbb{S}^n(\rho)$ is totally umbilical with $A=\dfrac{1}{\rho}I$, the j-th Newton transformation is given by $P_j=\dfrac{b_j}{n\rho^j}$, where $b_j=(j+1)\binom{n}{j+1}$. Then

$$L_jf=\frac{b_j}{\rho^j}\Delta f \qquad \text{for each } f\in C^\infty(M)\,.$$

Hence, for integers r, s such that $0 \le r \le s \le n-1$ and nonnegative real numbers a_j (with at least one nonzero) for all $1 \le j \le n$, we have

$$\mathcal{L}_{r,s} = \sum_{j=r}^{s} (j+1) a_j L_j = \sum_{j=r}^{s} \frac{(j+1)a_j b_j}{n\rho^j} \Delta \,.$$

Since the first positive eigenvalue for the Laplacian operator Δ in $\mathbb{S}^n(\rho)$ is given by $\lambda_1^{\Delta} = \dfrac{n}{\rho^2}$, we conclude that

$$\lambda_1^{\mathcal{L}_{r,s}} = \sum_{j=r}^{s} \frac{(j+1)a_j b_j}{\rho^{j+2}} = \sum_{j=r}^{s} (j+1) a_j b_j H_{j+2} \,.$$

Let us consider $(x - \mathbf{c})^{\top} = (x - \mathbf{c}) - \langle x - \mathbf{c}, N \rangle N \in \mathfrak{X}(M)$, where $(x - \mathbf{c})^{\top}$ denotes the component tangent of $x - \mathbf{c}$ along M^n. For every $j \in \{r, \dots, s\}$, using (2.1), it is not difficult to verify that

$$\mathrm{div} P_j (x - \mathbf{c})^{\top} = b_j \big(H_j + \langle x - \mathbf{c}, N \rangle H_{j+1} \big) \,.$$

Consequently,

$$\sum_{j=r}^{s} (j+1) a_j \Big[\mathrm{div} P_j (x - \mathbf{c})^{\top} \Big] = \sum_{j=r}^{s} (j+1) a_j b_j \big(H_j + \langle x - \mathbf{c}, N \rangle H_{j+1} \big) \,, \tag{3.9}$$

where $b_j = (j+1)\binom{n}{j+1} = (n-j)\binom{n}{j}$ and a_j are nonnegative real numbers (with at least one nonzero) for all $j \in \{r, \dots, s\}$.

At this point, we will assume that the hypersurface M^n is closed. So, from (3.9) we obtain the following Minkowski type integral formula

$$\sum_{j=r}^{s} (j+1) a_j b_j \int_M \big(H_j + \langle x - \mathbf{c}, N \rangle H_{j+1} \big) \, \mathrm{d}M = 0 \,. \tag{3.10}$$

In the next result, motivated by Remark 1, we apply Proposition 1 to obtain a Reilly type inequality for the Jacobi operator $\mathcal{L}_{r,s}$.

Lemma 3. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface and let $\mathbf{c}$ be its center of gravity. If either $H_{s+1} > 0$, for some integer number $s \in \{1, \dots, n-1\}$, or the Ricci curvature of M^n is positive (hence, necessarily embedded), then*

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M |x - \mathbf{c}|^2 \, \mathrm{d}M \le \sum_{j=r}^{s} (j+1) a_j b_j \int_M H_j \, \mathrm{d}M \,, \tag{3.11}$$

for all $r \in \{0,\ldots,s-1\}$*, where* $\lambda_1^{\mathcal{L}_{r,s}}$ *is the first positive eigenvalue of Jacobi operator* $\mathcal{L}_{r,s}$ *defined in* (1.1)*,* a_j *are nonnegative real numbers (with at least one nonzero) for all* $j \in \{r,\ldots,s\}$ *and* $b_j = (j+1)\binom{n}{j+1}$*. In particular, the equality occurs in* (3.11) *if and only if* $x(M)$ *is a round sphere of* $\mathbb{R}^{n+1}$ *centered at* $\mathbf{c}$*.*

Proof. Since either $H_{s+1} > 0$ or the Ricci curvature of M^n is positive, Lemma 1 and Lemma 2 guarantee that L_j is elliptic for $j \in \{1,\ldots,s\}$ and, hence, $\mathcal{L}_{r,s}$ is elliptic. Thus, it holds the following characterization of $\lambda_1^{\mathcal{L}}$

$$\lambda_1^{\mathcal{L}_{r,s}} = \inf\left\{\frac{-\int_M f\mathcal{L}_{r,s}(f)\,\mathrm{d}M}{\int_M f^2\,\mathrm{d}M} \;:\; \int_M f\,\mathrm{d}M = 0\right\}. \tag{3.12}$$

Let $\{e_1,\ldots,e_{n+1}\}$ be the canonical orthonormal basis of $\mathbb{R}^{n+1}$. For every $1 \le k \le n+1$, we consider the k-th coordinate function $f_k = \langle x-\mathbf{c}, e_k\rangle$. Thus, for every $1 \le k \le n+1$, from (3.1) we have that $\int_M f_k\,\mathrm{d}M = 0$. So, from (3.12) we get

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M f_k^2\,\mathrm{d}M \le -\int_M f_k\mathcal{L}_{r,s}(f_k)\,\mathrm{d}M\,. \tag{3.13}$$

Furthermore, from (3.4) we obtain

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M f_k^2\,\mathrm{d}M \le -\sum_{j=r}^{s}(j+1)a_jb_j \int_M f_k\langle N, e_k\rangle H_{j+1}\,\mathrm{d}M\,. \tag{3.14}$$

Now, summing on k of 1 until $n+1$ in (3.14) and taking into account that

$$\sum_{k=1}^{n+1} f_k^2 = |x-\mathbf{c}|^2 \qquad \text{and} \qquad \sum_{k=1}^{n+1} f_k\langle N, e_k\rangle = \langle N, x-\mathbf{c}\rangle\,,$$

we get

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M |x-\mathbf{c}|^2\,\mathrm{d}M \le -\sum_{j=r}^{s}(j+1)a_jb_j \int_M \langle N, x-\mathbf{c}\rangle H_{j+1}\,\mathrm{d}M\,. \tag{3.15}$$

Hence, from (3.15) and (3.10) we have

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M |x-\mathbf{c}|^2\,\mathrm{d}M \le \sum_{j=r}^{s}(j+1)a_jb_j \int_M H_j\,\mathrm{d}M\,.$$

If occurs the equality in (3.11), all of the above inequalities are, in fact, equalities and, in particular, from (3.13) we get

$$\mathcal{L}_{r,s}(f_k) + \lambda_1^{\mathcal{L}_{r,s}} f_k = 0\,,$$

for every $k = 1, \dots, n+1$, which happens if and only if $\mathcal{L}_{r,s}(x - \mathbf{c}) + \lambda_1^{\mathcal{L}_{r,s}}(x - \mathbf{c}) = 0$. In this case, Proposition 1 assures that $x(M)$ is a round sphere centered at $\mathbf{c}$. ∎

4. Upper estimates for $\lambda_1^{\mathcal{L}_{r,s}}$ in $\mathbb{R}^{n+1}$

In [3, Theorem 9], Alías and Malacarne obtained the following sharp estimate for the first positive eigenvalue $\lambda_1^{L_r}$ of linearized operator L_r concerning a closed hypersurface immersed in the Euclidean space $\mathbb{R}^{n+1}$

$$\lambda_1^{L_r}\left(\int_M H_s\,\mathrm{d}M\right)^2 \leq b_r \int_M H_r\,\mathrm{d}M \int_M H_{s+1}^2\,\mathrm{d}M\,, \qquad 0 \leq s \leq n-1\,,$$

occurring the equality if and only if M^n is a round sphere of $\mathbb{R}^{n+1}$.

In our next result, we extend the ideas of Alías and Malacarne [3] in order to get a sharp estimate for the first positive eigenvalue of the Jacobi operator $\mathcal{L}_{r,s}$ which was defined in (1.1).

Theorem 1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface and let $\mathbf{c}$ be its center of gravity. If either $H_{s+1} > 0$, for some integer number $s \in \{1, \dots, n-1\}$, or the Ricci curvature of M^n is positive (hence, necessarily embedded), then*

$$\lambda_1^{\mathcal{L}_{r,s}}\left(\int_M \sum_{i=r}^{s}(i+1)\widetilde{a}_i H_i\,\mathrm{d}M\right)^2$$

$$\leq \left(\sum_{j=r}^{s}(j+1)a_j b_j \int_M H_j\,\mathrm{d}M\right)\int_M\left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2\mathrm{d}M\,, \tag{4.1}$$

for all $r \in \{0, \dots, s-1\}$, where $\lambda_1^{\mathcal{L}_{r,s}}$ is the first positive eigenvalue of Jacobi operator $\mathcal{L}_{r,s}$ defined in (1.1), a_j and $\widetilde{a}_i$ are nonnegative real numbers (with at least one nonzero) for all $i, j \in \{r, \dots, s\}$ and $b_j = (j+1)\binom{n}{j+1}$. In particular, the equality in (4.1) holds if and only if $x(M)$ is a round sphere of $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$.

Proof. Let $\mathbf{c}$ the center of gravity of M defined in (3.1). If we multiply both sides of (3.11) by $\int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M$, we obtain

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M |x-\mathbf{c}|^2 \,\mathrm{d}M \int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M \\ \leq \sum_{j=r}^{s}(j+1)a_j b_j \int_M H_j \,\mathrm{d}M \int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M\,.$$

Using Cauchy-Schwarz inequality, the left side can be developed as follows

$$\begin{aligned} \lambda_1^{\mathcal{L}_{r,s}} \int_M |x-\mathbf{c}|^2 \,\mathrm{d}M \int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M & \\ \geq \lambda_1^{\mathcal{L}_{r,s}} \left(\int_M |x-\mathbf{c}| \left|\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right| \mathrm{d}M\right)^2 & \\ \geq \lambda_1^{\mathcal{L}_{r,s}} \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i \int_M \langle x-\mathbf{c}, N\rangle H_{i+1} \,\mathrm{d}M\right)^2 & \\ = \lambda_1^{\mathcal{L}_{r,s}} \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i \int_M H_i \,\mathrm{d}M\right)^2, & \end{aligned}$$

where in the last equality, it was used the Minkowski type integral formula (3.10). Hence,

$$\lambda_1^{\mathcal{L}_{r,s}} \left(\int_M \sum_{i=r}^{s}(i+1)\widetilde{a}_i H_i \,\mathrm{d}M\right)^2 \\ \leq \sum_{j=r}^{s}(j+1)a_j b_j \int_M H_j \,\mathrm{d}M \int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M\,.$$

Now if the equality occurs in (4.1), then the equality occurs also in (3.11), implying that M is a round sphere centered at $\mathbf{c}$. ∎

Proceeding, we also get the following result.

THEOREM 2. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface and let $\mathbf{c}$ be its center of gravity. Assume that, either $H_{s+1} > 0$,* *for some integer number $s \in \{1, \dots, n-1\}$, or the Ricci curvature of M^n is positive (hence, necessarily embedded). If H_{k+1} is constant for some $k \in \{r, \dots, s\}$ then*

$$\lambda_1^{\mathcal{L}_{r,s}} \leq \frac{1}{\mathrm{vol}(M)} \big(H_{k+1}\big)^{\frac{2}{k+1}} \left(\sum_{j=r}^{s} (j+1) a_j b_j \int_M H_j \, \mathrm{d}M \right). \tag{4.2}$$

where $\lambda_1^{\mathcal{L}_{r,s}}$ is the first positive eigenvalue of Jacobi operator $\mathcal{L}_{r,s}$ defined in (1.1), a_j are nonnegative real numbers (with at least one nonzero) for all $j \in \{r, \dots, s\}$ and $b_j = (j+1)\binom{n}{j+1}$. In particular, the equality in (4.2) holds if and only if $x(M)$ is a round sphere of $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$.

Proof. Taking

$$\widetilde{a}_i = \begin{cases} 0\,, & \text{for } i \neq k \in \{r, \dots, s\}, \\ \frac{1}{k+1}\,, & \text{for } i = k \in \{r, \dots, s\}, \end{cases}$$

in Theorem 1 and supposing H_{k+1} constant, for some $k \in \{r, \dots, s\}$, we obtain

$$\lambda_1^{\mathcal{L}_{r,s}} \left(\int_M H_k \, \mathrm{d}M \right)^2 \leq \mathrm{vol}(M) H_{k+1}^2 \left(\sum_{j=r}^{s} (j+1) a_j b_j \int_M H_j \, \mathrm{d}M \right). \tag{4.3}$$

Since $H_{s+1} > 0$, we have that $H_{k+1}^{\frac{1}{k+1}} \leq H_k^{\frac{1}{k}}$ (cf. [5, Proposition 2.3]). Hence, $H_{k+1}^{\frac{k}{k+1}} \leq H_k$ and consequently, from (4.3) we get inequality (4.2). Moreover, if equality occurs in (4.2), then in (4.1) we also have an equality and hence $x(M)$ is a round sphere of $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$. ∎

As a consequence of Theorem 2 we have the following

Corollary 1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface and let $\mathbf{c}$ be its center of gravity. Assume that, either $H_{s+1} > 0$, for some integer number $s \in \{1, \dots, n-1\}$, or the Ricci curvature of M^n is positive (hence, necessarily embedded). If H_{k+1} is constant for some $k \in \{r, \dots, s\}$ then*

$$\lambda_1^{\mathcal{L}_{r,s}} \leq \frac{1}{\mathrm{vol}(M)} \inf_M \, (H_m)^{\frac{2}{m}} \left(\sum_{j=r}^{s} (j+1) a_j b_j \int_M H_j \, \mathrm{d}M \right), \tag{4.4}$$

for any $m \in \{2, \dots, k+1\}$, *where* $\lambda_1^{\mathcal{L}_{r,s}}$ *is the first positive eigenvalue of Jacobi operator* $\mathcal{L}_{r,s}$ *defined in* (1.1), a_j *are nonnegative real numbers (with at least one nonzero) for all* $j \in \{r, \dots, s\}$ *and* $b_j = (j+1)\binom{n}{j+1}$. *In particular, the equality in* (4.4) *holds if and only if* $x(M)$ *is a round sphere of* $\mathbb{R}^{n+1}$ *centered at* $\mathbf{c}$.

Proof. Since $H_{k+1}^{\frac{1}{k+1}} \le H_m^{\frac{1}{m}}$, for all $m \in \{2, \dots, k+1\}$ (cf. [5, Proposition 2.3]), then from (4.2) we have

$$\begin{aligned}\lambda_1^{\mathcal{L}_{r,s}} &\le \frac{1}{\mathrm{vol}(M)} \inf_M \left(H_{k+1}\right)^{\frac{2}{k+1}} \left(\sum_{j=r}^{s} (j+1) a_j b_j \int_M H_j \, \mathrm{d}M\right) \\ &\le \frac{1}{\mathrm{vol}(M)} \inf_M \left(H_m\right)^{\frac{2}{m}} \left(\sum_{j=r}^{s} (j+1) a_j b_j \int_M H_j \, \mathrm{d}M\right),\end{aligned}$$

for any $m \in \{2, \dots, k+1\}$. When equality occurs in (4.4), the same happens in (4.2) and in this case $x(M)$ is a round sphere of $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$. ∎

We close this section with the following

THEOREM 3. *Let* $x : M^n \to \mathbb{R}^{n+1}$ *be an orientable closed connected hypersurface and let* $\mathbf{c}$ *be its center of gravity. If either* $H_{s+1} > 0$, *for some integer number* $s \in \{1, \dots, n-1\}$, *or the Ricci curvature of* M^n *is positive (hence, necessarily embedded), then*

$$\lambda_1^{\mathcal{L}_{r,s}} \left(\int_M \langle x - \mathbf{c}, N \rangle \, \mathrm{d}M\right)^2 \le \mathrm{vol}(M) \sum_{j=r}^{s} (j+1) a_j b_j \int_M H_j \, \mathrm{d}M \,, \tag{4.5}$$

for all $r \in \{0, \dots, s-1\}$, *where* $\lambda_1^{\mathcal{L}_{r,s}}$ *is the first positive eigenvalue of Jacobi operator* $\mathcal{L}_{r,s}$ *defined in* (1.1), a_j *are nonnegative real numbers (with at least one nonzero) for all* $j \in \{r, \dots, s\}$ *and* $b_j = (j+1)\binom{n}{j+1}$. *In particular, the equality occurs in* (4.5) *if and only if* $x(M)$ *is a round sphere of* $\mathbb{R}^{n+1}$ *centered at* $\mathbf{c}$. *Moreover, if* M^n *embedded in* $\mathbb{R}^{n+1}$, *then*

$$\lambda_1^{\mathcal{L}_{r,s}} \le \frac{\mathrm{vol}(M)}{(n+1)^2 \mathrm{vol}(\Omega)^2} \sum_{j=r}^{s} (j+1) a_j b_j \int_M H_j \, \mathrm{d}M \,, \tag{4.6}$$

with equality if and only if $x(M)$ *is a round sphere in* $\mathbb{R}^{n+1}$ *centered at* $\mathbf{c}$. *Here* Ω *is the compact domain in* $\mathbb{R}^{n+1}$ *bounded by* M^n *and* $\mathrm{vol}(\Omega)$ *denotes its* $(n+1)$*-dimensional volume.*

Proof. If we multiply both sides of (3.11) by $\int_M 1^2 \,\mathrm{d}M$, and use Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathrm{vol}(M)\sum_{j=r}^{s}(j+1)a_jb_j\int_M H_j\,\mathrm{d}M &\geq \lambda_1^{\mathcal{L}_{r,s}}\int_M |x-\mathbf{c}|^2\,\mathrm{d}M\int_M 1^2\,\mathrm{d}M\\ &\geq \lambda_1^{\mathcal{L}_{r,s}}\left(\int_M |x-\mathbf{c}|\,\mathrm{d}M\right)^2\\ &\geq \lambda_1^{\mathcal{L}_{r,s}}\left(\int_M \langle x-\mathbf{c},N\rangle\,\mathrm{d}M\right)^2,\end{aligned}$$

showing that (4.5) holds. Now, if the equality occurs in (4.5), then the equality also occurs in (3.11) and, hence, $x(M)$ is a round sphere in $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$.

Moreover, in the case in that M^n is embedded in $\mathbb{R}^{n+1}$, let Ω be a compact domain in $\mathbb{R}^{n+1}$ bounded by M^n so that $M = \partial\Omega$. According to the proof of [3, Theorem 10], let us consider the vector field $Y(p) = p - \mathbf{c}$ defined on Ω, as $\mathrm{div}(Y) = (n+1)$. So, it follows from divergence theorem that

$$(n+1)\mathrm{vol}(\Omega) = \int_M \mathrm{div}(Y)\,\mathrm{d}\Omega = \int_M \langle x-\mathbf{c},N\rangle\,\mathrm{d}M\,.$$

Therefore, from (4.5) we get

$$\lambda_1^{\mathcal{L}_{r,s}} \leq \frac{\mathrm{vol}(M)}{(n+1)^2\mathrm{vol}(\Omega)^2}\sum_{j=r}^{s}(j+1)a_jb_j\int_M H_j\,\mathrm{d}M\,.$$

∎

5. Upper estimates for $\lambda_1^{\mathcal{L}_{r,s}}$ in $\mathbb{S}^{n+1}$

In this last section, we will consider orientable closed connected hypersurface hypersurfaces $x : M^n \to \mathbb{S}^{n+1}$ immersed into the Euclidean sphere $\mathbb{S}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$. According to [3], we defined a center of gravity of M^n as a critical point of the smooth function $\mathcal{E} : \mathbb{S}^{n+1} \to \mathbb{R}$ given by

$$\mathcal{E}(\mathbf{p}) = \int_M \langle x,\mathbf{p}\rangle\,\mathrm{d}M\,, \qquad \mathbf{p}\in\mathbb{S}^{n+1}.$$

In this way, a point $\mathbf{c}\in\mathbb{S}^{n+1}$ is a center of gravity of M^n if, and only if,

$$d\mathcal{E}_{\mathbf{c}}(v) = \int_M \langle x,v\rangle\,\mathrm{d}M = \left\langle \int_M x\,\mathrm{d}M, v\right\rangle = 0\,,$$

for every $v \in T_{\mathbf{c}}\mathbb{S}^{n+1} = \mathbf{c}^{\perp} = \{p \in \mathbb{R}^{n+2} : \langle p, \mathbf{c}\rangle = 0\}$. Hence a center of gravity of M^n is given by

$$\mathbf{c} = \frac{1}{|\int_M x \, \mathrm{d}M|} \int_M x \, \mathrm{d}M \in \mathbb{S}^{n+1},$$

whenever $\int_M x \, \mathrm{d}M \neq 0 \in \mathbb{R}^{n+2}$.

For a fixed nonzero vector $a \in \mathbb{R}^{n+2}$, let us the smooth function $\langle x, a\rangle$ defined on M^n. Then, the gradient of the function $\langle x, a\rangle$ is given by

$$\nabla\langle x, a\rangle = a^{\top} = a - \langle N, a\rangle N - \langle x, a\rangle x \in \mathfrak{X}(M)\,,$$

where N is the orientation of $x : M^n \to \mathbb{S}^{n+1}$. Moreover,

$$\nabla_X \nabla\langle x, a\rangle = \langle N, a\rangle AX - \langle x, a\rangle X,$$

for all $X \in \mathfrak{X}(M)$ and,hence, from (2.1)

$$\begin{aligned}
\mathcal{L}_{r,s}(\langle x, a\rangle) &= \sum_{j=r}^{s}(j+1)a_j L_j(\langle x, a\rangle) \\
&= \sum_{j=r}^{s}(j+1)a_j \operatorname{tr}\big(P_j \circ \operatorname{Hess}(\langle x, a\rangle)\big) \\
&= \sum_{j=r}^{s}(j+1)a_j \big(\langle N, a\rangle \operatorname{tr}(A \circ P_j) - \langle x, a\rangle \operatorname{tr}(P_j)\big) \qquad (5.1)\\
&= \sum_{j=r}^{s}(j+1)a_j b_j \big(\langle N, a\rangle H_{j+1} - \langle x, a\rangle H_j\big)\,,
\end{aligned}$$

where $b_j = (j+1)\binom{n}{j+1} = (n-j)\binom{n}{j}$.

Proceeding with the above notation, in what follows we are able to establish an extension of Lemma 3 for the case that M^n is a hypersurface immersed in $\mathbb{S}^{n+1}$.

Lemma 4. *Let $x : M^n \to \mathbb{S}^{n+1}$ be an orientable closed connected hypersurface, which lies in an open hemisphere of $\mathbb{S}^{n+1}$, and let $\mathbf{c}$ be its center of gravity. If $H_{s+1} > 0$, for some integer number $s \in \{1, \ldots, n-1\}$, then*

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M \big(1 - \langle x, \mathbf{c}\rangle^2\big) \, \mathrm{d}M \leq \sum_{j=r}^{s}(j+1)a_j b_j \int_M H_j \, \mathrm{d}M\,, \qquad (5.2)$$

for all $r \in \{0, \ldots, s-1\}$, where $\lambda_1^{\mathcal{L}_{r,s}}$ is the first positive eigenvalue of Jacobi operator $\mathcal{L}_{r,s}$ defined in (1.1), a_j are nonnegative real numbers (with at least one nonzero) for all $j \in \{r, \ldots, s\}$ and $b_j = (j+1)\binom{n}{j+1}$. In particular, the equality occurs in (5.2) if and only if $x(M)$ is an geodesic sphere in $\mathbb{S}^{n+1}$ centered at $\mathbf{c}$.

Proof. Since $H_{s+1} > 0$, Lemma 1 guarantees that $\mathcal{L}_{r,s}$ is elliptic and, hence, it holds the characterization of its first positive eigenvalue given in (3.12). We consider the canonical basis $\{e_1 \ldots, e_{n+1}\} \subset \mathbb{R}^{n+2}$ of $T_c\mathbb{S}^{n+1} = \mathbf{c}^\perp = \{v \in \mathbb{R}^{n+2} : \langle v, \mathbf{c}\rangle = 0\}$ and for every $1 \le k \le n+1$, let us $f_k = \langle x, e_k\rangle$. Then, as before, $\int_M f_k \, \mathrm{d}M = 0$, for every $1 \le k \le n+1$, and from (5.1)

$$\mathcal{L}_{r,s}(f_k) = \sum_{j=r}^{s}(j+1)a_j b_j \big(\langle N, e_k\rangle H_{j+1} - \langle x, e_k\rangle H_j\big) . \tag{5.3}$$

Hence, from (3.12) we have

$$\begin{aligned}\lambda_1^{\mathcal{L}_{r,s}} \int_M f_k^2 \, \mathrm{d}M &\le -\int_M f_k \mathcal{L}_{r,s}(f_k) \, \mathrm{d}M \\ &= \sum_{j=r}^{s}(j+1)a_j b_j \int_M \big(f_k^2 H_j - f_k\langle N, e_k\rangle H_{j+1}\big) \, \mathrm{d}M .\end{aligned} \tag{5.4}$$

On the one hand,

$$x = \sum_{k=1}^{n+1} f_k e_k + \langle x, \mathbf{c}\rangle\mathbf{c} \quad \text{and} \quad N = \sum_{k=1}^{n+1}\langle N, e_k\rangle e_k + \langle \mathbf{c}, N\rangle\mathbf{c} ,$$

so that

$$\sum_{k=1}^{n+1} f_k\langle N, e_k\rangle = -\langle \mathbf{c}, N\rangle\langle x, \mathbf{c}\rangle \quad \text{and} \quad 1 - \langle x, \mathbf{c}\rangle^2 = \sum_{k=1}^{n+1} f_k^2 . \tag{5.5}$$

Summing on k of 1 until $n+1$ in (5.4) and using relations (5.5), we obtain

$$\begin{aligned}&\lambda_1^{\mathcal{L}_{r,s}} \int_M (1 - \langle x, \mathbf{c}\rangle^2) \, \mathrm{d}M \\ &\le \sum_{j=r}^{s}(j+1)a_j b_j \left(\int_M (1 - \langle x, \mathbf{c}\rangle^2) H_j \, \mathrm{d}M + \int_M \langle \mathbf{c}, N\rangle\langle x, \mathbf{c}\rangle H_{j+1} \, \mathrm{d}M\right) .\end{aligned} \tag{5.6}$$

Now, taking $a = \mathbf{c}$ in (5.1)

$$\mathcal{L}_{r,s}(\langle x, \mathbf{c}\rangle) = \sum_{j=r}^{s}(j+1)a_j b_j\big(\langle \mathbf{c}, N\rangle H_{j+1} - \langle x, \mathbf{c}\rangle H_j\big)\,, \tag{5.7}$$

multiply both sides of (5.7) by $\langle x, \mathbf{c}\rangle$, we obtain

$$\langle x, \mathbf{c}\rangle\mathcal{L}_{r,s}(\langle x, \mathbf{c}\rangle) = \sum_{j=r}^{s}(j+1)a_j b_j\big(\langle x, \mathbf{c}\rangle\langle \mathbf{c}, N\rangle H_{j+1} - \langle x, \mathbf{c}\rangle^2 H_j\big)\,. \tag{5.8}$$

Replacing (5.8) in (5.6), we get

$$\begin{aligned}\lambda_1^{\mathcal{L}_{r,s}}&\int_M (1-\langle x, \mathbf{c}\rangle^2)\,\mathrm{d}M\\ &\le \sum_{j=r}^{s}(j+1)a_j b_j\left(\int_M H_j\,\mathrm{d}M + \int_M \langle x, \mathbf{c}\rangle\mathcal{L}_{r,s}(\langle x, \mathbf{c}\rangle)H_{j+1}\,\mathrm{d}M\right).\end{aligned} \tag{5.9}$$

With a straightforward computation, we see that

$$\mathcal{L}_{r,s}\left(\langle x, \mathbf{c}\rangle^2\right) = \sum_{j=r}^{s}(j+1)a_j\left\langle \nabla\langle x, \mathbf{c}\rangle, P_j\left(\nabla\langle x, \mathbf{c}\rangle\right)\right\rangle + \langle x, \mathbf{c}\rangle\mathcal{L}_{r,s}(\langle x, \mathbf{c}\rangle)\,.$$

Integrating over M^n and using divergence theorem we obtain

$$\int_M \langle x, \mathbf{c}\rangle\mathcal{L}_{r,s}\left(\langle x, \mathbf{c}\rangle\right)\,\mathrm{d}M = -\sum_{j=r}^{s}(j+1)a_j\int_M \left\langle \mathbf{c}^\top, P_j(\mathbf{c}^\top)\right\rangle\,\mathrm{d}M\,, \tag{5.10}$$

where $\mathbf{c}^\top = \nabla\langle x, \mathbf{c}\rangle$. From (5.9) and (5.10), we get

$$\begin{aligned}\lambda_1^{\mathcal{L}_{r,s}}&\int_M \left(1-\langle x, \mathbf{c}\rangle^2\right)\,\mathrm{d}M\\ &\le \sum_{j=r}^{s}(j+1)a_j b_j\int_M H_j\,\mathrm{d}M - \sum_{j=r}^{s}(j+1)a_j\int_M \left\langle \mathbf{c}^\top, P_j(\mathbf{c}^\top)\right\rangle\,\mathrm{d}M\,.\end{aligned} \tag{5.11}$$

Since each operator L_j is elliptic, for $r \le j \le s$, from Lemma 1 we have that the operator $\widetilde{P} = \sum_{j=r}^{s}(j+1)a_jP_j$ is positive. Consequently, from (5.11) we get

$$\lambda_1^{\mathcal{L}_{r,s}}\int_M \left(1-\langle x, \mathbf{c}\rangle^2\right)\,\mathrm{d}M \le \sum_{j=r}^{s}(j+1)a_j b_j\int_M H_j\,\mathrm{d}M\,,$$

with the equality occurs if and only if $\mathbf{c}^\top = \nabla\langle x, \mathbf{c}\rangle = 0$, that is, if and only if $x(M)$ is a geodesic sphere $\mathbb{S}^{n+1}$ centered at the point $\mathbf{c}$. ∎

Before to present our last result, we observe that integrating (5.1) over M^n and using divergence theorem we obtain the following Minkowski type formula for hypersurfaces immersed in $\mathbb{S}^{n+1}$

$$\sum_{j=r}^{s}(j+1)a_jb_j\int_M \big(\langle N,a\rangle H_{j+1}\,\mathrm{d}M-\langle x,a\rangle H_j\big)\,\mathrm{d}M=0\,, \tag{5.12}$$

where $a\in\mathbb{R}^{n+2}$ is arbitrary.

As an application of Lemma 4, we derive the following Reilly type inequality for the first positive eigenvalue of the Jacobi operator $\mathcal{L}_{r,s}$ of a closed hypersurface in sphere, which extend [3, Theorem 16].

THEOREM 4. *Let $x: M^n\to\mathbb{S}^{n+1}$ orientable closed connected hypersurface, which lies in an open hemisphere of $\mathbb{S}^{n+1}$, and let $\mathbf{c}$ be its center of gravity. If $H_{s+1}>0$, for some integer number $s\in\{1,\dots,n-1\}$, then we have following inequalities*

$$\begin{aligned}&\lambda_1^{\mathcal{L}_{r,s}}\left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i\int_M H_i\langle x,\mathbf{c}\rangle\,\mathrm{d}M\right)^2\\ &\le\sum_{j=r}^{s}(j+1)a_jb_j\int_M H_j\,\mathrm{d}M\int_M\left(\sum_{i=r}^{s}(i+1)\widetilde{a}_iH_{i+1}\right)^2\mathrm{d}M\end{aligned} \tag{5.13}$$

and

$$\lambda_1^{\mathcal{L}_{r,s}}\left(\int_M\langle\mathbf{c},N\rangle\,\mathrm{d}M\right)^2\le\mathrm{vol}(M)\sum_{j=r}^{s}(j+1)a_jb_j\int_M H_j\,\mathrm{d}M\,, \tag{5.14}$$

for all $r\in\{0,\dots,s-1\}$, where $\lambda_1^{\mathcal{L}_{r,s}}$ is the first positive eigenvalue of Jacobi operator $\mathcal{L}_{r,s}$ defined in (1.1), a_j and $\widetilde{a}_i$ are nonnegative real numbers (with at least one nonzero) for all $i,j\in\{r,\dots,s\}$, $b_j=(j+1)\binom{n}{j+1}$ and $\mathrm{vol}(M)$ denotes the n-dimensional volume of M^n. In particular, if M is embedded in $\mathbb{S}^{n+1}$ then (5.13) results in

$$\lambda_1^{\mathcal{L}_{r,s}}\left(\int_\Omega\langle\mathbf{c},p\rangle\,\mathrm{d}\Omega(p)\right)^2\le\frac{\mathrm{vol}(M)}{(n+1)^2}\sum_{j=r}^{s}(j+1)a_jb_j\int_M H_j\,\mathrm{d}M\,, \tag{5.15}$$

where Ω is any one of the two compact domains of $\mathbb{S}^{n+1}$ bounded by M^n. Moreover, the equality occurs in one of these three inequalities if and only if $x(M)$ is a geodesic sphere in $\mathbb{S}^{n+1}$ centered at $\mathbf{c}$.

Proof. Multiply both sides of (5.2) by $\int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M$, we have

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M (1-\langle x,\mathbf{c}\rangle^2)\,\mathrm{d}M \int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M$$

$$\leq \sum_{j=r}^{s}(j+1)a_j b_j \int_M H_j\,\mathrm{d}M \int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M\,.$$

Using Cauchy-Schwarz inequality, the side left can be developed as in following way

$$\lambda_1^{\mathcal{L}_{r,s}} \int_M (1-\langle x,\mathbf{c}\rangle^2)\,\mathrm{d}M \int_M \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right)^2 \mathrm{d}M$$

$$\geq \lambda_1^{\mathcal{L}_{r,s}} \left(\int_M \sqrt{1-\langle x,\mathbf{c}\rangle^2}\left|\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right| \mathrm{d}M\right)^2. \tag{5.16}$$

On the other hand, $\mathbf{c} = \mathbf{c}^\top + \langle \mathbf{c}, N\rangle N + \langle x,\mathbf{c}\rangle x$, so that

$$1-\langle x,\mathbf{c}\rangle^2 = \left|\mathbf{c}^\top\right|^2 + \langle \mathbf{c},N\rangle^2 \geq \langle \mathbf{c},N\rangle^2,$$

which implies

$$\sqrt{1-\langle x,\mathbf{c}\rangle^2} \geq |\langle \mathbf{c},N\rangle|\,. \tag{5.17}$$

Occurring equality if and only if $\nabla\langle x,\mathbf{c}\rangle = \mathbf{c}^\top = 0$, that is, if and only if $x(M)$ is a geodesic sphere in $\mathbb{S}^{n+1}$ centered at $\mathbf{c}$.

Replacing (5.17) in (5.16) and using the Minkowski type formula (5.12) with $a = \mathbf{c}$, we obtain

$$\lambda_1^{\mathcal{L}_{r,s}} \left(\int_M \sqrt{1-\langle x,\mathbf{c}\rangle^2}\left|\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right| \mathrm{d}M\right)^2$$

$$\geq \lambda_1^{\mathcal{L}_{r,s}} \left(\int_M |\langle \mathbf{c},N\rangle|\left|\sum_{i=r}^{s}(i+1)\widetilde{a}_i H_{i+1}\right| \mathrm{d}M\right)^2$$

$$\geq \lambda_1^{\mathcal{L}_{r,s}} \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i \int_M \langle \mathbf{c},N\rangle H_{i+1}\,\mathrm{d}M\right)^2$$

$$= \lambda_1^{\mathcal{L}_{r,s}} \left(\sum_{i=r}^{s}(i+1)\widetilde{a}_i \int_M \langle x,\mathbf{c}\rangle H_i\,\mathrm{d}M\right)^2,$$

which proves (5.13).

For proof the of (5.14), we multiply both sides of (5.2) by $\mathrm{vol}(M) = \int_M 1^2 \,\mathrm{d}M$, using Cauchy-Schwarz inequality in (5.17), we have

$$\begin{aligned}\mathrm{vol}(M)\sum_{j=r}^{s}(j+1)a_jb_j\int_M H_j\,\mathrm{d}M &\geq \lambda_1^{\mathcal{L}_{r,s}}\int_M(1-\langle x,\mathbf{c}\rangle^2)\,\mathrm{d}M\int_M 1^2\,\mathrm{d}M\\ &\geq \lambda_1^{\mathcal{L}_{r,s}}\left(\int_M\sqrt{1-\langle x,\mathbf{c}\rangle^2}\,\mathrm{d}M\right)^2\\ &\geq \lambda_1^{\mathcal{L}_{r,s}}\left(\int_M\langle \mathbf{c},N\rangle\,\mathrm{d}M\right)^2,\end{aligned}$$

which shows (5.14). Moreover, if occurs the equality either in (5.13) or in (5.14), then occurs in (5.2), and $x(M)$ is a geodesic sphere in $\mathbb{S}^{n+1}$ centered at $\mathbf{c}$.

Now, if M^n is embedded in $\mathbb{S}^{n+1}$, following the same steps of [3, Theorem 16], let us consider the vector field Y on $\mathbb{S}^{n+1}$ defined by $Y(p) = \mathbf{c} - \langle \mathbf{c}, p\rangle p$, $p \in \mathbb{S}^{n+1}$. Observe that Y is a conformal vector field on $\mathbb{S}^{n+1}$ with singularities in $\mathbf{c}$ and $-\mathbf{c}$, and with spherical divergence given by

$$\mathrm{div}Y = -(n+1)\langle \mathbf{c}, p\rangle\,.$$

Moreover, if Ω denotes one of the two compact domains in $\mathbb{S}^{n+1}$ bounded by M^n so that $\partial\Omega = M$, then

$$(n+1)^2\left(\int_\Omega\langle \mathbf{c},p\rangle\,\mathrm{d}\Omega(p)\right)^2 = \left(\int_M\langle \mathbf{c},N\rangle\,\mathrm{d}M\right)^2. \tag{5.18}$$

Therefore, replacing (5.18) in (5.13), we obtain (5.15). ∎

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