EXTRACTA MATHEMATICAE Vol. **31**, Núm. 2, 123 – 144 (2016)

Weighted Spaces of Holomorphic Functions on Banach Spaces and the Approximation Property

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Presented by Manuel Maestre Received April 4, 2016

Abstract: In this paper, we study the linearization theorem for the weighted space $\mathcal{H}_w(U;F)$ of holomorphic functions defined on an open subset U of a Banach space E with values in a Banach space F. After having introduced a locally convex topology $\tau_{\mathcal{M}}$ on the space $\mathcal{H}_w(U;F)$, we show that $(\mathcal{H}_w(U;F),\tau_{\mathcal{M}})$ is topologically isomorphic to $(\mathcal{L}(\mathcal{G}_w(U);F),\tau_c)$ where $\mathcal{G}_w(U)$ is the predual of $\mathcal{H}_w(U)$ consisting of all linear functionals whose restrictions to the closed unit ball of $\mathcal{H}_w(U)$ are continuous for the compact open topology τ_0. Finally, these results have been used in characterizing the approximation property for the space $\mathcal{H}_w(U)$ and its predual for a suitably restricted weight w.

Key words: Holomorphic mappings, weighted spaces of holomorphic functions, linearization, approximation property.

AMS *Subject Class.* (2010): 46G20, 46E50, 46B28.

1. Introduction

Approximation properties for various classes of holomorphic functions have been studied earlier by using linearization techniques in [6], [7], [8], [18], etc. If E and F are Banach spaces and U is an open subset of E, then the linearization results help in identifying a given class of holomorphic functions defined on U with values in F, with the space of continuous linear mappings from a certain Banach space G to F; indeed, a holomorphic mapping is being identified with a linear operator through linearization results. This study for various classes of holomorphic mappings have been carried out by Beltran [2], Galindo, Garcia and Maestre [11], Mazet [17], Mujica [18, 19, 20] and several other mathematicians.

On the other hand, whereas the weighted spaces of holomorphic functions defined on an open subset of the finite dimensional space $\mathbb{C}^N$, $N \in \mathbb{N}$ (set of natural numbers) have been investigated in [3], [4], [5], [24], etc., the infinite dimensional case was considered by Garcia, Maestre and Rueda [12], Jorda [15], Rueda [25]. The present paper is an attempt to study approximation properties for weighted spaces of holomorphic mappings. Indeed, after having given preliminaries in Section 2, we prove in Section 3 a linearization theorem for the weighted space $\mathcal{H}_w(U;F)$ of holomorphic functions defined on U with values in F. As an application of this result, we show that E is topologically isomorphic to a complemented subspace of $\mathcal{G}_w(U)$ for those weights w for which $\mathcal{H}_w(U)$ contains all the polynomials. In case of a weight being given by an entire function with positive coefficients, we also obtain estimates for the norm of the topological isomorphism.

In Section 4 we define a locally convex topology $\tau_{\mathcal{M}}$ on the space $\mathcal{H}_w(U;F)$ and show the topological isomorphism between the spaces $(\mathcal{H}_w(U;F),\tau_{\mathcal{M}})$ and $(\mathcal{L}(\mathcal{G}_w(U);F),\tau_c)$ for a weight w on an open set U.

Finally, in Section 5 we consider the applications of results proved in Sections 3 and 4 to obtain characterizations of the approximation property for the space $\mathcal{H}_w(U)$ and its predual $\mathcal{G}_w(U)$; for instance, we prove that $\mathcal{H}_w(U)$ has the approximation property if and only if it satisfies the holomorphic analogue of Theorem 2.4(iv), *i.e*, for any Banach space F, each mapping in $\mathcal{H}_w(U;F)$ with relatively compact range belongs to the $\|\cdot\|_w$-closure of the subspace of $\mathcal{H}_w(U;F)$ consisting of finite dimensional holomorphic mappings. Besides, it is proved that for a suitably restricted w and U, $\mathcal{G}_w(U)$ has the approximation property if and only if E has the approximation property.

2. Preliminaries

Throughout this paper, the symbols $\mathbb{N},\mathbb{N}_0$ and $\mathbb{C}$ respectively denote the set of natural numbers, $\mathbb{N}\cup\{0\}$ and the complex plane. The letters E and F are used for complex Banach spaces. The symbols E' and E^* denote respectively the algebraic dual and topological dual of E. We denote by U a non-empty open subset of E; and by U_E and B_E, the open and closed unit ball of E. For a locally convex space X, we denote by X^*_β and X^*_c, the topological dual X^* of X equipped respectively with the strong topology, i.e., the topology of uniform convergence on all bounded subsets of X, and the compact open topology.

For each $m\in\mathbb{N}$, $\mathcal{L}(^mE;F)$ is the Banach space of all continuous m-linear mappings from E to F endowed with its natural sup norm. For m=1, we write $\mathcal{L}(E,F)$ for $\mathcal{L}(^mE;F)$. A mapping $P:E\to F$ is said to be a *continuous m-homogeneous polynomial* if there exists a continuous m-linear map $A\in\mathcal{L}(^mE;F)$ such that

$$P(x)=A(x,\dots,x),\quad x\in E.$$

In this case, we also write $P = \hat{A}$. The space of all continuous m-homogeneous polynomials from E to F is denoted by $\mathcal{P}(^mE;F)$ which is a Banach space endowed with the sup norm. A *continuous polynomial* P is a mapping from E into F which can be represented as a sum $P = P_0 + P_1 + \cdots + P_k$ with $P_m \in \mathcal{P}(^mE;F)$ for $m = 0, 1, \ldots, k$. The vector space of all continuous polynomials from E into F is denoted by $\mathcal{P}(E;F)$.

A polynomial $P \in \mathcal{P}(^mE;F)$ is said to be of *finite type* if it is of the form

$$P(x) = \sum_{j=1}^{k} \phi_j^m(x) y_j, \quad x \in E,$$

where $\phi_j \in E^*$ and $y_j \in F$, $1 \leq j \leq k$. We denote by $\mathcal{P}_f(^mE;F)$ the space of finite type polynomials from E into F. A continuous polynomial P from E into F is said to be of finite type if it has a representation as a sum $P = P_0 + P_1 + \cdots + P_k$ with $P_m \in \mathcal{P}_f(^mE;F)$ for $m = 0, 1, \ldots, k$. The vector space of continuous polynomials of finite type from E into F is denoted by $\mathcal{P}_f(E;F)$.

A mapping $f : U \to F$ is said to be *holomorphic*, if for each $\xi \in U$, there exists a ball $B(\xi, r)$ with center at ξ and radius $r > 0$, contained in U and a sequence $\{P_m\}_{m=1}^{\infty}$ of polynomials with $P_m \in \mathcal{P}(^mE;F)$, $m \in \mathbb{N}_0$ such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - \xi), \tag{2.1}$$

where the series converges uniformly for $x \in B(\xi, r)$. The series in (2.1) is called the Taylor series of f at ξ and in analogy with complex variable case, it is written as

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(\xi)(x - \xi), \tag{2.2}$$

where $P_m = \frac{1}{m!} \hat{d}^m f(\xi)$.

The space of all holomorphic mappings from U to F is denoted by $\mathcal{H}(U;F)$. It is usually endowed with the topology τ_0 of uniform convergence on compact subsets of U and $(\mathcal{H}(U;F), \tau_0)$ is a Fréchet space when U is an open subset of a finite dimensional Banach space. In case $U = E$, the class $\mathcal{H}(E;F)$ is the space of entire mappings from E into F. For $F = \mathbb{C}$, we write $\mathcal{H}(U)$ for $\mathcal{H}(U;\mathbb{C})$. We refer to [1], [9], [19] and [22] for notations and various results on infinite dimensional holomorphy.

If $f \in \mathcal{H}(U;F)$ and $n \in \mathbb{N}_0$, we write $S_n f(x) = \sum_{m=0}^{n} \frac{1}{m!}\hat{d}^m f(0)(x)$ and $C_n f(x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k f(x)$. It has been shown in [18] that

$$S_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(e^{it}x)D_n(t)dt \quad \text{and} \quad C_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(e^{it}x)K_n(t)dt,$$

where $D_n(t)$ and $K_n(t)$ are respectively the Dirichlet and Fejer kernels given as follows:

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n}\cos kt \quad \text{and} \quad K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t).$$

A subset A of U is called *U-bounded* if A is bounded and $\operatorname{dist}(A, \partial U) > 0$, where ∂U denotes the boundary of U. A mapping f in $\mathcal{H}(U;F)$ is of *bounded type* if it maps U-bounded sets to bounded sets. The space of holomorphic mappings of bounded type is denoted by $\mathcal{H}_b(U;F)$. The space $\mathcal{H}_b(U;F)$ endowed with the topology τ_b, the topology of uniform convergence on U-bounded sets, is a Fréchet space, cf. [1, p. 81]. For $U = U_E$, the following result is quoted from [27].

Theorem 2.1. *If $\{x_n\}$ is a sequence of distinct points in U_E such that*

$$\lim_{n\to\infty} \operatorname{dist}(\{x_n\}, \partial U_E) = 0$$

and $\{u_n\}$ is a sequence of vectors in F then there exists $f \in \mathcal{H}_b(U_E;F)$ such that

$$f(x_n) = u_n, \quad n = 1, 2, \dots$$

A *weight* w on U is a continuous and strictly positive function satisfying

$$0 < \inf_A w(x) \le \sup_A w(x) < \infty \tag{2.3}$$

for each U-bounded set A. A weight w defined on an open balanced subset U of E is said to be *radial* if $w(tx) = w(x)$ for all $x \in U$ and $t \in \mathbb{C}$, with $|t| = 1$; and on E it is said to be *rapidly decreasing* if $\sup_{x\in E} w(x)\|x\|^m < \infty$ for each $m \in \mathbb{N}_0$.

Corresponding to a weight function w, the weighted space of holomorphic functions is defined as

$$\mathcal{H}_w(U;F) = \Big\{ f \in \mathcal{H}(U;F) : \ \|f\|_w = \sup_{x\in U} w(x)\|f(x)\| < \infty \Big\}.$$

The space $(\mathcal{H}_w(U;F), \|\cdot\|_w)$ is a Banach space and B_w denotes its closed unit ball. For $F = \mathbb{C}$, we write $\mathcal{H}_w(U) = \mathcal{H}_w(U;\mathbb{C})$. It can be easily seen that the norm topology $\tau_{\|\cdot\|_w}$ on $\mathcal{H}_w(U;F)$ is finer than the topology induced by τ_0. In case, $\mathcal{P}(E) \subset \mathcal{H}_w(U)$, we have the following result from [12].

Proposition 2.2. *The topology $\tau_{\|\cdot\|_w}$ restricted to $\mathcal{P}(^mE)$ coincides with the norm topology.*

Since the closed unit ball B_w of $\mathcal{H}_w(U)$ is τ_0-compact by the Ascoli's theorem, the predual of $\mathcal{H}_w(U)$ is given by

$$\mathcal{G}_w(U) = \left\{\phi \in \mathcal{H}_w(U)' : \phi|B_w \text{ is } \tau_0 - \text{continuous}\right\}$$

by the Ng Theorem; cf. [23].

Further, we consider the locally convex topology τ_{bc} on $\mathcal{H}_w(U)$ for which a set $A \subset \mathcal{H}_w(U)$ is τ_{bc} open if and only if $A \cap B$ is open in $(B, B|\tau_0)$ for each $\|\cdot\|_w$-bounded subset B of $\mathcal{H}_w(U)$. Concerning this topology, we have the following result from [25].

Proposition 2.3. *Let U be an open subset of a Banach space E and w be a weight on U. Then*

(i) *$(\mathcal{H}_w(U), \|\cdot\|_w)$ and $(\mathcal{H}_w(U), \tau_{bc})$ have the same bounded sets.*

(ii) *$\mathcal{G}_w(U) = (\mathcal{H}_w(U), \tau_{bc})^*_\beta$.*

(iii) *$(\mathcal{H}_w(U), \tau_{bc}) = \mathcal{G}_w(U)^*_c$.*

An operator T in $\mathcal{L}(E;F)$ is said to have a *finite rank* if the range of T is finite dimensional and, an operator T in $\mathcal{L}(E;F)$ is called *compact* if $T(B_E)$ is a relatively compact subset of F. We denote by $\mathcal{F}(E;F)$ and $\mathcal{K}(E;F)$, respectively, the space of all finite rank operators and compact operators from E into F.

A Banach space E is said to have the *approximation property* if for every compact set K of E and $\epsilon > 0$, there exists an operator $T \in \mathcal{F}(E;E)$ such that

$$\sup_{x\in K} \|T(x) - x\| < \epsilon.$$

The following characterization of the approximation property due to Grothendieck, is given in [16].

Theorem 2.4. *For a Banach space E, the following are equivalent:*

(i) *E has the approximation property.*

(ii) *For every Banach space F, $\overline{\mathcal{F}(E;F)}^{\tau_c} = \mathcal{L}(E;F)$.*

(iii) *For every Banach space F, $\overline{\mathcal{F}(F;E)}^{\tau_c} = \mathcal{L}(F;E)$.*

(iv) *For every Banach space F, $\overline{\mathcal{F}(F;E)}^{\|\cdot\|} = \mathcal{K}(F;E)$.*

Proposition 2.5. *Let E be a Banach space. Then E^* has the approximation property if and only if $\overline{\mathcal{F}(E;F)}^{\|\cdot\|} = \mathcal{K}(E;F)$, for every Banach space F.*

Proposition 2.6. *Let E be a Banach space with the approximation property. Then each complemented subspace of E also has the approximation property.*

3. Linearization theorem for $\mathcal{H}_w(U;F)$ and its applications

In this section, we consider the linearization theorem for $\mathcal{H}_w(U;F)$ and some of its applications. Let us begin with

Theorem 3.1. (Linearization Theorem) *For an open subset U of a Banach space E and a weight w on U, there exists a Banach space $\mathcal{G}_w(U)$ and a mapping $\Delta_w \in \mathcal{H}_w(U;\mathcal{G}_w(U))$ with the following property: for each Banach space F and each mapping $f \in \mathcal{H}_w(U;F)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{G}_w(U);F)$ such that $T_f \circ \Delta_w = f$. The correspondence Ψ between $\mathcal{H}_w(U;F)$ and $\mathcal{L}(\mathcal{G}_w(U);F)$ given by*

$$\Psi(f) = T_f$$

is an isometric isomorphism. The space $\mathcal{G}_w(U)$ is uniquely determined up to an isometric isomorphism by these properties.

Proof. Though the proof of this result is similar to the one given in [2], we sketch the same for the sake of completeness.

Let B_w be the closed unit ball of $\mathcal{H}_w(U)$. Then it is τ_0-compact by Ascoli's Theorem. Hence by the Ng's Theorem, $\mathcal{H}_w(U)$ is a dual Banach space, its predual being given by

$$\mathcal{G}_w(U) = \{h \in \mathcal{H}_w(U)' : h|B_w \text{ is } \tau_0\text{-continuous}\}.$$

Further the mapping $J_U^w : \mathcal{H}_w(U) \to \mathcal{G}_w(U)^*$, $J_U^w(f) = \hat{f}$ with $\hat{f}(h) = h(f)$, $f \in \mathcal{H}_w(U)$ and $h \in \mathcal{G}_w(U)$, is an isometric isomorphism.

Now define $\Delta_w : U \to \mathcal{G}_w(U)$ as $\Delta_w(x) = \delta_x$, where $\delta_x(f) = f(x)$, $f \in \mathcal{H}_w(U)$.

Since for $x \in U$ and $f \in \mathcal{H}_w(U)$, $J_U^w(f) \circ \Delta_w(x) = J_U^w(f)(\delta_x) = f(x)$ and $J_U^w(\mathcal{H}_w(U)) = \mathcal{G}_w(U)^*$, Δ_w is weakly holomorphic and hence holomorphic, cf. [1, p.66]. In order to show that $\Delta_w \in \mathcal{H}_w(U; \mathcal{G}_w(U))$, fix $x_0 \in U$. Then for $f \in \mathcal{H}_w(U)$, $|\delta_{x_0}(f)| = |f(x_0)| \leq \frac{1}{w(x_0)} \|f\|_w$ implies $\|\delta_{x_0}\| \leq \frac{1}{w(x_0)}$. Hence $\|\Delta_w\|_w = \sup\limits_{x \in U} w(x)\|\delta_x\| \leq 1$. Consequently, $\Delta_w \in \mathcal{H}_w(U; \mathcal{G}_w(U))$.

Corresponding to f in $\mathcal{H}_w(U; F)$, we now define T_f. For the case $F = \mathbb{C}$, define $T_f = J_U^w(f)$. Then $T_f \circ \Delta_w(f) = f$ and $\|T_f\| = \|f\|_w$.

In case of an arbitrary Banach space F, we first define $T_f : \mathcal{G}_w(U) \to F^{**}$ as

$$T_f(h)(\phi) = h(\phi \circ f), \quad h \in \mathcal{G}_w(U),\ \phi \in F^*.$$

Note that T_f is, indeed, F-valued; for $T_f(\delta_x) = f(x) \in F$ and $\overline{span}\{\delta_x : x \in U\} = \mathcal{G}_w(U)$. Further,

$$\|f\|_w = \sup_{x \in U} w(x)\|f(x)\| = \sup_{x \in U} w(x)\|T_f(\delta_x)\| \leq \|T_f\|$$

and

$$\|T_f(h)(\phi)\| \leq \|h\|\|\phi\|\|f\|_w, \quad h \in \mathcal{G}_w(U),\ \phi \in F^*.$$

Thus $\|T_f\| = \|f\|_w$ and Ψ is an isometric isomorphism. ∎

Remark 3.2. If $(w\Delta_w)(x) = w(x)\Delta_w(x),\ x \in U$, then

$$J_U^w(B_w) = \big\{(w\Delta_w)(x) : x \in U\big\}^\circ.$$

Consequently, $(J_U^w(B_w))^\circ = B_{\mathcal{G}_w(U)} = \overline{\Gamma}\{(w\Delta_w)(x) : x \in U\}$, where $\overline{\Gamma}(A)$ denotes the absolutely convex closed hull of A.

In case the weight w is given by an entire function γ with positive coefficients, i.e., $w(x) = \frac{1}{\gamma(\|x\|)}$, $x \in E$, we write $\mathcal{H}_\gamma$ for $\mathcal{H}_w$; and the above linearization theorem takes the following form:

THEOREM 3.3. *Let γ be an entire function with positive coefficients. Then for an open subset U of a Banach space E and weight w, $w(x) = \frac{1}{\gamma(\|x\|)}$, $x \in U$, there exists a Banach space $G_\gamma(U)$ and a mapping $\Delta_\gamma \in \mathcal{H}_\gamma(U; G_\gamma(U))$, $\|\Delta_\gamma\| = 1$ with the following property: for each Banach space F and each* *mapping $f \in \mathcal{H}_\gamma(U;F)$, there is a unique operator $T_f \in \mathcal{L}(G_\gamma(U);F)$ such that $T_f \circ \Delta_\gamma = f$. The correspondence Ψ between $\mathcal{H}_\gamma(U;F)$ and $\mathcal{L}(G_\gamma(U);F)$ given by*

$$\Psi(f) = T_f$$

is an isometric isomorphism. The space $G_\gamma(U)$ is uniquely determined up to an isometric isomorphism by these properties.

Proof. It suffices to prove here that $\|\Delta_\gamma\| = 1$. Let $\gamma(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n > 0$ for each $n \in \mathbb{N}_0$. Fix $x_0 \in E$. Choose $\phi \in E^*$ with $\|\phi\| = 1$ and $|\phi(x_0)| = \|x_0\|$. Define $f : E \to \mathbb{C}$ as

$$f(x) = \sum_{n=1}^{\infty} a_n \phi^n(x), \quad x \in E.$$

Clearly, $f \in \mathcal{H}_\gamma(E)$ and $\|f\|_\gamma \leq 1$. Since $|f(x_0)| = \gamma(\|x_0\|)$, we have

$$\|\delta_{x_0}\| = \sup_{\|h\|_\gamma \leq 1} |h(x_0)| = \gamma(\|x_0\|).$$

Thus $\|\Delta_\gamma\| = 1$. ∎

Before we consider the applications of the above linearization theorem, let us prove results related to the inclusion of polynomials in the weighted space of holomorphic mappings.

Proposition 3.4. *Let w be a weight defined on an open subset U of a Banach space E. Then, for each $m \in \mathbb{N}$, the following are equivalent:*

(a) *$\mathcal{P}(^mE;F) \subset \mathcal{H}_w(U;F)$ for each Banach space F.*

(b) *$\mathcal{P}(^mE) \subset \mathcal{H}_w(U)$.*

Proof. (a)⇒(b). Immediate.
(b)⇒(a). Consider $Q \in \mathcal{P}(^mE;F)$. For $x \in U$, choose $\phi_x \in F^*$ such that $\|\phi_x\| = 1$ and $\phi_x(Q(x)) = \|Q(x)\|$. Write $A = \{\phi_x \circ Q : x \in U\}$. Then A is a $\|\cdot\|$-bounded subset of $\mathcal{P}(^mE)$ since $\|\phi_x \circ Q\| \leq \|Q\|$. Hence by Proposition 2.2, A is $\|\cdot\|_w$-bounded. Consequently,

$$\|Q\|_w = \sup_{x\in U} w(x)|\phi_x(Q(x))| \leq \sup_{x\in U} \sup_{y\in U} w(y)|\phi_x(Q(y))| < \infty.$$

Thus $Q \in \mathcal{H}_w(U;F)$ and (a) follows. ∎

PROPOSITION 3.5. *Let w be a weight on an open subset U of a Banach space E. Then*

(a) *If U is bounded, $\mathcal{P}(E) \subset \mathcal{H}_w(U)$ if and only if w is bounded.*

(b) *For $U = E$, $\mathcal{P}(E) \subset \mathcal{H}_w(E)$ if and only if w is rapidly decreasing.*

Proof. (a) Since constant functions are in $\mathcal{P}(E)$, the proof follows.
(b) This is a particular case of a result proved in [12, p. 6], by taking the family V consisting of a single weight. ∎

In the remaining part of this section, we consider weights w defined on an open subset U of E so that the space $\mathcal{P}(E,F)$ is contained in $\mathcal{H}_w(U,F)$, for which it suffices to consider the scalar case in view of Proposition 3.4.

PROPOSITION 3.6. *Let w be a weight defined on an open subset U of a Banach space E such that $\mathcal{P}(E) \subset \mathcal{H}_w(U)$. Then E is topologically isomorphic to a complemented subspace of $\mathcal{G}_w(U)$.*

Proof. Since the inclusion map I from U to E is a member of $\mathcal{H}_w(U;E)$, by Theorem 3.1, there exists $T \in \mathcal{L}(\mathcal{G}_w(U);E)$ and $\Delta_w \in \mathcal{H}_w(U;\mathcal{G}_w(U))$ such that

$$T \circ \Delta_w(x) = I_w(x) = x, \quad x \in U.$$

Fix $a \in U$ and write $S = d^1\Delta_w(a)$. Note that $S \in \mathcal{L}(E;\mathcal{G}_w(U))$. Further, by Cauchy's integral formula,

$$S(t) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{\Delta_w(a+\zeta t)}{\zeta^2}d\zeta, \quad t \in E,$$

where $r > 0$ is chosen so that $\{a + \zeta t : |\zeta| \le r\} \subset U$. Now

$$T \circ S(t) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{(a+\zeta t)}{\zeta^2}d\zeta = t, \quad t \in E.$$

This gives $\|S(t)\| \ge \frac{1}{\|T\|}\|t\|$ and so, S is injective and S^{-1} is continuous.

Define $P = S \circ T$. Then P is a projection map from $\mathcal{G}_w(U)$ into itself. Also $S(E) = P(\mathcal{G}_w(U))$. Hence S is a topological isomorphism between E and a complemented subspace of $\mathcal{G}_w(U)$. ∎

For the weight w as considered in Theorem 3.3, we have

Proposition 3.7. *Let γ be an entire function with positive coefficients and t_0 be a positive real satisfying the equation $\gamma(t) = t\gamma^{'}(t)$. Assume that U is an open subset of a Banach space E for which $\{x \in E : \|x\| \leq t_0\} \subset U$. Then there exists a topological isomorphism S between E and a complemented subspace of $G_\gamma(U)$ with $\|S\| = \frac{\gamma(t_0)}{t_0}$.*

Proof. Since the weight given by γ is bounded, $I \in H_\gamma(U; E)$. By Theorem 3.3, there exists $T \in \mathcal{L}(G_\gamma(U); E)$ and $\Delta_\gamma \in \mathcal{H}_\gamma(U; G_\gamma(U))$ such that $T \circ \Delta_\gamma = I$ and $\|T\| = \|I\|_\gamma$. But

$$\|T\| = \|I\|_\gamma = \sup_{x \in U} \frac{\|x\|}{\gamma(\|x\|)} = \frac{t_0}{\gamma(t_0)}. \tag{3.1}$$

Writing S for $d^1\Delta_\gamma(0)$, by Cauchy's inequality, we get

$$\|S\| = \|d^1\Delta_\gamma(0)\| \leq \frac{1}{t_0} \sup_{\|x\|=t_0} \|\Delta_\gamma(x)\| = \frac{1}{t_0} \sup_{\|x\|=t_0} \|\delta_x\| = \frac{\gamma(t_0)}{t_0}. \tag{3.2}$$

Now proceeding as in the proof of Proposition 3.4, we have

$$T \circ S(t) = t, \quad \forall t \in E.$$

Consequently, by (3.1) and (3.2), we get

$$\|t\| = \|T \circ S(t)\| \leq \frac{t_0}{\gamma(t_0)} \, \|S(t)\| \leq \|t\|, \quad t \in E.$$

Hence,

$$\|S\| = \frac{\gamma(t_0)}{t_0}.$$

∎

Illustrating the above result, we have

Example 3.8. Let $\gamma(z) = e^{\tau z}$, $\tau > 0$. One can easily find that $t_0 = \frac{1}{\tau}$. In this case $\|I\|_\gamma = \frac{1}{\tau e}$ and $\|S\| = \tau e$. If $\tau = \frac{1}{e}$, S becomes an isometric isomorphism.

For our next result, we make use of the following linearization theorem quoted from [18] and proved by using tensor product techniques for locally convex spaces in [26].

THEOREM 3.9. *Let E be a Banach space and $m \in \mathbb{N}$. Then there exists a Banach space $Q(^mE)$ and a polynomial $q_m \in \mathcal{P}(^mE; Q(^mE))$ such that for any Banach space F and each polynomial $P \in \mathcal{P}(^mE; F)$, there is a unique operator $T_P \in \mathcal{L}(Q(^mE); F)$ satisfying $T_P \circ q_m = P$. The correspondence $\Phi : \mathcal{P}(^mE; F) \to \mathcal{L}(Q(^mE); F)$, $\Phi(P) = T_P$ is an isometric isomorphism and the space $Q(^mE)$ is uniquely determined up to an isometric isomorphism.*

In the statement of the above result, the space $Q(^mE)$ is defined as the predual of $\mathcal{P}(^mE)$, i.e., $\{h \in \mathcal{P}(^mE)' : h|B_m \text{ is } \tau_0\text{-continuous}\}$, where B_m is the closed unit ball of $\mathcal{P}(^mE)$. The map $q_m : E \to Q(^mE)$ is given by $q_m(x) = \delta_x$, where $\delta_x(P) = P(x)$, $P \in \mathcal{P}(^mE)$ or equivalently $q_m(x) = x \otimes \cdots \otimes x$, cf. [10, p. 29]. For w and U as in Proposition 3.6, we prove

PROPOSITION 3.10. *The space $Q(^mE)$ is topologically isomorphic to a complemented subspace of $\mathcal{G}_w(U)$.*

Proof. Consider $q_m \in \mathcal{P}(^mE; Q(^mE))$. By Theorem 3.1, there exist $T_m \in \mathcal{L}(\mathcal{G}_w(U); Q(^mE))$ and $\Delta_w \in \mathcal{H}_w(U; G_w(U))$ such that $T_m \circ \Delta_w = q_m$. Let S_m be the m-th Taylor series coefficient of Δ_w around 'a', .i.e., $S_m = \frac{1}{m!}\widehat{d}^m \Delta_w(a)$. As $S_m \in \mathcal{P}(^mE; \mathcal{G}_w(U))$, by Theorem 3.9 there exists $R_m \in \mathcal{L}(Q(^mE); \mathcal{G}_w(U))$ such that $R_m \circ q_m = S_m$. Now,

$$T_m \circ R_m \circ q_m = T_m \circ S_m = \frac{1}{m!}\widehat{d}^m (T_m \circ \Delta_w)(a) = \frac{1}{m!}\widehat{d}^m q_m(a).$$

As $\overline{span}\{q_m(x) : x \in E\} = Q(^mE)$, it follows that $T_m \circ R_m(u) = u$, $u \in Q(^mE)$. Let $P_m = R_m \circ T_m$. Then P_m is a projection map from $\mathcal{G}_w(U)$ into itself and R_m is the topological isomorphism between $Q(^mE)$ and a complemented subspace of $\mathcal{G}_w(U)$. ∎

PROPOSITION 3.11. *For $m \in \mathbb{N}$, there exists a topological isomorphism R_m between the space $Q(^mE)$ and a complemented subspace of $G_\gamma(U)$, for any open subset U of E containing the set $\{x \in E : \|x\| \le r_0\}$, r_0 being a positive real number satisfying the equation $r\gamma'(r) - m\gamma(r) = 0$ and $r_0 > m$. Further $\|R_m\| = \frac{\gamma(r_0)}{r_0^m}$.*

Proof. As $q_m \in H_\gamma(U; Q(^mE))$, by Theorem 3.3, there exist $T_m \in \mathcal{L}(G_\gamma(U); Q(^mE))$ and $\Delta_\gamma \in \mathcal{H}_\gamma(U; G_\gamma(U))$ such that $T_m \circ \Delta_\gamma = q_m$. Since $\sup_{x\in U} \frac{\|x\|^m}{\gamma(\|x\|)} = \frac{r_0^m}{\gamma(r_0)}$, we have

$$\|q_m\|_\gamma = \|T_m\| = \frac{r_0^m}{\gamma(r_0)}. \tag{3.3}$$

Now by Cauchy's inequality, we get

$$\Big\|\frac{1}{m!}\widehat{d}^m\Delta_\gamma(0)\Big\| \le \frac{1}{r_0^m}\sup_{\|x\|=r_0}\|\Delta_\gamma(x)\| = \frac{\gamma(r_0)}{r_0^m}.$$

Continuing as in the proof of the above result, we have

$$T_m \circ R_m(u) = u, \quad u \in Q(^mE). \tag{3.4}$$

By using (3.3) and (3.4), we get

$$\|u\| = \|T_m \circ R_m(u)\| \le \frac{r_0^m}{\gamma(r_0)}\|R_m(u)\| \le \|u\|$$

for every $u \in Q(^mE)$. Thus $\|R_m\| = \frac{\gamma(r_0)}{r_0^m}$. ∎

Considering the function given in Example 3.8, we have the following, illustrating the above result

EXAMPLE 3.12. If $\gamma(z) = e^{\tau z}$, $\tau > 0$, we find $r_0 = \frac{m}{\tau}$ and, so $\|R_m\| = \frac{\tau^m e^m}{m^m}$.

Also, by using the same argument as in Proposition 3.11, one can easily check

EXAMPLE 3.13. For $n \in \mathbb{N}$, define $w : U_E \to (0,\infty)$ by $w(x) = (1 - \|x\|)^n$, $x \in U_E$. Then

$$\|R_m\| = (\frac{n}{m+n})^n$$

for any $m \in \mathbb{N}$.

4. THE TOPOLOGY $\tau_{\mathcal{M}}$

In this section we introduce a locally convex topology $\tau_{\mathcal{M}}$ on $\mathcal{H}_w(U;F)$ of which the particular cases have been considered in [18] and [25]. For a finite set A and $r > 0$, let us define

$$N(A,r) = \{f \in \mathcal{H}_w(U;F) : \inf_{x\in A} w(x)\sup_{y\in A}\|f(y)\| \le r\}.$$

Consider the class

$$\mathcal{U} = \left\{ \bigcap_{j=1}^{\infty} N(A_j,r_j) : (A_j) \text{ varies over all sequences of finite subsets of } U \text{ and } (r_j) \text{ varies over all positive sequences diverging to infinity} \right\}$$

It can be easily checked that each member of $\mathcal{U}$ is balanced, convex and absorbing. Thus it forms a fundamental neighborhood system at 0 for a locally convex topology, which we denote by $\tau_{\mathcal{M}}$. Equivalently, this topology is generated by the family

$$\Big\{p_{\overline{\alpha},\overline{A}} : \overline{\alpha} = (\alpha_j) \in c_0^+,\ \overline{A} = (A_j),\ A_j \text{ being finite subset of } U \text{ for each } j\Big\}$$

of seminorms given by

$$p_{\overline{\alpha},\overline{A}}(f) = \sup_{j\in\mathbb{N}} \Big(\alpha_j \inf_{x\in A_j} w(x) \sup_{y\in A_j} \|f(y)\|\Big).$$

These are the Minkowski functionals of members in $\mathcal{U}$. For $F = \mathbb{C}$, $\tau_{\mathcal{M}} = \tau_{bc}$, cf. [25, p. 350].

For our results in the sequel, we make use of the following

Lemma 4.1. *Let M be a compact subset of $\mathcal{G}_w(U)$. Then there exist sequences $\overline{\alpha} = (\alpha_j) \in c_0^+$ and $\overline{A} = (A_j)$ of finite subsets of U such that*

$$M \subset \overline{\Gamma}\Big(\bigcup_{j\geq 1} \Big\{\alpha_j \inf_{x\in A_j} w(x)\Delta_w(y) : y \in A_j\Big\}\Big).$$

Proof. Since M° is a τ_c-neighborhood of 0 in $\mathcal{G}_w(U)^*$, it is τ_{bc}-neighborhood of 0 by Proposition 2.3(iii). Consequently, there exist sequences $(\alpha_j) \in c_0^+$ and $\overline{A} = (A_j)$ of finite subsets of U such that $\{f \in \mathcal{H}_w(U) : p_{\overline{\alpha},\overline{A}}(f) \leq 1\} \subset M^\circ$, where $M^\circ = \{f \in \mathcal{H}_w(U) : \sup_{u\in M} |< f, u >| \leq 1\}$. Writing $B = \bigcup_{j\geq 1}\{\alpha_j \inf_{x\in A_j} w(x)\Delta_w(y) : y \in A_j\}$, we get $B^\circ \subset M^\circ$. Therefore, by the bipolar theorem, we have

$$M \subset \overline{\Gamma}\Big(\bigcup_{j\geq 1} \Big\{\alpha_j \inf_{x\in A_j} w(x)\Delta_w(y) : y \in A_j\Big\}\Big).$$

∎

Relating $\tau_{\mathcal{M}}$ with τ_0 and $\tau_{\|.\|_w}$, and bounded sets with respect to these topologies, we prove

Proposition 4.2. *For a weight w on an open subset U of a Banach space E, the following hold:*

(i) $\tau_0 \leq \tau_{\mathcal{M}} \leq \tau_{\|.\|_w}$ *on* $\mathcal{H}_w(U; F)$.

(ii) $\tau_{\mathcal{M}}$ *and* $\|\cdot\|_w$*-bounded sets are the same.*

(iii) $\tau_{\mathcal{M}}|\mathcal{B} = \tau_0|\mathcal{B}$ *for any* $\|\cdot\|_w$*-bounded set* $\mathcal{B}$.

Proof. (i) Let K be a compact subset of U. Then by Lemma 4.1, there exist sequences $(\alpha_j) \in c_0^+$ and $\overline{A} = (A_j)$ of finite subsets of U such that

$$\Delta_w(K) \subset \overline{\Gamma}\Big(\bigcup_{j \geq 1} \Big\{ \alpha_j \inf_{x \in A_j} w(x) \Delta_w(y) : y \in A_j \Big\} \Big).$$

Hence, for $f \in \mathcal{H}_w(U;F)$, we have

$$\sup_{x \in K} \|f(x)\| = \sup_{x \in K} \|T_f \circ \Delta_w(x)\| \leq p_{\overline{\alpha},\overline{A}}(f).$$

Thus $\tau_{\mathcal{M}} \geq \tau_0$ on $\mathcal{H}_w(U;F)$. The inequality $\tau_{\mathcal{M}} \leq \tau_{\|\cdot\|_w}$ clearly holds.

(ii) As every $\|\cdot\|_w$-bounded set is $\tau_{\mathcal{M}}$-bounded, it suffices to prove the other implication. Assume that there exists a $\tau_{\mathcal{M}}$-bounded set A which is not $\|\cdot\|_w$-bounded. Then for each $k \in \mathbb{N}$, there exist $f_k \in A$ such that

$$\|f_k\|_w > k^2.$$

Therefore, $w(x_k)\|f_k(x_k)\| > k^2$ for some sequence $\{x_k\} \subset U$. Consider the $\tau_{\mathcal{M}}$-continuous semi-norm p on $\mathcal{H}_w(U;F)$ defined by the sequences $\{\frac{1}{j}\}$ and $\{x_j\}$ obtained as above, namely

$$p(f) = \sup_{j \in \mathbb{N}} \frac{1}{j}\, w(x_j)\|f(x_j)\|.$$

Then $p(\frac{f_k}{k}) > 1$, for each k. This contradicts the $\tau_{\mathcal{M}}$-boundedness of A as $\frac{1}{k} \to 0$ and $\{f_k\} \subset A$, cf. [14, p. 161] .

(iii) Let $\mathcal{B}$ be a bounded set in $(\mathcal{H}_w(U;F), \|\cdot\|_w)$. Then there exists a constant $M > 0$ such that $\|f\|_w \leq M$, for every $f \in \mathcal{B}$. In order to show that $\tau_{\mathcal{M}}|\mathcal{B} \leq \tau_0|\mathcal{B}$, consider a $\tau_{\mathcal{M}}$-continuous semi-norm p given by

$$p(f) = \sup_{j \in \mathbb{N}} \Big(\alpha_j \inf_{x \in A_j} w(x) \sup_{y \in A_j} \|f(y)\| \Big), \quad f \in \mathcal{H}_w(U;F),$$

where $(\alpha_j) \in c_0^+$ and (A_j) is a sequence of finite subsets of U. Fix $\epsilon > 0$ arbitrarily. Then there exists $k_0 \in \mathbb{N}$ such that

$$\alpha_j < \frac{\epsilon}{2M}, \quad \forall j > k_0.$$

Write $K = \bigcup_{j \leq k_0} A_j$. Then K is a compact subset of U. For $f, g \in \mathcal{B}$,

$$p(f-g) < \epsilon \;\text{ whenever }\; p_K(f-g) < \delta,$$

where

$$\delta = \frac{\epsilon}{\|\overline{\alpha}\|_\infty \sup_{1\le j\le k_0} \big(\inf_{x\in A_j} w(x)\big)};$$

indeed

$$\sup_{j\le k_0}\Big(\alpha_j \inf_{x\in A_j} w(x) \sup_{y\in A_j} \|(f-g)(y)\|\Big) \le \|\overline{\alpha}\|_\infty \sup_{1\le j\le k_0}\Big(\inf_{x\in A_j} w(x)\Big) p_K(f-g).$$

This completes the proof as the other implication is obviously true. ∎

Proceeding on the lines similar to [25, Remark 3.32], it can be proved that the topology $\tau_{\mathcal{M}}$ may be strictly finer than τ_0 on $\mathcal{H}_w(U;F)$. However, for the sake of convenience of the reader, we give

Example 4.3. Let E be a Banach space and w be a bounded weight on U_E. Assume that $\tau_{\mathcal{M}} = \tau_0$ on $\mathcal{H}_w(U_E;F)$. Choose a sequence $\{x_n\}$ in U_E such that $\|x_n\| \to 1$ and $\{u_n\}$ in F with $\|u_n\| = n$, $n \in \mathbb{N}$. Then by Theorem 2.1, there exists a function $f \in \mathcal{H}_b(U;F)$ such that

$$f(x_n) = \frac{u_n}{w(x_n)}, \quad n \in \mathbb{N}.$$

Since $\|f\|_w = \sup_{x\in U} w(x)\|f(x)\| > n$ for all $n \in \mathbb{N}$, $f \notin \mathcal{H}_w(U_E;F)$. Consequently, the set

$$A = \Big\{ \sum_{m=0}^{N} \frac{1}{m!}\, \hat{d}^m f(0) : N = 0,1,2,\dots \Big\}$$

is not $\|\cdot\|_w$ bounded. But the convergence of the series $\sum_{m=0}^{\infty} \frac{1}{m!}\, \hat{d}^m f(0)$ to f in τ_0 topology yields that the set A is τ_0-bounded. As $\tau_{\mathcal{M}}$ and $\|\cdot\|_w$-bounded sets are the same by Proposition 4.2(ii), it follows that $\tau_{\mathcal{M}} \ne \tau_0$, i.e., $\tau_0 < \tau_{\mathcal{M}}$.

One can easily establish the following observation which we write as

Proposition 4.4. *Let (A_j) be a sequence of finite sets in E and $A = \bigcup_{j\in\mathbb{N}} A_j$. Then A is bounded if and only if the set $K = (\bigcup_{j\in\mathbb{N}} \alpha_j A_j) \bigcup \{0\}$ is compact for each $\overline{\alpha} = (\alpha_j) \in c_0$.*

Proof. Immediate. ∎

PROPOSITION 4.5. *Let E and F be Banach spaces. For a weight w on an open subset U of E with $\mathcal{P}(E) \subset \mathcal{H}_w(U)$, $\tau_{\mathcal{M}}$ coincides with τ_0 on $\mathcal{P}(^mE;F)$ for each $m \in \mathbb{N}$.*

Proof. Let p be a $\tau_{\mathcal{M}}$-continuous semi-norm on $\mathcal{H}_w(U;F)$. Then there exist sequences $\overline{\alpha} = (\alpha_j) \in c_0^+$ and $\overline{A} = (A_j)$ of finite subsets of U such that

$$p(f) = \sup_{j \in \mathbb{N}} \Big(\alpha_j \inf_{x \in A_j} w(x) \sup_{y \in A_j} \|f(y)\| \Big), \quad f \in \mathcal{H}_w(U;F).$$

Define $K = \bigcup_{j \in \mathbb{N}} \big\{ (\alpha_j \inf_{x \in A_j} w(x))^{\frac{1}{m}} y : y \in A_j \big\} \cup \{0\}$. For each $y \in U$, choose $\phi_y \in E^*$ with $\|\phi_y\| = 1$ and $\phi_y(y) = \|y\|$. Then the set $B = \{\phi_y^m : y \in U\}$ is a norm bounded subset of $\mathcal{P}(^mE)$ and hence $\|\cdot\|_w$-bounded by Proposition 2.2. Therefore

$$\sup_{j \in \mathbb{N}} \sup_{y \in A_j} w(y) \|y\|^m \leq \sup_{y \in U} \sup_{x \in U} w(x) \|\phi_y^m(x)\| < \infty.$$

Then by Proposition 4.4, K is a compact subset of E. Since

$$p(P) = \sup_{j \in \mathbb{N}} \sup_{y \in A_j} \Big\| P\Big(\big(\alpha_j \inf_{x \in A_j} w(x)\big)^{\frac{1}{m}} y \Big) \Big\| = p_K(P).$$

for any $P \in \mathcal{P}(^mE;F)$, the proof follows. ∎

Next, we prove

PROPOSITION 4.6. *Let E and F be Banach spaces. For a radial weight w on a balanced open subset U of E with $\mathcal{P}(E) \subset \mathcal{H}_w(U)$, the space $\mathcal{P}(E;F)$ is $\tau_{\mathcal{M}}$-dense in $\mathcal{H}_w(U;F)$.*

Proof. Recalling the notations $S_n(f)$ and $C_n(f)$, and their integral representations for $f \in \mathcal{H}_w(U;F)$ from Section 2, we have

$$\|C_n(f)(x)\| = \Big\| \frac{1}{\pi} \int_{-\pi}^{\pi} f(e^{it}x) K_n(t) dt \Big\| \leq \sup_{t \in [-\pi,\pi]} \|f(e^{it}x)\|$$

since $\int_{-\pi}^{\pi} K_n(t)dt = 1$, cf. [28, p. 45]. Consequently, for each $n \in \mathbb{N}_0$,

$$\|C_n(f)(x)\|_w \leq \sup_{x \in U} w(x) \sup_{|t|=1} \|f(tx)\| = \sup_{x \in U} \sup_{|t|=1} w(tx) \|f(tx)\| \leq \|f\|_w < \infty.$$

Thus, for given $f \in \mathcal{H}_w(U;F)$, the set $\{C_n(f) : n \in \mathbb{N}_0\}$ is $\|\cdot\|_w$-bounded in $\mathcal{H}_w(U;F)$. As $C_n f \to f$ in $(H(U;F), \tau_0)$, the result follows by Proposition 4.2(iii). ∎

Finally in this section, we consider an analogue of Theorem 3.1 on $\mathcal{H}_w(U;F)$ when it is equipped with the topology $\tau_{\mathcal{M}}$. This result will be useful for our study of approximation properties in the next section. Indeed, we prove

THEOREM 4.7. *Let E and F be Banach spaces, and w be a weight on an open subset U of E. Then the mapping*

$$\Psi : \big(\mathcal{H}_w(U;F), \tau_{\mathcal{M}}\big) \to \big(\mathcal{L}(\mathcal{G}_w(U);F), \tau_c\big)$$

is a topological isomorphism.

Proof. Let M be a compact subset of $\mathcal{G}_w(U)$. Then by Lemma 4.1, there exist sequences $(\alpha_j) \in c_0^+$ and $\overline{A} = (A_j)$ of finite subsets of U such that

$$M \subset \overline{\Gamma}\Big(\bigcup_{j\geq 1}\Big\{\alpha_j \inf_{x\in A_j} w(x)\Delta_w(y) : y \in A_j\Big\}\Big).$$

Hence for $f \in \mathcal{H}_w(U;F)$,

$$p_M(\Psi(f)) = \sup_{u\in M}\|T_f(u)\| \leq \sup_{j\in\mathbb{N}}\Big(\alpha_j \inf_{x\in A_j} w(x) \sup_{y\in A_j}\|f(y)\|\Big) = p_{\overline{\alpha},\overline{A}}(f).$$

Thus Ψ is $\tau_{\mathcal{M}} - \tau_c$ continuous.

In order to show the continuity of the inverse map Ψ^{-1}, let us note that

$$\sup_{j\in\mathbb{N}}\sup_{y\in A_j}\Big(\inf_{x\in A_j} w(x)\|\Delta_w(y)\|\Big) \leq 1.$$

Hence by Proposition 4.4, the set

$$K = \overline{\Gamma}\Big(\bigcup_{j\geq 1}\Big\{\alpha_j \inf_{x\in A_j} w(x)\Delta_w(y) : y \in A_j\Big\}\Big) \cup \{0\}$$

is a compact subset of $\mathcal{G}_w(U)$, which immediately yields the $\tau_c - \tau_{\mathcal{M}}$ continuity of the inverse mapping Ψ^{-1}. ∎

5. THE APPROXIMATION PROPERTIES

This section is devoted to the study of the approximation property for the space E, the weighted space $\mathcal{H}_w(U)$ of holomorphic mappings and its predual $\mathcal{G}_w(U)$. We write

$$\mathcal{H}_w(U) \otimes F = \{f \in \mathcal{H}_w(U;F) : f \text{ has finite dimensional range}\}$$

and

$$\mathcal{H}_w^c(U;F) = \{f \in \mathcal{H}_w(U;F) : wf \text{ has a relatively compact range}\}.$$

In the next proposition we establish the interplay between the properties of a mapping $f \in \mathcal{H}_w(U;F)$ and the corresponding operator $T_f \in \mathcal{L}(\mathcal{G}_w(U);F)$.

Proposition 5.1. *Let U be an open subset of a Banach space E and w be a weight on U. Then for any Banach space F,*

(a) *$f \in \mathcal{H}_w(U) \otimes F$ if and only if $T_f \in \mathcal{F}(\mathcal{G}_w(U);F)$,*

(b) *$f \in \mathcal{H}_w^c(U;F)$ if and only if $T_f \in \mathcal{K}(\mathcal{G}_w(U);F)$.*

Proof. (a) Note that for $(g_i)_{i=1}^n \subset \mathcal{H}_w(U)$ and $(y_i)_{i=1}^n \subset F$,

$$f(x) = \sum_{i=1}^n g_i(x) y_i \quad \Leftrightarrow \quad T_f(\delta_x) = \sum_{i=1}^n < \delta_x, g_i > y_i$$

for each $x \in U$. As $\mathcal{G}_w(U)^* = \mathcal{H}_w(U)$ and $\overline{span}\{\delta_x : x \in U\} = \mathcal{G}_w(U)$, the result follows.
(b) By Remark 3.2, $B_{\mathcal{G}_w(U)} = \overline{\Gamma}(w\Delta_w)(U)$, the result follows from

$$(wf)(U) = T_f\big((w\Delta_w)(U)\big) \subset T_f\big(\overline{\Gamma}(w\Delta_w)(U)\big) = \overline{\Gamma}\big((wf)(U)\big).$$

∎

Proposition 5.2. *Let w be a weight on an open subset U of a Banach space E. Then $\overline{\mathcal{F}(\mathcal{G}_w(U);F)}^{\|\cdot\|} = \mathcal{K}(\mathcal{G}_w(U);F)$ if and only if $\overline{\mathcal{H}_w(U) \otimes F}^{\|\cdot\|_w} = \mathcal{H}_w^c(U;F)$ for each Banach space F.*

Proof. Assume that $\overline{\mathcal{F}(\mathcal{G}_w(U);F)}^{\|\cdot\|} = \mathcal{K}(\mathcal{G}_w(U);F)$. Consider $f \in \mathcal{H}_w^c(U;F)$. Then $T_f \in \mathcal{K}(\mathcal{G}_w(U);F)$ by Proposition 5.1(b). Hence there exists a net $(T_\alpha) \subset \mathcal{F}(\mathcal{G}_w(U);F)$ such that $T_\alpha \xrightarrow{\|\cdot\|} T_f$. Now, corresponding to each α, we have $f_\alpha \in \mathcal{H}_w(U) \otimes F$ such that $T_{f_\alpha} = T_\alpha$ by Proposition 5.1(a). Apply Theorem 3.1 to get $f_\alpha \xrightarrow{\|\cdot\|_w} f$, thereby proving $\overline{\mathcal{H}_w(U) \otimes F}^{\|\cdot\|_w} = \mathcal{H}_w(U;F)$. Conversely, for $T \in \mathcal{K}(\mathcal{G}_w(U);F)$, there exists $f \in \mathcal{H}_w^c(U;F)$ such that $T = T_f$ by Proposition 5.1(b). Then there exists a net $\{f_\alpha\} \subset \mathcal{H}_w(U) \otimes F$ such that $f_\alpha \xrightarrow{\|\cdot\|_w} f$. Thus $(T_{f_\alpha}) \subset \mathcal{F}(\mathcal{G}_w(U);F)$ by Proposition 5.1(a) and $T_\alpha \xrightarrow{\|\cdot\|} T_f = T$ by Proposition 3.1. ∎

PROPOSITION 5.3. *Let w be a weight on an open subset U of a Banach space E. Then* $\overline{\mathcal{F}(\mathcal{G}_w(U);F)}^{\tau_c} = \mathcal{L}(\mathcal{G}_w(U);F)$ *if and only if* $\overline{\mathcal{H}_w(U)\otimes F}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U;F)$ *for each Banach space F.*

Proof. The proof follows analogously by using Theorem 4.7 and Proposition 5.1(b). ∎

Characterizing the approximation property for the space E, we have

THEOREM 5.4. *Let E be a Banach space. Then for each Banach space F, the following are equivalent:*

(i) *E has the approximation property.*

(ii) $\overline{\mathcal{H}_w(V)\otimes E}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(V;E)$, *for each open subset V of F and weight w on V.*

(iii) $\overline{\mathcal{H}_w(V)\otimes E}^{\|\cdot\|_w} = \mathcal{H}^c_w(V;E)$, *for each open subset V of F and weight w on V.*

Proof. (i) $\Rightarrow$ (ii): Assume that E has the approximation property. Then by Theorem 2.4, $\overline{\mathcal{F}(\mathcal{G}_w(U);E)}^{\tau_c} = L(\mathcal{G}_w(U);E)$. Thus $\overline{\mathcal{H}_w(V)\otimes E}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(V;E)$ by Proposition 5.3.

(ii) $\Rightarrow$(i): We claim that $\overline{\mathcal{F}(F;E)}^{\tau_c} = L(F;E)$ for each Banach space F. Let $A \in \mathcal{L}(F;E)$. Applying Proposition 3.4 , there exist operators $S \in \mathcal{L}(F;\mathcal{G}_w(U_F))$ and $T \in \mathcal{L}(\mathcal{G}_w(U_F);F)$ such that $T\circ S(y) = y,\ y \in F$. Since $\overline{\mathcal{G}_w(U_F)^*\otimes E}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U_F;E)$ by (ii), in view of Proposition 5.3 there exists a net $(A_\alpha) \subset \mathcal{F}(\mathcal{G}_w(U_F);E)$ such that $A_\alpha \xrightarrow{\tau_c} A\circ T$. Thus $A_\alpha\circ S \xrightarrow{\tau_c} A\circ T\circ S = A$. As $A_\alpha\circ S \subset \mathcal{F}(F;E)$, our claim holds and (i) follows by Theorem 2.4.

(i) $\Rightarrow$ (iii): Again using Theorem 2.4, $\overline{\mathcal{F}(\mathcal{G}_w(U);E)}^{\|\cdot\|} = \mathcal{K}(\mathcal{G}_w(U);E)$ by (i). Therefore $\overline{\mathcal{H}_w(U)\otimes F}^{\|\cdot\|_w} = \mathcal{H}^c_w(U;F)$ by Proposition 5.2.

(iii) $\Rightarrow$(i): Let $A \in \mathcal{K}(F;E)$ and T, S be the operators as above. Then $A\circ T \in \mathcal{K}(\mathcal{G}_w(U_F);E)$. By hypothesis and Proposition 5.2, there exists a sequence $(A_n) \subset \mathcal{F}(\mathcal{G}_w(U_F);E)$ such that $A_n \xrightarrow{\|\cdot\|} A\circ T$. Thus $A_n\circ S \xrightarrow{\|\cdot\|} A$ and we have, $\overline{\mathcal{F}(F;E)}^{\|\cdot\|} = \mathcal{K}(F;E)$. This proves (i). ∎

Next, we characterize the approximation property for the weighted space $\mathcal{H}_w(U)$.

Theorem 5.5. *For an open subset U of a Banach space E, $\mathcal{H}_w(U)$ has the approximation property if and only if $\mathcal{H}_w(U) \otimes F$ is $\|\cdot\|_w$-dense in $\mathcal{H}_w^c(U;F)$ for each Banach space F.*

Proof. By Proposition 2.5, $\mathcal{G}_w(U)^*$ has the approximation property if and only if $\mathcal{F}(\mathcal{G}_w(U);F)$ is $\|\cdot\|$-dense in $\mathcal{K}(\mathcal{G}_w(U);F)$ for each Banach space F. As $\mathcal{H}_w(U) = \mathcal{G}_w(U)^*$, the result follows by Proposition 5.2. ∎

We now cite the following known result, cf. [18]; along with the proof for convenience.

Proposition 5.6. *If a Banach space E has the approximation property, then for every Banach space F and $m \in \mathbb{N}$, $\overline{\mathcal{P}_f(^mE;F)}^{\tau_c} = \mathcal{P}(^mE;F)$.*

Proof. Let $P \in \mathcal{P}(^mE;F)$. Then for a compact subset K of E and $\epsilon > 0$, there exists a $\delta > 0$ such that $\|P(x) - P(y)\| < \epsilon$ whenever $x \in K$ and $y \in E$ with $\|y - x\| < \delta$. Since E has the approximation property, there is a $T \in \mathcal{F}(E;E)$ such that $\sup_{x\in K} \|T(x) - x\| < \delta$. Thus, $\sup_{x\in K} \|P \circ T(x) - P(x)\| < \epsilon$. ∎

Making use of the above proposition, we finally prove

Theorem 5.7. *Let E be a Banach space and w be a radial weight on a balanced open subset U of E such that $H_w(U)$ contains all the polynomials. Then the following assertions are equivalent:*

(i) *E has the approximation property.*

(ii) *$\overline{\mathcal{P}_f(E;F)}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U;F)$ for each Banach space F.*

(iii) *$\overline{\mathcal{H}_w(U) \otimes F}^{\tau_{\mathcal{M}}} = \mathcal{H}_w(U;F)$ for each Banach space F.*

(iv) *$\mathcal{G}_w(U)$ has the approximation property.*

Proof. (i) $\Rightarrow$ (ii): Let p be a $\tau_{\mathcal{M}}$ continuous semi-norm on $\mathcal{H}_w(U;F)$. Then for $f \in \mathcal{H}_w(U;F)$, there exists $P \in \mathcal{P}(E;F)$ such that $p(f-P) < \frac{\epsilon}{2}$ by Proposition 4.6. Let $P = P_0+P_1+\cdots+P_k$, $P_m \in \mathcal{P}(^mE;F)$, $0 \le m \le k$. Then by using Proposition 5.6 and Proposition 4.5, there exist Q_m in $\mathcal{P}_f(^mE;F)$, $0 \le m \le k$ such that

$$p(P_m - Q_m) < \frac{\epsilon}{2(k+1)}.$$

Write $Q = Q_0 + Q_1 + \cdots + Q_k$. Clearly $Q \in \mathcal{P}_f(E;F)$ and $p(f-Q) < \epsilon$.

(ii) $\Rightarrow$ (iii): It suffices to prove that $\mathcal{P}_f(E;F) \subset \mathcal{H}_w(U) \otimes F$. Consider $P \in \mathcal{P}_f(E;F)$. Then there exist $\phi_j \in E^*$ and $y_j \in F$, $1 \leq j \leq k$ such that

$$P = \sum_{j=1}^{k} \phi_j^m \otimes y_j \,.$$

Now, $\phi_j^m \in \mathcal{H}_w(U)$ for each $1 \leq j \leq k$ as w is bounded. Thus $P \in \mathcal{H}_w(U) \otimes F$.

(iii) $\Rightarrow$ (iv): Note that $\Delta_w \in \overline{\mathcal{H}_w(U) \otimes \mathcal{G}_w(U)}^{\tau_{\mathcal{M}}}$ by taking $F = \mathcal{G}_w(U)$ in (iii). Now $\overline{\mathcal{H}_w(U) \otimes \mathcal{G}_w(U)}^{\tau_{\mathcal{M}}}$ can be identified with $\overline{\mathcal{F}(\mathcal{G}_w(U); \mathcal{G}_w(U))}^{\tau_c}$ via the map Ψ by Proposition 5.1(a) and Theorem 4.7 . Since $T_{\Delta_w} \circ \Delta_w = \Delta_w$, we get $\Psi(\Delta_w) = I$, the identity map on $\mathcal{G}_w(U)$. Thus $I \in \overline{\mathcal{F}(\mathcal{G}_w(U); \mathcal{G}_w(U))}^{\tau_c}$.

(iv) $\Rightarrow$(i) follows from Proposition 2.6 and Proposition 3.6. ∎

Acknowledgements

The second author acknowledges the Council of Scientific and Industrial Research INDIA for a Research Fellowship.

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