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Resolutions of Cohomology Algebras and other Struggles with Integer Coefficients

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Abstract: There is a well known homotopy Π-algebra resolution of a space by wedges of spheres. An attempt to construct the Eckmann-Hilton dual gives a nice resolution for $\mathbb{F}_p$ coefficients which can then be used in a spectral sequence. For $\mathbb{Z}$ coefficients the dual construction has several compounding problems illustrating that integral cohomology becomes relatively problematic when we try to include primary operations.

Key words: primary cohomology operations, integer coefficients, free resolution, derived functors.

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1. Introduction

The cohomology groups of a topological space X are well defined over coefficients in any abelian group, however, there is a richer, natural structure that includes the primary cohomology operations. It is within this extra structure that cohomology over integer coefficients becomes problematic when compared to cohomology over coefficients in a finite field or the rationals. Over the integers we may think of the primary structure as the integral version of the Steenrod algebra, however, the viewpoint taken here is to consider these cohomology algebras as the Eckmann-Hilton dual of the well known Π-algebras of homotopy theory [20, 34, 2, 13].

To observe the Eckmann-Hilton duality we will consider the reduced spectral cohomology over the Eilenberg-Mac Lane spectrum of a ring. This ordinary, reduced theory is defined as the homotopy classes of maps into Eilenberg-Mac Lane spaces (E-M spaces), $\tilde{H}^n(X;R) = [X, K(R,n)]$.

The 'naturality' of the operations increases the precision of calculations by allowing a map between spaces to induce a morphism between cohomology algebras rather than simply cohomology groups (see [33] for a survey of applications of additional structure). Primary operations are operations that are globally natural and are induced by a universal arrow between products of E-M spaces, known as generalised E-M spaces (GEMs).

For coefficients in a field, the Künneth formula shows that cup product and composition are the only primary operations and the Cartan formula tells us how these distribute. For integer coefficients, there is no known analogue of the Cartan formula. Moreover, there are binary operations with no known formulation for the universal arrows representing them (see Example 3.5). Consequently, for integer coefficients we have only a partial formulation for the primary operation structure on cohomology groups.

With no explicit formulation for the primary structure, we might take inspiration from the Eckmann-Hilton dual in which the primary homotopy structure can be encoded in a category of operations Π. Functors from Π to pointed sets are called Π-algebras and the image of a Π-algebra are homotopy groups with the primary homotopy operations acting on them. We can encode the primary cohomology operations in the category $\mathcal{H}(\mathbb{Z})$, of products of integral Eilenberg-Mac Lane spaces and homotopy classes of maps. We will call the Eckmann-Hilton dual of a Π-algebra an $\mathcal{H}(\mathbb{Z})$-algebra, which will be a functor from $\mathcal{H}(\mathbb{Z})$ to the category of pointed sets. Generalizing we can define an $\mathcal{H}(R)$-algebra to be a functor from the category $\mathcal{H}(R)$, of products of Eilenberg-Mac Lane spaces over a ring R and homotopy classes of maps, to the category of pointed sets. For $R = \mathbb{F}_p$ it is well known that an $\mathcal{H}(\mathbb{F}_p)$-algebra is an unstable algebra over the mod p Steenrod algebra [8, 4].

Taking further inspiration from the theory of Π-algebras, we might attempt to study the relation between the $\mathcal{H}(R)$-algebra of X and X itself, using a free cosimplicial resolution and a spectral sequence. In homotopy theory, Stover [34] constructs a free simplicial resolution $X_\bullet$ which is homotopy equivalent to a wedge of spheres in each simplicial dimension. Taking the p-th homotopy of this simplicial space gives a simplicial group and the homotopy groups of that simplicial group fits into the E^2 page of the Bousfield-Friedlander spectral sequence [16]. The sequence commutes with primary operations and by design of the Stover resolution, the sequence collapses on the third page, to the Π-algebra of X.

A simplicial resolution of an $\mathcal{H}(R)$-algebra is a simplicial $\mathcal{H}(R)$-algebra which is weakly equivalent to the constant simplicial $\mathcal{H}(R)$-algebra. A free simplicial resolution additionally has a free $\mathcal{H}(R)$-algebra in each simplicial dimension. A free cosimplicial resolution of a space is a cosimplicial space, $X^\bullet$, with $H^*(X^\bullet)$ a free simplicial resolution of the $\mathcal{H}(R)$-algebra of X. The model category structure, defining the weak equivalences on simplicial $\mathcal{H}(R)$-algebras, is given by Blanc and Peschke for $\mathcal{H}(R)$ containing finite products of E-M spaces [8].

Although a dual Stover construction will work fairly well for coefficients in a finite field, there is a difficulty for integer coefficients. Unlike the finite fields, $\mathbb{Z}$ is not algebraically compact, consequently, maps out of an infinite product of E-M spaces over $\mathbb{Z}$ do not factor through a finite sub-product. In order that the cosimplicial resolution be free, $\mathcal{H}(\mathbb{Z})$ needs to contain infinite products and we need to allow infinitary primary operations. Fortunately, we are still able to define a model category structure for $\mathcal{H}(\mathbb{Z})$-algebras but the free resolutions are larger than for field coefficients [26].

We would like to employ a second quadrant cohomology spectral sequence $E_2^{-p,q} = \pi_p H^q(X^\bullet; R) \Rightarrow H^{q-p}(X; R)$ which commutes with primary cohomology operations. The spectral sequence exists [18, 21, 5] although convergence is not guaranteed. The spectral sequence would actually converge to $H^*(\mathrm{Tot}(X^\bullet); R)$, where $\mathrm{Tot}(X^\bullet)$ is the total space associated to the cosimplicial space $X^\bullet$ [17]. Then, to meet our purposes, we also require that $\mathrm{Tot}(X^\bullet)$ has the same cohomology algebra as X. This requires $\mathrm{Tot}(X^\bullet)$ to have the same R-cohomology type as the R-completion of X, which, according to Bousfield [15], will occur if $H^*(X^\bullet; R)$ is acyclic over all R-module coefficients. For integer coefficients, this requires an expansion of the category $\mathcal{H}(\mathbb{Z})$ to include all GEMs, that is, (countable) products of Eilenberg-Mac Lane spaces, $K(M, n)$, for arbitrary $\mathbb{Z}$-modules M, whereby $\mathcal{H}(\mathbb{Z})$-algebras will no longer be well defined [29].

We expect integer cohomology to contain more topological information than cohomology over a quotient ring of $\mathbb{Z}$. When we look at primary operations the additional information contained over integral coefficients is not well understood. In fact, the research has focused, with great success, on cohomology over finite fields and seems to have abandoned the integers since Kochman's work [27]. Perhaps the research has not been abandoned, it's just that no results have been achieved to report. One caveat for Eckmann-Hilton duality should be "integers for homotopy, finite fields for cohomology".

After some notation in Section 2, Section 3 looks at the problem of defining the universal arrows for primary integral operations. Section 4 develops the encoding of operations in a category and defines $\mathcal{H}(R)$-algebras. Section 5 gives the construction of free resolutions by simplicial $\mathcal{H}(R)$-algebras and cosimplicial spaces. The need to consider infinitary integral operations is also explained in this section. Section 6 looks at the spectral sequence for calculating $\mathcal{H}(R)$-algebras and the obstacles occurring for integer coefficients. Section 7 is a brief conclusion.

2. Notation

We assume that all statements and results are for some fixed but arbitrary ring R. We will denote E-M spaces by $K^n = K(R, n)$. Hence the reduced cohomology groups of a space with coefficients in R are given by $\tilde{H}^n(X) = [X, K^n]$. For the various constructions given we will then discuss the obstacles for $R = \mathbb{Z}$ compared to $R = \mathbb{F}_p$.

For convenience we will consider the sub-category, $\mathcal{T}_*$, of pointed topological spaces which are simply-connected, CW-complexes. For spaces $X, Y \in \mathcal{T}_*$, we write $[X, Y]$ for the homotopy classes of maps from X to Y. Pre-composition by a map $f : X \to Y$ is denoted $f^*(g) = gf$, for $g : Y \to Z$, and post-composition denoted by f_*. The category of pointed sets is denoted $\mathcal{SET}_*$ and graded pointed sets by $Gr\mathcal{SET}_*$.

For a product $\prod_{i=1}^n X_i$, let pr_i denote the canonical projection onto the factor X_i, and for maps $x_i : Y \to X$ let $\{x_1, x_2, \ldots, x_n\} : Y \to \prod_{i=1}^n X_i$ denote the canonical product map.

Let $s\mathcal{C}$ denote the category of simplicial objects over a category $\mathcal{C}$ and $c\mathcal{C}$ that of cosimplicial objects.

Given functors $u, v : \mathcal{C} \to \mathcal{C}$, an object $X \in \mathcal{C}$ and natural transformations $\mu : u \to v$, composition is defined as $u\mu : uu \to uv, (u\mu)_X := u(\mu_X)$ and $\mu u : uu \to vu, (\mu u)_X := \mu_{uX}$.

3. Primary operations and universal arrows

In this section we give a brief overview of primary cohomology operations and explain where the integral theory differs from cohomology over $\mathbb{F}_p$.

Definition 3.1. A cohomology operation of type (G, n, A, m) is a family of functions $\theta_X : H^n(X; G) \to H^m(X; A)$, one for each space X, satisfying the naturality condition $f^*\theta_Y = \theta_X f^*$ for any map $f : X \to Y$.

Operations exist for any abelian group coefficients, G and A, whether finitely generated or not and any integers $n \leq m$. The set of all (G, n, A, m) operations is denoted $\theta(G, n, A, m)$ and it follows from the naturality condition that $\theta(G, n, A, m) \cong H^m(K(G, n); A)$ so that the cohomology classes of E-M spaces are often called primary cohomology operations [33, 22].

The cohomology operations are generally functions on the underlying sets of the cohomology groups, however, they can be semi-additive or additive. Additive operations induce homomorphisms on cohomology groups and, for any spectral cohomology theory, there is a range called the stable range in which operations form families of additive operations. For $G = A = \mathbb{Z}/2$, the stable range are called Steenrod squares,

$$Sq^i : H^n(X; \mathbb{Z}/2) \longrightarrow H^{n+1}(X; \mathbb{Z}/2)\,, \qquad 0 \leq i \leq n\,,$$

which, under composition form the mod-2 Steenrod algebra.

Similarly, if $G = A = \mathbb{Z}/p$ for an odd prime p, there are stable operations called the reduced powers,

$$P^i : H^n(X; \mathbb{Z}/p) \longrightarrow H^{n+2i(p-1)}(X; \mathbb{Z}/p)\,, \qquad 0 \leq i \leq \frac{n}{2}\,,$$

which, under composition form the mod-p Steenrod algebra.

If $G = A = \mathbb{Z}/p$ for any prime p there are the Bockstein operations, $\beta : H^n(X; \mathbb{Z}/p) \to H^{n+1}(X; \mathbb{Z}/p)$ and if $G = \mathbb{Z}/p^k$, for p prime and $k \geq 1$, and $A = \mathbb{Z}/p^{k+1}$, there are the Pontrjagin p-th powers, $\beta_p : H^{2n}(X; \mathbb{Z}/p^k) \to H^{2np}(X; \mathbb{Z}/p^{k+1})$.

Remark 3.2. For the case where G is a finitely generated abelian group and $A = \mathbb{Z}/p$ for any prime, from the results of Cartan [19] all cohomology operations are generated (by compositioins) from the Steenrod squares, the reduced powers, the Bockstein operations and the Pontrjagin p-th powers [33].

This can equivalently be shown using the Serre spectral sequence [30]. However, if $A = \mathbb{Z}$, the process is much more difficult because of mixed torsion occuring on the E^2 page, and no complete calculation has been given for the operations $H^m(K(\mathbb{Z}, n); \mathbb{Z})$ (see Remark 3.4).

We should also note that if $G = A = \mathbb{Q}$, rational cohomology is an algebra. Hence, there are only the $\mathbb{Q}$-vector space structure for $n = m$ and a power operation $x \mapsto x^2$ in even degrees, satisfying Definition 3.1 [28].

Now, if $G = A = R$, for R a commutative ring, there exists the binary operation of cup product

$$\cup_X : H^n(X; R) \times H^m(X; R) \longrightarrow H^{n+m}(X; R)\,,$$

satisfying the naturality condition, $f^*\cup_Y = \cup_X(f^* \times f^*)$ for any map $f : X \to Y$ (see diagram (1)), which gives the graded cohomology groups a graded ring structure and the map$f$ induces a graded ring homomorphism. This introduces the concept of an n-ary operation and we now use $\mathbb{F}_p$ for the finite field rather than the cyclic group $\mathbb{Z}/p$, so cup products will exist.

DEFINITION 3.3. An n-ary operation $\theta : \tilde{H}^{m_1}(X) \times \cdots \times \tilde{H}^{m_n}(X) \to \tilde{H}^q(X)$, $n \in \mathbb{N}$, is primary if, given any spaces X and Y and any map $f : X \to Y$, the following naturality diagram commutes

$$\begin{array}{ccc} \tilde{H}^{m_1}(X) \times \cdots \times \tilde{H}^{m_n}(X) & \xrightarrow{\theta} & \tilde{H}^q(X) \\ \uparrow{\scriptstyle f^* \times \cdots \times f^*} & & \uparrow{\scriptstyle f^*} \\ \tilde{H}^{m_1}(Y) \times \cdots \times \tilde{H}^{m_n}(Y) & \xrightarrow{\theta} & \tilde{H}^q(Y) . \end{array} \tag{1}$$

There may be other operations (such as secondary or higher) that satisfy diagram (1) for some X, Y and m_i, but only primary operations satisfy this diagram universally. Since all spaces are simply connected there is no action of $\pi_1(X)$ on the cohomology groups of X. It follows directly from (1) that the homotopy class

$$\theta(pr_1, \ldots, pr_n) : K^{m_1} \times \cdots \times K^{m_n} \longrightarrow K^q$$

is a universal arrow and the operation $\theta(x_1, x_2, \ldots, x_n)$ is given by the composition

$$X \xrightarrow{\{x_1,\ldots,x_n\}} K^{m_1} \times \cdots \times K^{m_n} \xrightarrow{\theta(pr_1,\ldots,pr_n)} K^q$$

The proof [31] is Eckmann-Hilton dual to that for homotopy operations [36].

Composition with a representative element $\theta \in \tilde{H}^m(K^n)$ are the only possible unary operations (for a fixed coefficient ring R). The universal arrow is θ and the operation $\theta_* : \tilde{H}^n(X) \to \tilde{H}^m(X)$.

The group addition is given by identifying K^n with the loop space ΩK^{n+1}, up to homotopy. Then addition $+ : K^n \times K^n \to K^n$ is given by concatenation of loops.

Other binary operations are given by the Künneth formula applied to a product of two Eilenberg-Mac Lane spaces. Iteration of the Künneth formula shows all finitary operations are generated by the binary and unary operations. The Künneth formula is derived from a short exact sequence which splits non-naturally giving

$$\tilde{H}^n\left(K^r \times K^s\right) \cong \bigoplus_{i+j=n} \tilde{H}^i(K^r) \otimes \tilde{H}^j(K^s) \bigoplus_{i+j=n+1} \operatorname{Tor}\left(\tilde{H}^i(K^r), \tilde{H}^j(K^s)\right) \tag{2}$$

Elements $\alpha \otimes \beta \in \tilde{H}^i(K^r) \otimes \tilde{H}^j(K^s)$ are included into $\tilde{H}^n(K^r \times K^s)$ by the "external cup product" given by $\alpha_*(pr_r) \cup \beta_*(pr_s)$ where $\cup$ is the standard cup product (see [23, p. 210 and p. 278]). If we denote the universal arrow for cup product by $h_\cup$ then the universal arrow for the binary operation $\alpha \otimes \beta$ is $h_{\cup *}(\{\alpha_*(pr_r), \beta_*(pr_s)\})$.

To give all binary operations and hence, by iteration, all n-ary operations, it remains to describe elements of the summands

$$\bigoplus_{i+j=n+1} \mathrm{Tor}\left(\tilde{H}^i(K^r), \tilde{H}^j(K^s)\right).$$

Here it should be noted that integer coefficients differ from coefficients in $\mathbb{F}_p$. For coefficients in $\mathbb{F}_p$ there are no Tor terms [23, Theorem 3.16] so that all primary operations are combinations of compositions and cup products as for $\alpha \otimes \beta$ above. In fact, as a refinement to Remark 3.2, by explicitely listing generators of the groups $H^n(K(\mathbb{Z}/p, m); \mathbb{Z}/q)$, Cartan showed that the only operations over $\mathbb{F}_p$ are freely generated by the stable reduced powers and cup product [19].

In [22, Section 12] Eilenberg and Mac Lane defined a certain type of secondary operation, called cross-cap products, and showed these were in bijective correspondence with the Tor groups of the homology Künneth formula. The construction dualises from chain to cochain complexes and we can define a cross-cap product, $\alpha \bar{\times} \beta$ on elements $\alpha \in \tilde{H}^i(K^r)$ and $\beta \in \tilde{H}^j(K^s)$ of cohomology groups with torsion (for details see [31]). Cross-cap products are generalized by secondary Massey products [35]. These cross-caps correspond to elements of

$$\bigoplus_{i+j=n+1} \mathrm{Tor}(\tilde{H}^i(K^r), \tilde{H}^j(K^s))$$

by considering these Tor groups as cokernels of the external cup products, hence cosets

$$\tilde{H}^{i+j-1}(K^r \times K^s) \Big/ \bigoplus_{i+j=n+1} \left[\tilde{H}^i(K^r) \otimes \tilde{H}^{j-1}(K^s) + \tilde{H}^{i-1}(K^r) \otimes \tilde{H}^j(K^s)\right].$$

This means the cross-cap product $\alpha \bar{\times} \beta$ is not uniquely defined and consequently cannot be identified with a unique universal arrow. Moreover, since the Künneth short exact sequence splits non-naturally, there is no natural way to identify a representative arrow in each equivalence class corresponding to the cross-cap products. That these secondary operations correspond to cosets of $\tilde{H}^n(K^r \times K^s)$ in equation (2) is the closest description we have of an unknown type of primary operation for integer coefficients.

Remark 3.4. Appendix 8 contains a table of primary, unary cohomology operations over $\mathbb{Z}$ coefficients and a short discussion of differences with operations over $\mathbb{F}_p$ coefficients. In particular, integer operations do not generate freely and compositions can be unstable and multiple.

The calculations shown in Table 1 can be used to demonstrate the existence of the unknown primary operations in

Example 3.5. The binary operations (for $\mathbb{Z}$ coefficients)

$$\begin{aligned}\tilde{H}^{14}\left(K^3 \times K^3\right) &\cong \tilde{H}^3(K^3) \otimes \tilde{H}^{11}(K^3) \oplus \operatorname{Tor}\left(\tilde{H}^6(K^3), \tilde{H}^9(K^3)\right) \\ &\cong \mathbb{Z}/3 \oplus \mathbb{Z}/2 \end{aligned} \tag{3}$$

are generated by two types of universal arrows. The first is an external cup product universal arrow of order 3 and the second, of unknown formulation and order 2, corresponding to the generator of $\mathbb{Z}/2$.

4. $\mathcal{H}(R)$-algebras

Since being able to list all integral cohomology operations and relations between them is not possible, we may wish to take another approach in which this information is encoded in a category and does not need to be known explicitly. This was the point of view, adopted in the 1990's, for the study of primary homotopy operations and their relations acting on homotopy groups to give a Π-algebra [20, 34].

Let $\mathcal{H}(R)$ be the category of finite products of Eilenberg-Mac Lane spaces over R, including the trivial product $*$, with pointed homotopy classes of maps. For any $X \in \mathcal{T}_*$ consider the functor of homotopy classes of pointed maps $[X, \ ] : \mathcal{H}(R) \to \mathcal{SET}_*$. The image of these functors (with all copies of $\tilde{H}^n(X)$, $n \geq 0$, identified as one isomorphism class) give reduced cohomology groups and the morphisms of $\mathcal{H}(R)$ induce primary operations on those groups. Such functors, as well as their image in $\mathcal{SET}_*$ will be called topological $\mathcal{H}(R)$-algebras. Figure 1 indicates the identification of functor with graded cohomology algebra.

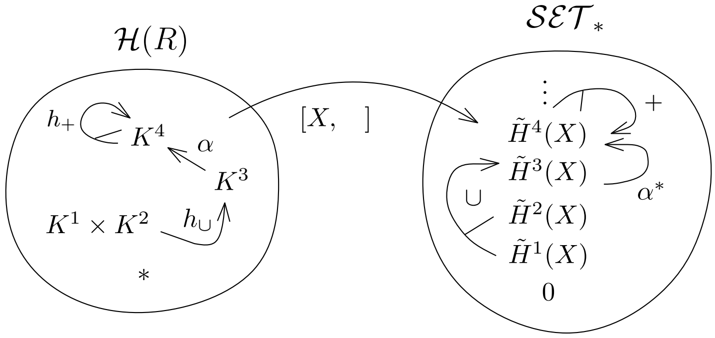


Figure 1: Topological $\mathcal{H}(R)$-algebras

Similarly to Π-algebras we can abstract the notion of topological $\mathcal{H}(R)$-algebras by

Definition 4.1. An $\mathcal{H}(R)$-algebra is a functor $Z : \mathcal{H}(R) \to \mathcal{SET}_*$ sending the point to the singleton set, 0, and with the property that the map $Z\left(\prod_{i=1}^{N} K^{n_i}\right) \to \prod_{i=1}^{N} Z(K^{n_i})$ is a natural isomorphism of abelian groups.

The category of $\mathcal{H}(R)$-algebras will be denoted ${\mathcal{SET}_*}^{\mathcal{H}(R)}$. There are several definitions of algebraic structures describing $\mathcal{H}(R)$-algebras. They could be considered a variety of algebras [29], universal graded algebras [11], models of the algebraic theory $\mathcal{H}(R)$ [12] or models of the finite product sketch $\mathcal{H}(R)$ [8]. Mac Lane states that ${\mathcal{SET}_*}^{\mathcal{H}(R)}$ contains all limits and colimits and that there exists a free functor, left adjoint to the underlying (forgetful) functor to $\mathcal{SET}_*$ [29].

Remark 4.2. Borceux [12] shows that the cohomology $\mathcal{H}(R)$-algebras of the objects of $\mathcal{H}(R)$ are free $\mathcal{H}(R)$-algebras.

In addition, Blanc and Stover show that simplicial objects over any category of universal graded algebras, such as ${\mathcal{SET}_*}^{\mathcal{H}(R)}$, has a closed simplicial model category structure [11]. Blanc and Peschke give a resolution model category structure on simplicial models of a finite product sketch arising from an adjunction of free and forgetful functors. Therefore we have a definition of weak equivalence between simplicial $\mathcal{H}(R)$-algebras.

Remark 4.3. In addition, the model category structure is enriched, forming a simplicial model category with tensor and cotensor products [8].

5. Standard constructions and resolutions

A standard construction of simplicial and cosimplicial functors from a comonad and a monad respectively, is given by Huber in [24].

Definition 5.1. A **comonad**, $\langle T, \mu, \eta \rangle$, in a category $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations, the counit $\mu : T \to 1$ and comultiplication $\eta : T \to T^2$ such that the following diagrams commute [29]:

$$\begin{array}{ccc} T & \xrightarrow{\eta} & T^2 \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle T\eta} \\ T^2 & \xrightarrow[\eta T]{} & T^3 \end{array} \quad , \qquad \begin{array}{ccccc} T & = & T & = & T \\ \| & & \downarrow{\scriptstyle\eta} & & \| \\ 1T & \xleftarrow{\mu T} & T^2 & \xrightarrow[T\mu]{} & T1 \end{array} .$$

Any comonad generates a simplicial functor $V_\bullet = (V_n, d_n^i, s_n^i)$, $n \geq -1$ with a sequence of functors $V_n : \mathcal{C} \to \mathcal{C}$, face maps $d_n^i : V_n \to V_{n-1}$ and degeneracy maps $s_n^i : V_n \to V_{n+1}$, $0 \leq i \leq n$. This is achieved by letting $T^0 = 1$ and $T^{n+1} = TT^n$ then letting $V_n = T^{n+1}$, $d_n^i = T^i \mu T^{n-i}$ and $s_n^i = T^i \eta T^{n-i}$. One may also consider the simplicial functors $V_\bullet = (V_n, d_n^i, s_n^i)$, $n \geq 0$ with an augmentation $\mu : V_0 \to 1$ with $\mu d_0^0 = \mu d_0^1$.

The construction of a cosimplicial functor from a monad is categorically dual to the construction above [24]. A monad $\langle T, \epsilon, \delta \rangle$ with $T : \mathcal{C} \to \mathcal{C}$, $\epsilon : 1 \to T$ (the unit) and $\delta : T^2 \to T$ (the multiplication), is used to build a cosimplicial functor $V^\bullet = (V^n, d_i^n, s_i^n)$, $n \geq -1$ as a sequence of functors $V^n : \mathcal{C} \to \mathcal{C}$, coface maps $d_i^n : V^{n-1} \to V^n$ and codegeneracy maps $s_i^n : V^{n+1} \to V^n$, $0 \leq i \leq n$. This is achieved by letting $T^0 = 1$ and $T^{n+1} = TT^n$ then letting $V^n = T^{n+1}$, $d_i^n = T^i \epsilon T^{n-i}$ and $s_i^n = T^i \delta T^{n-i}$. One may also consider the cosimplicial functors $F^\bullet = (V^n, d_i^n, s_i^n)$, $n \geq 0$ with an augmentation $\epsilon : 1 \to V^0$ with $d_0^0 \epsilon = d_1^0 \epsilon$.

5.1. Simplicial $\mathcal{H}(R)$-algebras.

Every adjunction between categories gives rise to both a comonad and a monad [29]. Let $F : Gr\mathcal{SET}_* \to \mathcal{SET}_*{}^{\mathcal{H}(R)}$ and U be the free and underlying functors for $\mathcal{H}(R)$-algebras with unit of adjunction $\alpha : 1 \to UF$ and counit of adjunction $\beta : FU \to 1$. A comonad, $\langle T, \epsilon, \delta \rangle$, is formed by letting the endofunctor be $T = FU$, the counit $\epsilon = \beta$ and the comultiplication $\delta = F\alpha U$.

From this comonad we would like to construct a simplicial $\mathcal{H}(R)$-algebra, $T_\bullet Z$, for any given $\mathcal{H}(R)$-algebra, Z, and would hope that $T_\bullet Z$ can be used as a free algebraic resolution of any Z in the sense of homotopical algebra. This requires a free object in each simplicial dimension and a homotopy equivalence with the constant simplicial $\mathcal{H}(R)$-algebra over Z.

The underlying graded set of the $\mathcal{H}(R)$-algebra, Z, may contain sets of infinite cardinality. Consequently, the first simplicial dimension of the resolution, that is, TZ, will be the $\mathcal{H}(R)$-algebra of an infinite product of E-M spaces. By Remark 4.2, this $\mathcal{H}(R)$-algebra will be free if it is an object of $\mathcal{H}(R)$.

For $R = \mathbb{F}_p$ coefficients we have that

$$H^*(P;\mathbb{F}_p) \cong \lim_{\overleftarrow{\alpha}} H^*(P_\alpha;\mathbb{F}_p) \tag{4}$$

where P_α are finite subproducts of the infinite product P [25, Proposition 2.1]. This means we can replace TZ with the $\mathcal{H}(\mathbb{F}_p)$-algebra of a finite subproduct and $\mathcal{H}(\mathbb{F}_p)$ need only contain finite products of E-M spaces. In addition, by Remark 3.2, all finitary operations in $\mathcal{H}(\mathbb{F}_p)$ are iterations of the Künneth formula on the Steenrod operations, reduced powers and Bocksteins, so $\mathcal{H}(\mathbb{F}_p)$ is well understood.

Because $\mathbb{Z}$ is not algebraically compact property (4) does not hold and maps out of infinite products of E-M spaces over $\mathbb{Z}$ do not factor through a finite subproduct and we must extend our definition of primary operations to cover "infinary" operations and hence let $\mathcal{H}(\mathbb{Z})^\infty$ contain arbitrary products of integral E-M spaces. This then brings up the question of whether the model category structure on $\mathcal{SET}_*{}^{\mathcal{H}(R)}$ for $\mathcal{H}(R)$ containing finite products is still valid for $\mathbb{Z}$ coefficients.

Sketch categories containing infinite products have many of the properties of finite sketch categories under certain restrictions. A good overview is given in [1]. It turns out that even when a sketch category contains infinite products the category of models will be locally presentable [12, 5.6.8]. As such, the category of $\mathcal{H}(\mathbb{Z})^\infty$-algebras has all limits and colimits [12, 5.5.8]. Then the model category structure of [8] can be extended to simplicial $\mathcal{H}(\mathbb{Z})^\infty$-algebras.

According to this model category structure the homotopy groups $\pi_n(T_\bullet Z)$, $n \geq 0$, of a simplicial $\mathcal{H}(R)$-algebra are the n-th homotopy groups of the underlying simplicial graded group. It can then be shown [26] that

$$\pi_q(T_\bullet Z) \cong \begin{cases} 0 & \text{if } q \neq 0 \\ Z & \text{if } q = 0 \end{cases}$$

so that $T_\bullet Z$ has the property of a free simplicial resolution of Z.

5.2. Resolution by cosimplicial space. We would also like to construct a simplicial resolution of a topological $\mathcal{H}(R)$-algebra as the $\mathcal{H}(R)$-algebra of a cosimplicial space. This will allow the use of spectral sequences in the calculation of topological $\mathcal{H}(R)$-algebras. That is, for a given space X we would like to construct an augmented cosimplicial space, $X^\bullet \leftarrow X$ such that the augmented simplicial group $\tilde{H}^q(X^\bullet) \to \tilde{H}^q(X)$ satisfies

$$\pi_p H^q(X^\bullet) = \begin{cases} H^q(X) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

for all q, with the homomorphism $\pi_0 \tilde{H}^q(X^\bullet) \cong \tilde{H}^q(X)$ induced by the augmentation. We would also require $\tilde{H}^*(X^\bullet)$ to have a free $\mathcal{H}(R)$-algebra in each simplicial dimension which, by Remark 4.2, requires $X^\bullet$ to have an object of $\mathcal{H}(R)$ in each cosimplicial dimension. Figure 2 shows the simplicial $\mathcal{H}(R)$-algebra, $\tilde{H}^*(X^\bullet)$ where each column in $\mathcal{SET}_*$ is an $\mathcal{H}(R)$-algebra and each row a simplicial (abelian) group.

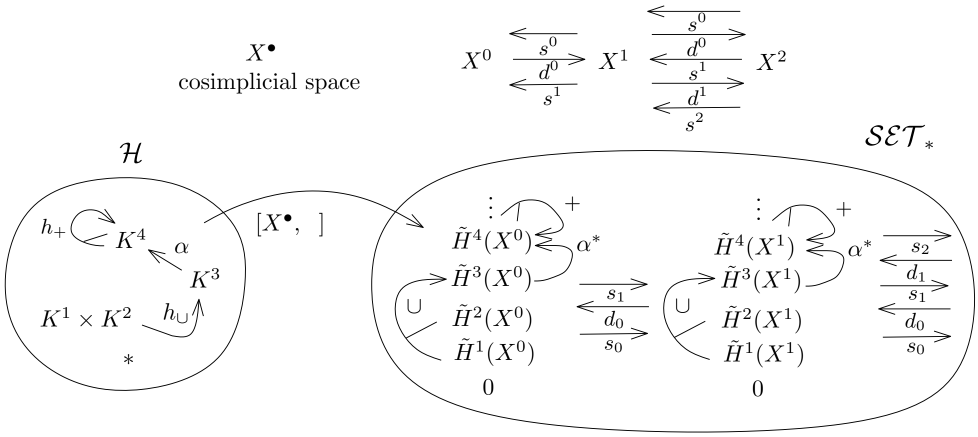


Figure 2: Simplicial $\mathcal{H}(R)$-algebra of a cosimplicial space

A cosimplical space can indeed be constructed, homotopic to a product of E-M spaces in each cosimplicial dimension, dual to Stover's pushout construction on wedges of spheres and discs [34].

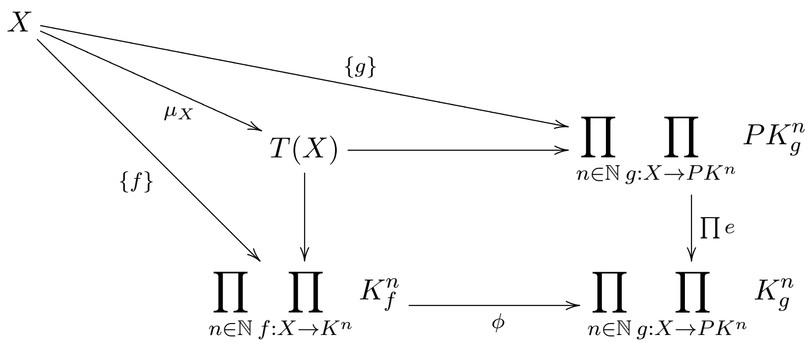
Let PK^n be the path space over K^n and e be the evaluation map sending a based path to it's end point. If a map $f : X \to K^n$ is null-homotopic there are maps $g : X \to PK^n$ such that $f \cong e_*(g)$ [3, Proposition 1.4.9]. Dually to Stover's construction, we define an endofunctor $T : \mathcal{T}_* \to \mathcal{T}_*$ such that, for a space X, $T(X)$ is the pullback of diagram (6). ϕ is the projection onto the subfactors $K^n_{e_*(g)}$ such that $f = e_*(g)$, followed by indentifying the indices $e_*(g)$ and g:

$$\begin{array}{ccc} TX & \longrightarrow & \prod_{n\in\mathbb{N}} \prod_{g:X\to PK^n} PK^n_g \\ \downarrow & & \downarrow{\scriptstyle \prod e} \\ \prod_{n\in\mathbb{N}} \prod_{f:X\to K^n} K^n_f & \xrightarrow{\phi} & \prod_{n\in\mathbb{N}} \prod_{g:X\to PK^n} K^n_g \end{array} \tag{6}$$

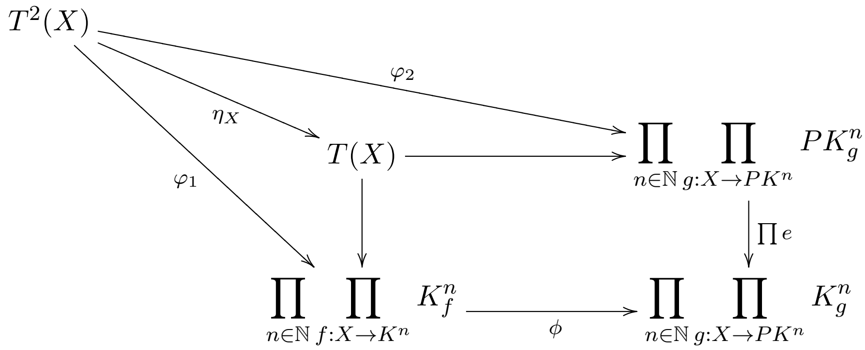
Effectively TX has a factor of K^n for each map $f : X \to K^n$. If $f = e_*(g)$ is a null-homotopic map then the points of K^n_f are identified with the end point of the paths in PK^n_g. Contracting any path-space leaves the set of loops on K^n. Since $\Omega K^n \cong K^{n-1}$ it follows that TX is homotopy equivalent to a product of Eilenberg-Mac Lane spaces and hence will have a free $\mathcal{H}(R)$-algebra.

Remark 5.2. The space TX will be less connected than X because of the loop spaces in the product. Consequently, at some cosimplicial dimension the product space will cease to be simply connected.

The unit for the monad is the natural transformation $\mu : 1 \to T$ defined on any space X as the unique map given by the universal property of a pullback:



The multiplication is a natural transformation $\eta : T^2 \to T$ defined for any $X \in \mathcal{T}_*$ as follows:



The map φ_1 is defined by projecting the factors $K^n_{pr_f}$ onto the factor K^n_f and similarly, φ_2 is defined by projecting the factors $PK^n_{pr_g}$ onto the factor PK^n_g.

From the monad, $\langle T, \mu, \eta \rangle$ we can then form a cosimplicial functor by the standard construction and apply this to a space X to give a cosimplicial space $X^\bullet$.

For the simplicial $\mathcal{H}(R)$-algebra, $\tilde{H}^*(X^\bullet)$, dually to Stover [34], we can show that defining $\pi_p H^q(X^\bullet)$ as the p-th homotopy group of the underlying simplicial group $\tilde{H}^q(X^\bullet)$ we have formed a free resolution [26]. That is, each simplicial dimension is a free $\mathcal{H}(R)$-algebra and equations (5) hold.

Remark 5.3. It is in the proof that the simplicial groups are acyclic that the path spaces must be included in the pullback construction T in diagram (6), to provide a null-homotopy for $d_i^{n+1} pr_f$, $1 \leq i \leq n+1$, so that for every f in the n-th Moore chain complex pr_f is in the $(n+1)$-th Moore complex with $d_0^{n+1} pr_f = f$.

6. Calculating with the cosimplicial resolution

If we had a convergent spectral sequence for a cosimplical space

$$E_2^{-p,q} = \pi_p H^q(X^\bullet) \;\Rightarrow\; H^{q-p}(\mathrm{Tot}(X^\bullet)),$$

then we would be able to calculate the cohomology of more complicated spaces, such as $\mathrm{map}(Y, X)$. This would be achieved by identifying the E_2 page as derived functors of $\mathrm{map}(Y, \quad)$ applied dimensionwise to a free cosimplicial resolution $X^\bullet$ [7].

Bökstedt and Ottosen [5] were similarly motivated to construct spectral sequences for calculating the cohomology of $\mathbb{F}_p$-completions of the string space

$(\Lambda X)_{h\mathbb{T}}$ of a simply-connected space, X of finite type. Using a free cosimplicial resolution $RX^\bullet \to X$ by simplicial $\mathbb{F}_p$-modules [17], they dualize the construction of Bousfield's homology spectral sequence [14]. Using $(\Lambda \quad)_{h\mathbb{T}}$ applied dimensionwise to $RX^\bullet$ gives a spectral sequence

$$E_2^{-p,q} = \big(\pi_p H^*((\Lambda RX^\bullet)_{h\mathbb{T}}; \mathbb{F}_p)^q \;\Rightarrow\; H^*(\mathrm{Tot}\big(((\Lambda RX^\bullet)_{h\mathbb{T}}); \mathbb{F}_p\big)$$

and associated convergence criteria. For a resolution by $\mathbb{F}_p$-modules, $\mathrm{Tot}(RX^\bullet)$ is the $\mathbb{F}_p$-completion of a space X [17], so the spectral sequence converges to the $\mathbb{F}_p$-cohomology algebra $H^*((\Lambda X)_{h\mathbb{T}}); \mathbb{F}_p)$.

To explain the E_2 terms, $(\pi_p H^*((\Lambda RX^\bullet)_{h\mathbb{T}}; \mathbb{F}_p)^q$ are the collection of $\mathbb{F}_p$-modules $\pi_p H^q((\Lambda RX^q)_{h\mathbb{T}}; \mathbb{F}_p)$ with the mod p Steenrod operations acting columnwise. Bökstedt and Ottosen [5] used the known endofunctor l, on the category of unstable algebras over the Steenrod algebra, to show that $E_2^{-p,q}$ is the derived functor $H_p(H^*(X); l)^q$ where the homology of $H^*(X)$ with coeficients in the functor l is defined by

$$H_p(H^*(X), l) = \pi_p l(RX^\bullet).$$

When dualizing Bousfield's homology spectral sequence for use with integer coefficients we must remember that we would want convergence to the cohomology $\mathcal{H}(\mathbb{Z})$-algebra of the space X or its $\mathbb{Z}$-completion. To do so, it is important that the cosimplicial space $X^\bullet$ be fibrant so $\mathrm{Tot}(X^\bullet)$ is the inverse limit of a tower of fibrations from which a convergent spectral sequence could be defined [17, §6]. Fibrancy is ensured by the group object property of GEM's [17, X §4.9]. However, unfortunately for integer coefficients $\mathrm{Tot}(X^\bullet)$ need not have the same cohomology as X itself.

In order to have the same cohomology, $\mathrm{Tot}(X^\bullet)$ must be homotopy equivalent to the R-completion of X and this is known to occur only when $\tilde{H}^*(X^\bullet)$ is an acyclic simplicial group for cohomology in any R-module coefficients [15, §7]. We have seen in Section 5 that by constructing $X^\bullet$ with GEM's over $\mathbb{Z}$, $\tilde{H}^*(X^\bullet)$ is acyclic over $\mathbb{Z}$ coefficients. However, by Remark 5.3, $\tilde{H}^*(X^\bullet)$ is not acyclic for cohomology in other $\mathbb{Z}$-modules, for instance $\mathbb{Z}/2$, because the pullback construction (6) supplies null homotopies only for $\mathbb{Z}$ coefficients. On the other hand, for $\mathbb{F}_p$ coefficients the construction (6) will give $\tilde{H}^*(X^\bullet)$ acyclic for cohomology in any $\mathbb{F}_p$-module since any $\mathbb{F}_p$-module is a direct sum of copies of $\mathbb{F}_p$. Hence, for $\mathbb{F}_p$ coefficients $\mathrm{Tot}(X^\bullet)$ is the $\mathbb{F}_p$-completion of X.

To ensure $\tilde{H}^*(X^\bullet)$ is acyclic over all $\mathbb{Z}$-modules we could expand construction (6) to be a pullback over indexed products of E-M spaces and path spaces over all $\mathbb{Z}$-module coefficients. This requires the construction to contain arbitrary products over a proper class of E-M spaces. As such, the construction does not exist in $\mathcal{T}_*$, since topological spaces must in particular be sets, not proper classes. Furthermore, by Remark 4.2, the GEM in each cosimplicial dimension must be an object in $\mathcal{H}(\mathbb{Z})$. Functors from the resulting sketch category $\mathcal{H}(\mathbb{Z})$ may not have a well defined image in $\mathcal{SET}_*$ [29].

Remark 6.1. It is possible, however, to restrict the GEM in each cosimplicial dimension, and hence the category $\mathcal{H}(\mathbb{Z})$, to products indexed by a set of bounded (infinite) cardinality, λ, and still achieve acyclicity over all $\mathbb{Z}$-modules. The construction of Blanc and Sen [9] is complex and requires the topological $\mathcal{H}(\mathbb{Z})$-algebra $\tilde{H}^*(X;\mathbb{Z})$, that is, the cardinality λ depends on the space X. Primarily the construction relies on the enrichment of $\mathcal{H}(\mathbb{Z})$ of Remark 4.3, which also enriches the resultant $\mathcal{H}(\mathbb{Z})$-algebras. It turns out that this enrichment can identify sufficient information, whilst putting an upper bound on λ, to ensure $\mathrm{Tot}(X^\bullet)$ is the $\mathbb{Z}$-completion of X [9, Corollary 4.28].

Finally then, it becomes a question of whether the spectral sequence

$$E_2^{-p,q} = \pi_p H^q(X^\bullet) \;\Rightarrow\; H^{q-p}(\mathrm{Tot}(X^\bullet))$$

can be shown to converge. A general result does not exist in the literature. What can be said is that wherever the well known Bousfield-Kan resolution $RX^\bullet \to X$ by R-modules [17] can be used in a spectral sequence, the restricted resolution by GEMs of Remark 6.1 can be used.

One advantage of resolutions by GEMs is recognizing the E_2 page of spectral sequences as derived functors. For coefficients in a general R, the Yoneda Lemma gives an embedding of the free representable $\mathcal{H}(R)$-algebras into $\mathcal{H}(R)^{op}$ [31]. Applying a functor dimensionwise to a cosimplical resolution by GEMs is equivalent to a simplicial functor from $\mathcal{H}(R)$-algebras and then taking (co)homotopy gives the E_2 page of spectral sequences as a derived functor.

7. Conclusion

In their introduction to part I of [17], Bousfield and Kan justify the importance of completions and localizations as being able to decompose a homotopy type into $\mathbb{F}_p$ type and, together with coherence information over the rationals, be able to reconstruct it. They point out that many problems can be solved with $\mathbb{F}_p$ information alone. In this review of properties of integral cohomology we have seen that often more can be obtained with $\mathbb{F}_p$ cohomology, since integral cohomology is both overly complex (Section 3) and unsuited to many of the powerful tools available for calculation (Section 5). These problems with the primary integral operations do not seem to get a mention in homology texts which usually switch from $\mathbb{Z}$ to $\mathbb{Z}/2$ coefficients, without explanation, when introducing the higher structure of operations.

Since Eckmann-Hilton duality is not a categorical duality it should not be expected to hold universally. The construction of a free simplicial resolution of a space using wedges of spheres and discs has been of great benefit to homotopy theory and algebraic topology generally however, the dual resolution is more complex and has not yet delivered the same benefits when working with integer coefficients.

It is true that, in comparison, the cosimplicial resolution by simplicial R-modules of Bousfield and Kan [17, §4] is smaller and remains simply connected for a simply connected space (see Remark 5.2), even for $\mathbb{Z}$ coefficients. However, the resolution by products of GEMs contains the additional information of higher order operations [9], which can be used to identify enough information to ensure $\mathrm{Tot}(X^\bullet)$ is the R-completion of X, for any commutative ring R. The additional information has been sufficient to reconstruct X, up to R-completion from $\tilde{H}^*(X;R)$, but, once again, this important result has only been achieved for $R = \mathbb{F}_p$ or $\mathbb{Q}$ [9, 10].

Resolutions by products of E-M spaces over $\mathbb{F}_p$ coefficients have been used to solve the realisation problem of which unstable coalgebras over the Steenrod algebra can be realised as the cohomology $\mathcal{H}(\mathbb{F}_p)$-algebras of a space, $\tilde{H}^*(X;\mathbb{F}_p)$ [6]. This work is also dual to work done in homotopy using Stover's resolution but does not extend to $\mathbb{Z}$ as it requires (co)homology to consist of vector spaces rather than modules.

The attempt to Eckmann-Hilton dualise Stover's seminal construction has been achieved with considerable effort for integer coefficients. However, for $\mathbb{F}_p$ coefficients, construction (6) is quite straightforward and new results, such as the realisation problem, are commencing to be attained. This survey should suffice to convince the reader that any extension of results for $\mathbb{F}_p$ coefficients to $\mathbb{Z}$ coefficients are likely to be difficult to achieve.

8. Appendix: Unary integral cohomology operations

Table 1 shows results of the (Leray)-Serre spectral sequence calculation of the groups

$$\tilde{H}^m(K^n)\,, \qquad 2 \le m \le 14\,,\ \ 2 \le n \le 7$$

(extended from [32] to include $m = 14$). Angle brackets $\langle x\rangle$ indicate that the class x generates the summand. The cup product structure is given by the s.s. and written as juxtaposition so $b^2 = bb = b \cup b$. The generators of other summands have been tested for decomposability into compositions or cross-cap products using dimensional arguments or known relations. The generator

m \ n	2	3	4	5	6	7
14	$\mathbb{Z}\langle a^7\rangle$	0	$\mathbb{Z}/2\langle g^2\rangle$	$\mathbb{Z}/3\langle n\rangle \oplus \mathbb{Z}/5\langle o\rangle$	0	$\mathbb{Z}/2\langle t^2\rangle \oplus \mathbb{Z}/2\langle u\rangle$
13	0	$\mathbb{Z}/2\langle bd\rangle$	$\mathbb{Z}/5\langle j\rangle \oplus \mathbb{Z}/3\langle fh\rangle$	$\mathbb{Z}/2\langle k\Omega^{-1}g\rangle$	$\mathbb{Z}/2\langle s\rangle$	0
12	$\mathbb{Z}\langle a^6\rangle$	$\mathbb{Z}/2\langle b^4\rangle \oplus \mathbb{Z}/5\langle e\rangle$	$\mathbb{Z}\langle f^3\rangle$	$\mathbb{Z}/2\langle m\rangle$	$\mathbb{Z}\langle p^2\rangle$	
11	0	$\mathbb{Z}/3\langle bc\rangle$	$\mathbb{Z}/2\langle fg\rangle \oplus \mathbb{Z}/2\langle ?\rangle$	0	$\mathbb{Z}/2\langle q\rangle \oplus \mathbb{Z}/3\langle r\rangle$	
10	$\mathbb{Z}\langle a^5\rangle$	$\mathbb{Z}/2\langle d\rangle$	0	$\mathbb{Z}/2\langle k^2\rangle \oplus \mathbb{Z}/3\langle l\rangle$		
9	0	$\mathbb{Z}/2\langle b^3\rangle$	$\mathbb{Z}/3\langle h\rangle$	0		
8	$\mathbb{Z}\langle a^4\rangle$	$\mathbb{Z}/3\langle c\rangle$	$\mathbb{Z}\langle f^2\rangle$			
7	0	0	$\mathbb{Z}/2\langle g\rangle$			$\langle t\rangle$
6	$\mathbb{Z}\langle a^3\rangle$	$\mathbb{Z}/2\langle b^2\rangle$			$\langle p\rangle$	
5	0	0		$\langle k\rangle$		
4	$\mathbb{Z}\langle a^2\rangle$		$\langle f\rangle$			
3	0	$\langle b\rangle$				
2	$\mathbb{Z}\langle a\rangle$					

Table 1: The groups $\tilde{H}^m(K(\mathbb{Z}, n); \mathbb{Z})$

of one summand of $\tilde{H}^{11}(K^4)$ remains undetermined as either indecomposable or $f^2 \circ \Omega^{-4} g$, hence is given as $\langle ? \rangle$. The stable range, where

$$\langle \Omega x \rangle = \tilde{H}^m(K^n) \cong \tilde{H}^{m+1}(K^{n+1}) = \langle x \rangle \,, \qquad n \leq m \leq 2n-1 \,,$$

is left blank, although the fundamental class is denoted by a new letter rather than as $\Omega^{-i}a$, where Ω^{-1} is the transgression.

There are some notable differences to cohomology groups,

$$\tilde{H}^m(K(\mathbb{F}_p, n); \mathbb{F}_p) \,,$$

of Eilengerg-Mac Lane spaces over $\mathbb{F}_p$. We see that there are non-stable compositions for integer coefficients, moreover, $H^{11}(K^6)$ shows that there can be more than one stable composition of a given degree, possibly of different order.

$\tilde{H}^{14}(K^3) \cong 0$ would be a surprising result if we had coefficients in a field since in those coefficients cup product generates freely [19] whereas with the integers we have $b^2 c = 0$. Some relation is involved here forcing a cup product of non-trivial generators to be trivial. We may also expect elements created by compositions $\Omega^{-10} g \circ (bc)$, $\Omega^{-3} r \circ (b^3)$ and $\Omega^{-3} q \circ (b^3)$ within $\tilde{H}^{14}(K^3) \cong 0$. There is also the external cup product and “unknown” primary operation of Example (3.5) which could act by universal example on two copies of the fundamental class.

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