EXTRACTA MATHEMATICAE Vol. **32**, Núm. 1, 25 – 54 (2017)

Trace Inequalities of Lipschitz Type for Power Series of Operators on Hilbert Spaces

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Presented by Alfonso Montes Received May 23, 2016

Abstract: Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. We show, amongst other that, if $T, V \in \mathcal{B}_1(H)$, the Banach space of all trace operators on H, are such that $\|T\|_1, \|V\|_1 < R$, then $f(V)$, $f(T)$, $f'((1-t)T + tV) \in \mathcal{B}_1(H)$ for any $t \in [0,1]$ and

$$\operatorname{tr}[f(V)] - \operatorname{tr}[f(T)] = \int_0^1 \operatorname{tr}\left[(V-T)f'\big((1-t)T + tV\big)\right] \mathrm{d}t.$$

Several trace inequalities are established. Applications for some elementary functions of interest are also given.

Key words: Banach algebras of operators on Hilbert spaces, Power series, Lipschitz type inequalities, Jensen's type inequalities, Trace of operators, Hilbert-Schmidt norm.

AMS *Subject Class.* (2010): 47A63, 47A99.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a separable complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [4] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e., there is no constant $L > 0$ such that

$$\big\| \, |A| - |B| \, \big\| \le L \, \|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [17], [18] and Kato in [22], the following inequality holds

$$\| \, |A| - |B| \, \| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log \left(\frac{\|A\| + \|B\|}{\|A - B\|} \right) \right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^* C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\| \, |A| - |B| \, \|_{HS} \leq \sqrt{2} \, \|A - B\|_{HS} \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [4] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\| \, |A| - |B| \, \| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3 \right) \tag{1.3}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \qquad \text{and} \qquad a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2 .$$

In [3] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\| \tag{1.4}$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \geq a I_H > 0$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5], [19] and the references therein.

By the help of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $a_n \geq 0$, then $f_a = f$.

We notice that if

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n=\ln\frac{1}{1+z}\,,\qquad z\in D(0,1)\,;\tag{1.5}$$

$$g(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}=\cos z\,,\qquad z\in\mathbb{C}\,;$$

$$h(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=\sin z\,,\qquad z\in\mathbb{C}\,;$$

$$l(z)=\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{1+z}\,,\qquad z\in D(0,1)\,;$$

where $D(0,1)$ is the open disk centered in 0 and of radius 1, then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(z)=\sum_{n=1}^{\infty}\frac{1}{n!}z^n=\ln\frac{1}{1-z}\,,\qquad z\in D(0,1)\,;\tag{1.6}$$

$$g_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z\,,\qquad z\in\mathbb{C}\,;$$

$$h_a(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z\,,\qquad z\in\mathbb{C}\,;$$

$$l_a(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z}\,,\qquad z\in D(0,1)\,.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \,, \qquad z \in \mathbb{C}\,; \tag{1.7}$$

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}\,, \qquad z \in D(0,1)\,;$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!} z^{2n+1}\,, \qquad z \in D(0,1)\,;$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}\,, \qquad z \in D(0,1)\,;$$

$${}_2F_1(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n\,, \qquad \alpha,\beta,\gamma > 0\,,\; z \in D(0,1)\,;$$

where Γ is *Gamma function.*

We recall the following result that provides a quasi-Lipschitzian condition for functions defined by power series and operator norm $\|\cdot\|$ [14]:

Theorem 1. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then*

$$\|f(T) - f(V)\| \le f_a'\big(\max\{\|T\|, \|V\|\}\big)\, \|T - V\|\,. \tag{1.8}$$

If $\|T\|, \|V\| \le M < R$, then from (1.8) we have the simpler inequality

$$\|f(T) - f(V)\| \le f_a'(M)\, \|T - V\| \tag{1.9}$$

In the recent paper [13] we improved the inequality (1.8) as follows:

Theorem 2. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If $T, V \in \mathcal{B}(H)$ are such that $\|T\|, \|V\| < R$, then*

$$\|f(T) - f(V)\| \le \|T - V\| \int_0^1 f_a'\big(\|(1-t)T + tV\|\big)\, \mathrm{d}t\,. \tag{1.10}$$

In order to obtain similar results for the trace of bounded linear operators on complex infinite dimensional Hilbert spaces we need some preparations as follows.

2. Some preliminary facts on trace for operators

Let $\left(H, \langle\cdot,\cdot\rangle\right)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty\,. \tag{2.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{2.2}$$

showing that the definition (2.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{2.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|\,x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (2.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 3. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{2.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) *we have the inequalities*
$$\|A\| \leq \|A\|_2 \tag{2.5}$$
for any $A \in \mathcal{B}_2(H)$ and
$$\|AT\|_2\,,\ \|TA\|_2 \leq \|T\|\,\|A\|_2 \tag{2.6}$$
for any $A \in \mathcal{B}_2(H)$ and $T \in \mathcal{B}(H)$;

(iii) *$\mathcal{B}_2(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.,*
$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H)\,;$$

(iv) *$\mathcal{B}_{fin}(H)$, the space of operators of finite rank, is a dense subspace of $\mathcal{B}_2(H)$;*

(v) *$\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the algebra of compact operators on H.*

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if
$$\|A\|_1 := \sum_{i\in I} \langle |A|\, e_i, e_i\rangle < \infty\,. \tag{2.7}$$
The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If $A \in \mathcal{B}(H)$, then the following are equivalent:*

(i) *$A \in \mathcal{B}_1(H)$;*

(ii) *$|A|^{1/2} \in \mathcal{B}_2(H)$;*

(iii) *A (or $|A|$) is the product of two elements of $\mathcal{B}_2(H)$.*

The following properties are also well known:

Theorem 4. *With the above notations:*

(i) *We have*
$$\|A\|_1 = \|A^*\|_1 \quad \text{and} \quad \|A\|_2 \leq \|A\|_1 \tag{2.8}$$
for any $A \in \mathcal{B}_1(H)$.

(ii) *$\mathcal{B}_1(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.,*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H)\,.$$

(iii) *We have $\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H)$.*

(iv) *We have*

$$\|A\|_1 = \sup\left\{ |\langle A, B\rangle_2| \,:\, B \in \mathcal{B}_2(H),\ \|B\| \le 1\right\}.$$

(v) *$(\mathcal{B}_1(H), \|\cdot\|_1)$ is a Banach space.*

(vi) *We have the following isometric isomorphisms*

$$\mathcal{B}_1(H) \cong K(H)^* \quad \text{and} \quad \mathcal{B}_1(H)^* \cong \mathcal{B}(H)\,,$$

where $K(H)^*$ is the dual space of $K(H)$ and $\mathcal{B}_1(H)^*$ is the dual space of $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle\,, \tag{2.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 5. *We have:*

(i) *if $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}\,; \tag{2.10}$$

(ii) *if $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \quad \text{and} \quad |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|\,; \tag{2.11}$$

(iii) *$\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;*

(iv) *if A, $B \in \mathcal{B}_2(H)$ then AB, $BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;*

(v) *$\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.*

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*)\,,$$
$$\|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any A, $B \in \mathcal{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [28]

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{2.12}$$

where $\alpha \in (0,1)$ and $A, B \in \mathcal{B}(H)$ with $|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{2.13}$$

with $A, B \in \mathcal{B}_2(H)$.

If $A \ge 0$ and $P \in \mathcal{B}_1(H)$ with $P \ge 0$, then

$$0 \le \operatorname{tr}(PA) \le \|A\| \operatorname{tr}(P)\,. \tag{2.14}$$

Indeed, since $A \ge 0$, then $\langle Ax, x\rangle \ge 0$ for any $x \in H$. If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, then

$$0 \le \left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle \le \|A\|\left\|P^{1/2}e_i\right\|^2 = \|A\|\langle Pe_i, e_i\rangle$$

for any $i \in I$. Summing over $i \in I$ we get

$$0 \le \sum_{i\in I}\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle \le \|A\|\sum_{i\in I}\langle Pe_i, e_i\rangle = \|A\|\operatorname{tr}(P)\,,$$

and since

$$\begin{aligned}\sum_{i\in I}\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle &= \sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i, e_i\right\rangle \\ &= \operatorname{tr}\left(P^{1/2}AP^{1/2}\right) = \operatorname{tr}(PA)\,,\end{aligned}$$

we obtain the desired result (2.14).

This obviously imply the fact that, if A and B are selfadjoint operators with $A \le B$ and $P \in \mathcal{B}_1(H)$ with $P \ge 0$, then

$$\operatorname{tr}(PA) \le \operatorname{tr}(PB)\,. \tag{2.15}$$

Now, if A is a selfadjoint operator, then we know that

$$|\langle Ax, x\rangle| \leq \langle |A|\, x, x\rangle \qquad \text{for any } x \in H.$$

This inequality follows from Jensen's inequality for the convex function $f(t) = |t|$ defined on a closed interval containing the spectrum of A.

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, then

$$\begin{aligned} |\operatorname{tr}(PA)| &= \left|\sum_{i\in I}\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle\right| \leq \sum_{i\in I}\left|\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle\right| \\ &\leq \sum_{i\in I}\left\langle |A|\, P^{1/2}e_i, P^{1/2}e_i\right\rangle = \operatorname{tr}(P\,|A|)\,, \end{aligned} \tag{2.16}$$

for any A a selfadjoint operator and $P \in \mathcal{B}_1(H)$ with $P \geq 0$.

For the theory of trace functionals and their applications the reader is referred to [31].

For some classical trace inequalities see [9], [11], [26] and [35], which are continuations of the work of Bellman [7]. For related works the reader can refer to [1], [8], [9], [20], [23], [24], [25], [29] and [32].

3. Trace inequalities

We have the following representation result:

Theorem 6. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If $T, V \in \mathcal{B}_1(H)$ are such that $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$, then $f(V), f(T), f'\big((1-t)T + tV\big) \in \mathcal{B}_1(H)$ for any $t \in [0,1]$ and*

$$\operatorname{tr}[f(V)] - \operatorname{tr}[f(T)] = \int_0^1 \operatorname{tr}\left[(V-T)f'\big((1-t)T+tV\big)\right] \mathrm{d}t\,. \tag{3.1}$$

Proof. We use the identity (see for instance [6, p. 254])

$$A^n - B^n = \sum_{j=0}^{n-1} A^{n-1-j}(A-B)B^j \tag{3.2}$$

that holds for any $A, B \in \mathcal{B}(H)$ and $n \geq 1$.

For $T, V \in \mathcal{B}(H)$ we consider the function $\varphi : [0,1] \to \mathcal{B}(H)$ defined by $\varphi(t) = [(1-t)T + tV]^n$. For $t \in (0,1)$ and $\varepsilon \neq 0$ with $t + \varepsilon \in (0,1)$ we have from (3.2) that

$$\begin{aligned}\varphi(t+\varepsilon) - \varphi(t) &= [(1-t-\varepsilon)T + (t+\varepsilon)V]^n - [(1-t)T + tV]^n \\ &= \varepsilon \sum_{j=0}^{n-1} [(1-t-\varepsilon)T + (t+\varepsilon)V]^{n-1-j} (V-T) [(1-t)T + tV]^j .\end{aligned}$$

Dividing with $\varepsilon \neq 0$ and taking the limit over $\varepsilon \to 0$ we have in the norm topology of $\mathcal{B}$ that

$$\begin{aligned}\varphi'(t) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\varphi(t+\varepsilon) - \varphi(t)] \\ &= \sum_{j=0}^{n-1} [(1-t)T + tV]^{n-1-j} (V-T) [(1-t)T + tV]^j .\end{aligned} \tag{3.3}$$

Integrating on $[0,1]$ we get from (3.3) that

$$\int_0^1 \varphi'(t)\,\mathrm{d}t = \sum_{j=0}^{n-1} \int_0^1 [(1-t)T + tV]^{n-1-j} (V-T) [(1-t)T + tV]^j \,\mathrm{d}t ,$$

and since

$$\int_0^1 \varphi'(t)\,\mathrm{d}t = \varphi(1) - \varphi(0) = V^n - T^n,$$

then we get the following equality of interest in itself

$$V^n - T^n = \sum_{j=0}^{n-1} \int_0^1 [(1-t)T + tV]^{n-1-j} (V-T) [(1-t)T + tV]^j \,\mathrm{d}t \tag{3.4}$$

for any $T, V \in \mathcal{B}(H)$ and $n \geq 1$.

If $T, V \in \mathcal{B}_1(H)$ and we take the trace in (3.4) we get

$$\begin{aligned}
&\operatorname{tr}(V^n) - \operatorname{tr}(T^n) \\
&\quad = \sum_{j=0}^{n-1} \int_0^1 \operatorname{tr}\left([(1-t)T + tV]^{n-1-j} (V-T) [(1-t)T + tV]^j \right) \mathrm{d}t \\
&\quad = \sum_{j=0}^{n-1} \int_0^1 \operatorname{tr}\left([(1-t)T + tV]^{n-1} (V-T) \right) \mathrm{d}t \qquad (3.5) \\
&\quad = n \int_0^1 \operatorname{tr}\left([(1-t)T + tV]^{n-1} (V-T) \right) \mathrm{d}t \\
&\quad = n \int_0^1 \operatorname{tr}\left((V-T) [(1-t)T + tV]^{n-1} \right) \mathrm{d}t
\end{aligned}$$

for any $n \geq 1$.

Let $m \geq 1$. Then by (3.5) we have have

$$\begin{aligned}
\operatorname{tr}\left(\sum_{n=0}^{m} a_n V^n \right) - \operatorname{tr}\left(\sum_{n=0}^{m} a_n T^n \right) &= \sum_{n=0}^{m} a_n \left[\operatorname{tr}\left(V^n\right) - \operatorname{tr}\left(T^n\right) \right] \\
&= \sum_{n=1}^{m} a_n \left[\operatorname{tr}\left(V^n\right) - \operatorname{tr}\left(T^n\right) \right] \qquad (3.6) \\
&= \sum_{n=1}^{m} n a_n \int_0^1 \operatorname{tr}\left((V-T) [(1-t)T + tV]^{n-1} \right) \mathrm{d}t \\
&= \int_0^1 \operatorname{tr}\left((V-T) \sum_{n=1}^{m} n a_n [(1-t)T + tV]^{n-1} \right) \mathrm{d}t
\end{aligned}$$

for any $T, V \in \mathcal{B}_1(H)$.

Since $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$ with $T, V \in \mathcal{B}_1(H)$ then the series $\sum_{n=0}^{\infty} a_n V^n$, $\sum_{n=0}^{\infty} a_n T^n$ and $\sum_{n=1}^{\infty} n a_n [(1-t)T + tV]^{n-1}$ are convergent in $\mathcal{B}_1(H)$ and

$$\sum_{n=0}^{\infty} a_n V^n = f(V), \qquad \sum_{n=0}^{\infty} a_n T^n = f(T)$$

and

$$\sum_{n=1}^{\infty} n a_n \, (1-t)T + tV]^{n-1} = f'\big((1-t)T + tV\big)$$

where $t \in [0,1]$. Moreover, we have

$$f(V)\,,\ f(T)\,,\ f'\big((1-t)T+tV\big)\ \in \mathcal{B}_1(H)$$

for any $t \in [0,1]$.

By taking the limit over $m \to \infty$ in (3.6) we get the desired result (3.1). ∎

In addition to the power identity (3.5), for any $T, V \in \mathcal{B}_1(H)$ we have other equalities as follows

$$\operatorname{tr}[\exp(V)] - \operatorname{tr}[\exp(T)] = \int_0^1 \operatorname{tr}\big((V-T)\exp((1-t)T+tV)\big)\,\mathrm{d}t\,, \tag{3.7}$$

$$\operatorname{tr}[\sin(V)] - \operatorname{tr}[\sin(T)] = \int_0^1 \operatorname{tr}\big((V-T)\cos((1-t)T+tV)\big)\,\mathrm{d}t\,, \tag{3.8}$$

$$\operatorname{tr}[\sinh(V)] - \operatorname{tr}[\sinh(T)] = \int_0^1 \operatorname{tr}\big((V-T)\cosh((1-t)T+tV)\big)\,\mathrm{d}t\,. \tag{3.9}$$

If $T, V \in \mathcal{B}_1(H)$ with $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < 1$ then

$$\begin{aligned} &\operatorname{tr}\left[(1_H - V)^{-1}\right] - \operatorname{tr}\left[(1_H - T)^{-1}\right] \\ &\qquad = \int_0^1 \operatorname{tr}\left((V-T)(1_H-(1-t)T-tV)^{-2}\right)\mathrm{d}t\,, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\operatorname{tr}\left[\ln(1_H - V)^{-1}\right]\mathrm{s} - \operatorname{tr}\left[\ln(1_H - T)^{-1}\right] \\ &\qquad = \int_0^1 \operatorname{tr}\left((V-T)(1_H-(1-t)T-tV)^{-1}\right)\mathrm{d}t\,. \end{aligned} \tag{3.11}$$

We have the following result:

Corollary 1. *With the assumptions in Theorem* 6 *we have the inequalities*

$$\begin{aligned}\left|\operatorname{tr}[f(V)]-\operatorname{tr}[f(T)]\right| &\le \min\left\{\|V-T\|\int_0^1\left\|f'((1-t)T+tV)\right\|_1\mathrm{d}t\,,\right. \\ &\qquad\left.\|V-T\|_1\int_0^1\left\|f'((1-t)T+tV)\right\|\mathrm{d}t\right\} \\ &\le \min\left\{\|V-T\|\int_0^1 f_a'\big(\|(1-t)T+tV\|_1\big)\,\mathrm{d}t,\right. \\ &\qquad\left.\|V-T\|_1\int_0^1 f_a'\big(\|(1-t)T+tV\|\big)\,\mathrm{d}t\right\},\end{aligned} \tag{3.12}$$

where $\|\cdot\|$ is the operator norm and $\|\cdot\|_1$ is the 1-norm introduced for trace class operators.

Proof. From (3.1), we have by taking the modulus

$$\left|\operatorname{tr}[f(V)]-\operatorname{tr}[f(T)]\right| \le \int_0^1\left|\operatorname{tr}\big((V-T)f'((1-t)T+tV)\big)\right|\mathrm{d}t\,. \tag{3.13}$$

Utilising the inequality (2.11) twice, for any $t\in[0,1]$ we get

$$\begin{aligned}\left|\operatorname{tr}\big((V-T)f'\big((1-t)T+tV\big)\big)\right| &\le \|V-T\|\left\|f'\big((1-t)T+tV\big)\right\|_1, \\ \left|\operatorname{tr}\big((V-T)f'\big((1-t)T+tV\big)\big)\right| &\le \|V-T\|_1\left\|f'\big((1-t)T+tV\big)\right\|.\end{aligned}$$

By integrating these inequalities, we get the first part of (3.12).

We have, by the use of $\|\cdot\|_1$ properties that

$$\begin{aligned}\left\|f'\big((1-t)T+tV\big)\right\|_1 &= \left\|\sum_{n=1}^{\infty} na_n\left[(1-t)T+tV\right]^{n-1}\right\|_1 \\ &\le \sum_{n=1}^{\infty} n\left|a_n\right|\left\|\left[(1-t)T+tV\right]^{n-1}\right\|_1 \\ &\le \sum_{n=1}^{\infty} n\left|a_n\right|\left\|(1-t)T+tV\right\|_1^{n-1} \\ &= f_a'\big(\|(1-t)T+tV\|_1\big)\end{aligned}$$

for any $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$.

This proves the first part of the second inequality.

Since $\|X\| \leq \|X\|_1$ for any $X \in \mathcal{B}_1(H)$ then $\|(1-t)T + tV\| < R$ for any $T, V \in \mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1 < R$ which shows that $f_a'\big(\,\|(1-t)T + tV\|\,\big)$ is well defined.

The second part of the second inequality follows in a similar way and the details are omitted. ∎

Remark 1. We observe that f_a' is monotonic nondecreasing and convex on the interval $[0, R)$ and since the function $\psi(t) := \|(1-t)T + tV\|$ is convex on $[0,1]$ we have that $f_a' \circ \psi$ is also convex on $[0,1]$. Utilising the Hermite-Hadamard inequality for convex functions (see for instance [16, p. 2]) we have the sequence of inequalities

$$\begin{aligned}\int_0^1 f_a'\big(\,\|(1-t)T + tV\|\,\big)\,\mathrm{d}t &\leq \frac{1}{2}\left[f_a'\left(\left\|\frac{T+V}{2}\right\|\right) + \frac{f_a'(\|T\|) + f_a'(\|V\|)}{2}\right] \\ &\leq \frac{1}{2}\left[f_a'(\|T\|) + f_a'(\|V\|)\right] \\ &\leq \max\left\{f_a'(\|T\|), f_a'(\|V\|)\right\}. \end{aligned} \tag{3.14}$$

We also have

$$\begin{aligned}\int_0^1 f_a'\big(\,\|(1-t)T + tV\|\,\big)\,\mathrm{d}t &\leq \int_0^1 f_a'\big((1-t)\,\|T\| + t\,\|V\|\,\big)\,\mathrm{d}t \\ &\leq \frac{1}{2}\left[f_a'\left(\frac{\|T\| + \|V\|}{2}\right) + \frac{f_a'(\|T\|) + f_a'(\|V\|)}{2}\right] \\ &\leq \frac{1}{2}\left[f_a'(\|T\|) + f_a'(\|V\|)\right] \\ &\leq \max\left\{f_a'(\|T\|), f_a'(\|V\|)\right\}. \end{aligned} \tag{3.15}$$

We observe that if $\|V\| \neq \|T\|$, then by the change of variable $s = (1-t)\,\|T\| + t\,\|V\|$ we have

$$\begin{aligned}\int_0^1 f_a'\big((1-t)\,\|T\| + t\,\|V\|\,\big)\,\mathrm{d}t &= \frac{1}{\|V\| - \|T\|}\int_{\|T\|}^{\|V\|} f_a'(s)\,\mathrm{d}s \\ &= \frac{f_a(\|V\|) - f_a(\|T\|)}{\|V\| - \|T\|}. \end{aligned}$$

If $\|V\| = \|T\|$, then

$$\int_0^1 f_a'\big((1-t)\,\|T\| + t\,\|V\|\,\big)\,\mathrm{d}t = f_a'(\|T\|)\,.$$

Utilising these observations we then get the following divided difference inequality for $T \neq V$

$$\int_0^1 f_a'\big(\,\|(1-t)T + tV\|\,\big)\,\mathrm{d}t \;\le \begin{cases} \frac{f_a(\|V\|)-f_a(\|T\|)}{\|V\|-\|T\|} & \text{if } \|V\| \neq \|T\|\,, \\ f_a'(\|T\|) & \text{if } \|V\| = \|T\|\,. \end{cases} \tag{3.16}$$

Similar comments apply for the 1-norm $\|\cdot\|_1$ when $T, V \in \mathcal{B}_1(H)$.

If we use the first part in the inequalities (3.12) and the above remarks, then we get the following string of inequalities

$$\begin{aligned} \big|\,\mathrm{tr}\,[f(V)] \;-\; \mathrm{tr}\,[f(T)]\,\big| &\le \|V-T\| \int_0^1 \big\|f'\big((1-t)T+tV\big)\big\|_1\,\mathrm{d}t \\ &\le \|V-T\| \int_0^1 f_a'\big(\,\|(1-t)T+tV\|_1\,\big)\,\mathrm{d}t \qquad\qquad (3.17) \\ &\le \|V-T\| \times \begin{cases} \frac{1}{2}\left[f_a'\Big(\,\big\|\frac{T+V}{2}\big\|_1\Big) + \frac{f_a'(\|T\|_1)+f_a'(\|V\|_1)}{2}\right], \\ \begin{cases} \frac{f_a(\|V\|_1)-f_a(\|T\|_1)}{\|V\|_1-\|T\|_1} & \text{if } \|V\|_1 \neq \|T\|_1\,, \\ f_a'(\|T\|_1) & \text{if } \|V\|_1 = \|T\|_1\,, \end{cases} \end{cases} \\ &\le \frac{1}{2}\,\|V-T\|\,\big[f_a'(\|T\|_1) + f_a'(\|V\|_1)\big] \\ &\le \|V-T\| \max\big\{f_a'(\|T\|_1), f_a'(\|V\|_1)\big\} \end{aligned}$$

provided $T, V \in \mathcal{B}_1(H)$ with $\|T\|_1\,, \|V\|_1 < R$.

If $\|T\|_1\,, \|V\|_1 \le M < R$, then we have from (3.17) the simple inequality

$$\big|\,\mathrm{tr}\,[f(V)] - \mathrm{tr}\,[f(T)]\,\big| \le \|V-T\|\, f_a'(M)\,.$$

A similar sequence of inequalities can also be stated by swapping the norm $\|\cdot\|$ with $\|\cdot\|_1$ in (3.17). We omit the details.

If we use the inequality (3.17) for the exponential function, then for any $T, V \in \mathcal{B}_1(H)$ we have the inequalities

$$
\begin{aligned}
\left| \operatorname{tr}[\exp(V)] - \operatorname{tr}[\exp(T)] \right| &\le \|V-T\| \int_0^1 \left\| \exp\left((1-t)T + tV\right) \right\|_1 \mathrm{d}t \\
&\le \|V-T\| \int_0^1 \exp\left(\|(1-t)T + tV\|_1 \right) \mathrm{d}t \qquad (3.18) \\
&\le \|V-T\| \times \begin{cases} \frac{1}{2}\left[\exp\left(\left\| \frac{T+V}{2} \right\|_1 \right) + \frac{\exp(\|T\|_1) + \exp(\|V\|_1)}{2} \right], \\ \begin{cases} \frac{\exp(\|V\|_1) - \exp(\|T\|_1)}{\|V\|_1 - \|T\|_1} & \text{if } \|V\|_1 \ne \|T\|_1, \\ \exp(\|T\|_1) & \text{if } \|V\|_1 = \|T\|_1, \end{cases} \end{cases} \\
&\le \frac{1}{2} \|V-T\| \left[\exp(\|T\|_1) + \exp(\|V\|_1) \right] \\
&\le \|V-T\| \max\left\{ \exp(\|T\|_1), \exp(\|V\|_1) \right\}.
\end{aligned}
$$

If $\|T\|_1, \|V\|_1 < 1$, then we have the inequalities

$$
\begin{aligned}
&\left| \operatorname{tr}\left[\ln(1_H - V)^{-1} \right] - \operatorname{tr}\left[\ln(1_H - T)^{-1} \right] \right| \qquad (3.19) \\
&\le \|V-T\| \int_0^1 \left\| (1_H - (1-t)T - tV)^{-1} \right\|_1 \mathrm{d}t \\
&\le \|V-T\| \int_0^1 (1 - \|(1-t)T + tV\|_1)^{-1} \mathrm{d}t \\
&\le \|V-T\| \times \begin{cases} \frac{1}{2}\left[\left(1 - \left\| \frac{T+V}{2} \right\|_1 \right)^{-1} + \frac{(1-\|T\|_1)^{-1} + (1-\|V\|_1)^{-1}}{2} \right], \\ \begin{cases} \frac{\ln(1-\|V\|_1)^{-1} - \ln(1-\|T\|_1)^{-1}}{\|V\|_1 - \|T\|_1} & \text{if } \|V\|_1 \ne \|T\|_1, \\ (1-\|T\|_1)^{-1} & \text{if } \|V\|_1 = \|T\|_1, \end{cases} \end{cases} \\
&\le \frac{1}{2} \|V-T\| \left[(1-\|T\|_1)^{-1} + (1-\|V\|_1)^{-1} \right] \\
&\le \|V-T\| \max\left\{ (1-\|T\|_1)^{-1}, (1-\|V\|_1)^{-1} \right\}.
\end{aligned}
$$

The following result for the Hilbert-Schmidt norm $\|\cdot\|_2$ also holds:

THEOREM 7. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If $T, V \in \mathcal{B}_2(H)$ are* *such that* $\operatorname{tr}(|T|^2), \operatorname{tr}(|V|^2) < R^2$, *then* $f(V), f(T), f'((1-t)T+tV) \in \mathcal{B}_2(H)$ *for any* $t \in [0,1]$ *and*

$$\operatorname{tr}[f(V)] - \operatorname{tr}[f(T)] = \int_0^1 \operatorname{tr}((V-T)f'\big((1-t)T+tV\big))\,\mathrm{d}t\,. \tag{3.20}$$

Moreover, we have the inequalities

$$\begin{aligned}
\big|\operatorname{tr}[f(V)] - \operatorname{tr}[f(T)]\big| &\le \|V-T\|_2 \int_0^1 \big\|f'\big((1-t)T+tV\big)\big\|_2\,\mathrm{d}t\,,\\
&\le \|V-T\|_2 \int_0^1 f_a'\big(\,\|(1-t)T+tV\|_2\,\big)\,\mathrm{d}t \qquad (3.21)\\
&\le \|V-T\|_2 \times \begin{cases} \frac{1}{2}\left[f_a'\Big(\left\|\frac{T+V}{2}\right\|_2\Big) + \frac{f_a'(\|T\|_2)+f_a'(\|V\|_2)}{2}\right], \\ \begin{cases} \frac{f_a(\|V\|_2)-f_a(\|T\|_2)}{\|V\|_2-\|T\|_2} & \text{if } \ \|V\|_2 \ne \|T\|_2\,, \\ f_a'(\|T\|_2) & \text{if } \ \|V\|_2 = \|T\|_2\,, \end{cases} \end{cases}\\
&\le \frac{1}{2}\|V-T\|_2\left[f_a'(\|T\|_2)+f_a'(\|V\|_2)\right]\\
&\le \|V-T\|_2 \max\left\{f_a'(\|T\|_2), f_a'(\|V\|_2)\right\}.
\end{aligned}$$

Proof. The proof of the first part of the theorem follows in a similar manner to the one from Theorem 6.

Taking the modulus in (3.20) and using the Schwarz inequality for trace (2.13) we have

$$\begin{aligned}
|\operatorname{tr}[f(V)] - \operatorname{tr}[f(T)]| &\le \int_0^1 \big|\operatorname{tr}((V-T)f'\big((1-t)T+tV\big)\big|\,\mathrm{d}t \qquad (3.22)\\
&\le \int_0^1 \|V-T\|_2 \big\|f'\big((1-t)T+tV\big)\big\|_2\,\mathrm{d}t\,.
\end{aligned}$$

The rest follows in a similar manner as in the case of 1-norm and the details are omitted. ∎

We notice that similar examples to (3.18) and (3.19) may be stated where both norms $\|\cdot\|$ and $\|\cdot\|_1$ are replaced by $\|\cdot\|_2$.

We also observe that, if $T, V \in \mathcal{B}_2(H)$ with $\|T\|_2, \|V\|_2 \le K < R$, then we have from (3.17) the simple inequality

$$\big|\operatorname{tr}[f(V)] - \operatorname{tr}[f(T)]\big| \le \|V-T\|_2\, f_a'(K)\,.$$

4. Norm inequalities

We have the following norm inequalities:

Theorem 8. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$.*

(i) *If $T, V \in \mathcal{B}_1(H)$ are such that $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$, then we have the norm inequalities*

$$\|f(V)-f(T)\|_1 \le \begin{cases} \|V-T\|_1 \int_0^1 f_a'\big(\|(1-t)T+tV\|\big)\,dt\,, \\ \|V-T\| \int_0^1 f_a'\big(\|(1-t)T+tV\|_1\big)\,dt\,. \end{cases} \tag{4.1}$$

(ii) *If $T, V \in \mathcal{B}_2(H)$ are such that $\operatorname{tr}(|T|^2), \operatorname{tr}(|V|^2) < R^2$, then we also have the norm inequalities*

$$\|f(V)-f(T)\|_2 \le \begin{cases} \|V-T\|_2 \int_0^1 f_a'\big(\|(1-t)T+tV\|\big)\,dt\,, \\ \|V-T\| \int_0^1 f_a'\big(\|(1-t)T+tV\|_2\big)\,dt\,. \end{cases} \tag{4.2}$$

Proof. We use the equality

$$V^n - T^n = \sum_{j=0}^{n-1} \int_0^1 [(1-t)T+tV]^{n-1-j} (V-T) [(1-t)T+tV]^j \,dt\,, \tag{4.3}$$

for any $T, V \in \mathcal{B}(H)$ and $n \ge 1$.

(i) If $T, V \in \mathcal{B}_1(H)$ are such that $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$, then by taking the $\|\cdot\|_1$ norm and using its properties, for any $n \ge 1$ we have successively

$$\begin{aligned} \|V^n - T^n\|_1 &\le \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)T+tV]^{n-1-j} (V-T) [(1-t)T+tV]^j \right\|_1 dt \\ &\le \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)T+tV]^{n-1-j} (V-T) \right\|_1 \left\| [(1-t)T+tV]^j \right\| dt \\ &\le \sum_{j=0}^{n-1} \int_0^1 \|V-T\|_1 \left\| [(1-t)T+tV]^{n-1-j} \right\| \left\| [(1-t)T+tV]^j \right\| dt \end{aligned}$$

$$\leq \|V-T\|_1 \sum_{j=0}^{n-1} \|(1-t)T+tV\|^{n-1-j} \|(1-t)T+tV\|^{j} \,\mathrm{d}t$$

$$= n \|V-T\|_1 \int_0^1 \|(1-t)T+tV\|^{n-1} \,\mathrm{d}t\,. \tag{4.4}$$

Let $m \geq 1$. By (4.4) we have

$$\begin{aligned}
\left\| \sum_{n=0}^{m} a_n V^n - \sum_{n=0}^{m} a_n T^n \right\|_1 &= \left\| \sum_{n=1}^{m} a_n (V^n - T^n) \right\|_1 \qquad (4.5)\\
&\leq \sum_{n=1}^{m} |a_n| \, \|V^n - T^n\|_1 \\
&\leq \|V-T\|_1 \sum_{n=1}^{m} |a_n| \, n \int_0^1 \|(1-t)T+tV\|^{n-1} \,\mathrm{d}t \\
&= \|V-T\|_1 \int_0^1 \left(\sum_{n=1}^{m} n \, |a_n| \, \|(1-t)T+tV\|^{n-1} \right) \mathrm{d}t\,.
\end{aligned}$$

Also, we observe that

$$\begin{aligned}
\|(1-t)T+tV\| &\leq \|(1-t)T+tV\|_1 \\
&\leq (1-t)\|T\|_1 + t\|V\|_1 < R
\end{aligned}$$

for any $t \in [0,1]$, which implies that the series $\sum_{n=1}^{\infty} n\,|a_n|\, \|(1-t)T+tV\|^{n-1}$ is convergent and

$$\sum_{n=1}^{\infty} n\,|a_n|\,\|(1-t)T+tV\|^{n-1} = f_a'\big(\|(1-t)T+tV\|\big)$$

for any $t \in [0,1]$.

Since the series $\sum_{n=0}^{\infty} a_n V^n$ and $\sum_{n=0}^{\infty} a_n T^n$ are convergent in $\big(\mathcal{B}_1(H), \|\cdot\|_1\big)$, then by letting $m \to \infty$ in the inequality (4.5) we get the first inequality in (4.1).

For any $n \geq 1$ we also have

$$
\begin{aligned}
\|V^n - T^n\|_1 &\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)T + tV]^{n-1-j} (V-T) [(1-t)T + tV]^j \right\|_1 \mathrm{d}t \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \left\| [(1-t)T + tV]^{n-1-j} (V-T) \right\| \left\| [(1-t)T + tV]^j \right\|_1 \mathrm{d}t \\
&\leq \sum_{j=0}^{n-1} \int_0^1 \|V-T\| \left\| [(1-t)T + tV]^{n-1-j} \right\| \left\| [(1-t)T + tV]^j \right\|_1 \mathrm{d}t \\
&\leq \|V-T\| \sum_{j=0}^{n-1} \|(1-t)T + tV\|^{n-1-j} \|(1-t)T + tV\|_1^j \,\mathrm{d}t \\
&\leq \|V-T\| \sum_{j=0}^{n-1} \|(1-t)T + tV\|_1^{n-1-j} \|(1-t)T + tV\|_1^j \,\mathrm{d}t \\
&= n \|V-T\| \int_0^1 \|(1-t)T + tV\|_1^{n-1} \,\mathrm{d}t \,,
\end{aligned}
$$

which by a similar argument produces the second inequality in (4.1).

(ii) Follows in a similar way by utilizing the inequality $\|TA\|_2 \leq \|T\| \|A\|_2$ that holds for $T \in \mathcal{B}(H)$ and $A \in \mathcal{B}_2(H)$. The details are omitted. ∎

Remark 2. From the first inequality in (4.1) we have the sequence of inequalities

$$
\begin{aligned}
\|f(V) - f(T)\|_1 &\leq \|V-T\|_1 \int_0^1 f_a'\big(\|(1-t)T + tV\| \big) \,\mathrm{d}t \qquad (4.6) \\
&\leq \|V-T\|_1 \times \begin{cases} \frac{1}{2}\left[f_a'\left(\left\|\frac{T+V}{2}\right\|\right) + \frac{f_a'(\|T\|) + f_a'(\|V\|)}{2} \right], \\ \begin{cases} \frac{f_a(\|V\|) - f_a(\|T\|)}{\|V\| - \|T\|} & \text{if } \|V\| \neq \|T\|, \\ f_a'(\|T\|) & \text{if } \|V\| = \|T\|, \end{cases} \end{cases} \\
&\leq \frac{1}{2} \|V-T\|_1 \left[f_a'(\|T\|) + f_a'(\|V\|) \right] \\
&\leq \|V-T\|_1 \max\left\{ f_a'(\|T\|), f_a'(\|V\|) \right\}
\end{aligned}
$$

for $T, V \in \mathcal{B}_1(H)$ such that $\operatorname{tr}(|T|), \operatorname{tr}(|V|) < R$ and a similar result by swapping in the right hand side of (4.6) the norm $\|\cdot\|$ with $\|\cdot\|_1$. In particular, if $\operatorname{tr}(|T|), \operatorname{tr}(|V|) \le M < R$, then we have the simpler inequality

$$\|f(V) - f(T)\|_1 \le f_a'(M) \|V - T\|_1 . \tag{4.7}$$

If $T, V \in \mathcal{B}_2(H)$ are such that $\operatorname{tr}(|T|^2), \operatorname{tr}(|V|^2) < R^2$, then we have the norm inequalities

$$\begin{aligned} \|f(V) - f(T)\|_2 &\le \|V - T\|_2 \int_0^1 f_a'\big(\|(1-t)T + tV\| \big)\, \mathrm{d}t \qquad (4.8)\\ &\le \|V - T\|_2 \times \begin{cases} \frac{1}{2}\left[f_a'\left(\left\|\frac{T+V}{2}\right\|\right) + \frac{f_a'(\|T\|) + f_a'(\|V\|)}{2} \right], \\ \begin{cases} \frac{f_a(\|V\|) - f_a(\|T\|)}{\|V\| - \|T\|} & \text{if } \|V\| \ne \|T\|, \\ f_a'(\|T\|) & \text{if } \|V\| = \|T\|, \end{cases} \end{cases} \\ &\le \frac{1}{2} \|V - T\|_2 \left[f_a'(\|T\|) + f_a'(\|V\|) \right] \\ &\le \|V - T\|_2 \max\left\{ f_a'(\|T\|), f_a'(\|V\|) \right\} \end{aligned}$$

and a similar result by swapping in the right hand side of (4.6) the norm $\|\cdot\|$ with $\|\cdot\|_2$. In particular, if $\operatorname{tr}(|T|^2), \operatorname{tr}(|V|^2) \le K^2 < R^2$, then we have the simpler inequality

$$\|f(V) - f(T)\|_2 \le f_a'(K) \|V - T\|_2 . \tag{4.9}$$

5. Applications for Jensen's difference

We have the following representation:

Lemma 1. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If either $T, V \in \mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1 < R$, or $T, V \in \mathcal{B}_2(H)$ with $\|T\|_2, \|V\|_2 < R$ then $f(V)$, $f(T)$, $f\big(\frac{V+T}{2}\big) \in \mathcal{B}_1(H)$ or $f(V)$, $f(T)$, $f\big(\frac{V+T}{2}\big) \in \mathcal{B}_2(H)$, respectively and*

$$\begin{aligned} &\frac{\operatorname{tr}[f(V)] + \operatorname{tr}[f(T)]}{2} - \operatorname{tr}\left[f\left(\frac{V+T}{2} \right) \right] \qquad (5.1)\\ &= \frac{1}{4} \int_0^1 \operatorname{tr}\left((V - T) \left[f'\left((1-t)\frac{V+T}{2} + tV \right) - f'\left((1-t)\frac{V+T}{2} + tT \right) \right] \right) \mathrm{d}t . \end{aligned}$$

Proof. The first part of the theorem follows from Theorem 6.

From the identity (3.1), for $T, V \in \mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1 < R$ we have

$$\begin{aligned} \operatorname{tr}[f(V)] - \operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right] & \qquad\qquad (5.2) \\ &= \int_0^1 \operatorname{tr}\left[\left(V - \frac{V+T}{2}\right) f'\left((1-t)\frac{V+T}{2} + tV\right)\right] \mathrm{d}t \\ &= \frac{1}{2}\int_0^1 \operatorname{tr}\left[(V-T) f'\left((1-t)\frac{V+T}{2} + tV\right)\right] \mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr}[f(T)] - \operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right] & \qquad\qquad (5.3) \\ &= \int_0^1 \operatorname{tr}\left[\left(T - \frac{V+T}{2}\right) f'\left((1-t)\frac{V+T}{2} + tT\right)\right] \mathrm{d}t \\ &= \frac{1}{2}\int_0^1 \operatorname{tr}\left[(T-V) f'\left((1-t)\frac{V+T}{2} + tT\right)\right] \mathrm{d}t \\ &= -\frac{1}{2}\int_0^1 \operatorname{tr}\left[(V-T) f'\left((1-t)\frac{V+T}{2} + tT\right)\right] \mathrm{d}t\,. \end{aligned}$$

If we add the above inequalities (5.2) and (5.3) and divide by 2 we get the desired result (5.1). ∎

Theorem 9. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If $T, V \in \mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1 < R$, then*

$$\begin{aligned} \left| \frac{\operatorname{tr}[f(V)] + \operatorname{tr}[f(T)]}{2} - \operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right| & \qquad\qquad (5.4) \\ &\le \frac{1}{2}\|V-T\|^2 \int_0^1 \left|t - \frac{1}{2}\right| f_a''\left(\|(1-t)T + tV\|_1\right) \mathrm{d}t \\ &\le \frac{1}{24}\|V-T\|^2 \left[f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right) + \frac{f_a''(\|V\|_1) + f_a''(\|T\|_1)}{2}\right] \\ &\le \frac{1}{12}\|V-T\|^2 \left[f_a''(\|V\|_1) + f_a''(\|T\|_1)\right] \\ &\le \frac{1}{6}\|V-T\|^2 \max\left\{f_a''(\|V\|_1), f_a''(\|T\|_1)\right\}. \end{aligned}$$

Proof. Taking the modulus in (5.1), for $T, V \in \mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1 < R$ we have

$$\left|\frac{\operatorname{tr}[f(V)] + \operatorname{tr}[f(T)]}{2} - \operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right| \tag{5.5}$$
$$\leq \frac{1}{4}\int_0^1 \left|\operatorname{tr}\left((V-T)\left[f'\left((1-t)\frac{V+T}{2}+tV\right) - f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right)\right| \mathrm{d}t\,.$$

Using the properties of trace, for $T, V \in \mathcal{B}_1(H)$ with $\|T\|_1, \|V\|_1 < R$ and $t \in [0,1]$ we have

$$\left|\operatorname{tr}\left((V-T)\left[f'\left((1-t)\frac{V+T}{2}+tV\right) - f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right)\right| \tag{5.6}$$
$$\leq \|V-T\| \left\|\left[f'\left((1-t)\frac{V+T}{2}+tV\right) - f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right\|_1 .$$

From (4.1), for $A, B \in \mathcal{B}_1(H)$ with $\|A\|_1, \|B\|_1 < R$ we have

$$\begin{aligned}\|f(A) - f(B)\|_1 &\leq \|A-B\| \int_0^1 f'_a\left(\|(1-t)B + tA\|_1\right) \mathrm{d}t \\ &\leq \frac{1}{2}\|A-B\|\left[f'_a\left(\left\|\frac{A+B}{2}\right\|_1\right) + \frac{f'_a(\|A\|_1) + f'_a(\|B\|_1)}{2}\right] \\ &\leq \frac{1}{2}\|A-B\|\left[f'_a(\|A\|_1) + f'_a(\|B\|_1)\right] \\ &\leq \|A-B\| \max\left\{f'_a(\|A\|_1), f'_a(\|B\|_1)\right\}.\end{aligned} \tag{5.7}$$

Applying the second and third inequalities in (5.7) for f', $A = (1-t)\frac{V+T}{2} + tV$ and $B = (1-t)\frac{V+T}{2} + tT$ we get

$$\left\|\left[f'\left((1-t)\frac{V+T}{2}+tV\right) - f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right\|_1 \tag{5.8}$$
$$\leq \frac{1}{2}t\|V-T\|\left[f''_a\left(\left\|\frac{V+T}{2}\right\|_1\right)\right.$$
$$\left. + \frac{f''_a\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right) + f''_a\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)}{2}\right]$$
$$\leq \frac{1}{2}t\|V-T\|$$
$$\times\left[f''_a\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right) + f''_a\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)\right]$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$ and $t\in[0,1]$.

Since f_a'' is convex and monotonic nondecreasing and $\|\cdot\|_1$ is convex, then

$$\frac{f_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)+f_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)}{2} \tag{5.9}$$
$$\le(1-t)\,f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)+t\,\frac{f_a''(\|V\|_1)+f_a''(\|T\|_1)}{2}$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$ and $t\in[0,1]$.

From (5.8) and (5.9) we get

$$\left\|\left[f'\left((1-t)\frac{V+T}{2}+tV\right)-f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right\|_1 \tag{5.10}$$
$$\le\frac{1}{2}\,t\,\|V-T\|$$
$$\times\left[f_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)+f_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)\right]$$
$$\le t\,\|V-T\|\left[(1-t)f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)+t\,\frac{f_a''\left(\|V\|_1\right)+f_a''(\|T\|_1)}{2}\right]$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$ and $t\in[0,1]$.

Integrating (5.10) over t on $[0,1]$ we get

$$\int_0^1\left\|\left[f'\left((1-t)\frac{V+T}{2}+tV\right)-f'\left((1-t)\frac{V+T}{2}+tT\right)\right]\right\|_1\mathrm{d}t$$
$$\le\frac{1}{2}\,\|V-T\|\times\left[\int_0^1 tf_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)\mathrm{d}t\right.$$
$$\left.+\int_0^1 tf_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\mathrm{d}t\right)\right]$$
$$\le\|V-T\|\left[f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)\int_0^1 t(1-t)\,\mathrm{d}t\right.$$
$$\left.+\frac{f_a''(\|V\|_1)+f_a''(\|T\|_1)}{2}\int_0^1 t^2\,\mathrm{d}t\right]$$
$$=\frac{1}{6}\,\|V-T\|\left[f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)+\frac{f_a''(\|V\|_1)+f_a''(\|T\|_1)}{2}\right],$$

which together with (5.5) and (5.6) produce the inequality

$$\begin{aligned}
&\left|\frac{\operatorname{tr}[f(V)]+\operatorname{tr}[f(T)]}{2}-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right| \qquad (5.11)\\
&\le \frac{1}{8}\|V-T\|^2\times\left[\int_0^1 tf_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)\mathrm{d}t\right.\\
&\qquad\left.+\int_0^1 tf_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\mathrm{d}t\right)\right]\\
&\le \frac{1}{24}\|V-T\|^2\left[f_a''\left(\left\|\frac{V+T}{2}\right\|_1\right)+\frac{f_a''(\|V\|_1)+f_a''(\|T\|_1)}{2}\right].
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\int_0^1 tf_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)\mathrm{d}t \qquad (5.12)\\
&\qquad=\int_0^1 tf_a''\left(\left\|\frac{1-t}{2}T+\frac{1+t}{2}V\right\|_1\right)\mathrm{d}t\,,
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 tf_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)\mathrm{d}t \qquad (5.13)\\
&\qquad=\int_0^1 tf_a''\left(\left\|\frac{1-t}{2}V+\frac{1+t}{2}T\right\|_1\right)\mathrm{d}t\,.
\end{aligned}$$

Using the change of variable $u=\frac{1+t}{2}$, then we get

$$\begin{aligned}
&\int_0^1 tf_a''\left(\left\|\frac{1-t}{2}T+\frac{1+t}{2}V\right\|_1\right)\mathrm{d}t \qquad (5.14)\\
&\qquad=2\int_{\frac12}^1(2u-1)f_a''\big(\|(1-u)T+uV\|\big)\,\mathrm{d}u\,.
\end{aligned}$$

Also, by changing the variable $v=\frac{1-t}{2}$, we get

$$\begin{aligned}
&\int_0^1 tf_a''\left(\left\|\frac{1-t}{2}V+\frac{1+t}{2}T\right\|_1\right)\mathrm{d}t \qquad (5.15)\\
&\qquad=2\int_0^{\frac12}(1-2v)f_a''\big(\|(1-v)T+vV\|\big)\,\mathrm{d}v\,.
\end{aligned}$$

Utilising the equalities (5.12)-(5.15) we obtain

$$\begin{aligned}
&\int_0^1 t f_a''\left(\left\|(1-t)\frac{V+T}{2}+tV\right\|_1\right)\mathrm{d}t+\int_0^1 t f_a''\left(\left\|(1-t)\frac{V+T}{2}+tT\right\|_1\right)\mathrm{d}t\\
&\qquad=2\int_{\frac{1}{2}}^1(2t-1)f_a''\big(\|(1-t)T+tV\|\big)\,\mathrm{d}t\\
&\qquad\quad+2\int_0^{\frac{1}{2}}(1-2t)f_a''\big(\|(1-t)T+tV\|\big)\,\mathrm{d}t\\
&\qquad=2\int_0^1|2t-1|\,f_a''\big(\|(1-t)T+tV\|\big)\,\mathrm{d}t\\
&\qquad=4\int_0^1\left|t-\frac{1}{2}\right|f_a''\big(\|(1-t)T+tV\|\big)\,\mathrm{d}t
\end{aligned}$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1<R$.

Making use of (5.11) we deduce the first two inequalities in (5.4).

The rest is obvious. ∎

Corollary 2. *Let $f(z):=\sum_{n=0}^{\infty}a_nz^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1\le M<R$, then*

$$\left|\frac{\operatorname{tr}[f(V)]+\operatorname{tr}[f(T)]}{2}-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right|\le\frac{1}{8}\|V-T\|^2 f_a''(M)\,. \tag{5.16}$$

The constant $\frac{1}{8}$ is best possible in (5.16).

Proof. From the first inequality in (5.4) we have

$$\begin{aligned}
&\left|\frac{\operatorname{tr}[f(V)]+\operatorname{tr}[f(T)]}{2}-\operatorname{tr}\left[f\left(\frac{V+T}{2}\right)\right]\right|\\
&\qquad\le\frac{1}{2}\|V-T\|^2\int_0^1\left|t-\frac{1}{2}\right|f_a''\big(\|(1-t)T+tV\|_1\big)\,\mathrm{d}t\\
&\qquad\le\frac{1}{2}\|V-T\|^2 f_a''(M)\int_0^1\left|t-\frac{1}{2}\right|\mathrm{d}t\;=\;\frac{1}{8}\|V-T\|^2 f_a''(M)
\end{aligned}$$

for $T,V\in\mathcal{B}_1(H)$ with $\|T\|_1,\|V\|_1\le M<R$, and the inequality is proved.

If we consider the scalar case and take $f(z)=z^2$, $V=a$, $T=b$ with $a,b\in\mathbb{R}$ then we get in both sides of (5.16) the same quantity $\frac{1}{4}(b-a)^2$. ∎

Remark 3. A similar result holds by swapping the norm $\|\cdot\|$ with $\|\cdot\|_1$ in the right hand side of (5.4). The case of Hilbert-Schmidt norm may also be stated, however the details are not presented here.

If we write the inequality (5.4) for the exponential function, then we get

$$\left|\frac{\operatorname{tr}[\exp(V)]+\operatorname{tr}[\exp(T)]}{2}-\operatorname{tr}\left[\exp\left(\frac{V+T}{2}\right)\right]\right| \tag{5.17}$$
$$\leq \frac{1}{2}\|V-T\|^2\int_0^1\left|t-\frac{1}{2}\right|\exp\left(\|(1-t)T+tV\|_1\right)\mathrm{d}t$$
$$\leq \frac{1}{24}\|V-T\|^2\left[\exp\left(\left\|\frac{V+T}{2}\right\|_1\right)+\frac{\exp(\|V\|_1)+\exp(\|T\|_1)}{2}\right]$$
$$\leq \frac{1}{12}\|V-T\|^2\left[\exp(\|V\|_1)+\exp(\|T\|_1)\right]$$
$$\leq \frac{1}{6}\|V-T\|^2\max\left\{\exp(\|V\|_1),\exp(\|T\|_1)\right\}$$

for any for $T, V \in \mathcal{B}_1(H)$.

If $T, V \in \mathcal{B}_1(H)$ with $\|V\|_1, \|T\|_1 \leq M$, then

$$\left|\frac{\operatorname{tr}[\exp(V)]+\operatorname{tr}[\exp(T)]}{2}-\operatorname{tr}\left[\exp\left(\frac{V+T}{2}\right)\right]\right| \tag{5.18}$$
$$\leq \frac{1}{8}\|V-T\|^2\exp(M).$$

If we write the inequality (5.4) for the function $f(z)=(1-z)^{-1}$, then we get

$$\left|\frac{\operatorname{tr}\left[(1_H-V)^{-1}\right]+\operatorname{tr}\left[(1_H-T)^{-1}\right]}{2}-\operatorname{tr}\left[\left(1_H-\frac{V+T}{2}\right)^{-1}\right]\right| \tag{5.19}$$
$$\leq \|V-T\|^2\int_0^1\left|t-\frac{1}{2}\right|\left(1-\|(1-t)T+tV\|_1\right)^{-3}\mathrm{d}t$$
$$\leq \frac{1}{12}\|V-T\|^2\times\left[\left(1-\left\|\frac{V+T}{2}\right\|_1\right)^{-3}+\frac{(1-\|V\|_1)^{-3}+(1-\|T\|_1)^{-3}}{2}\right]$$
$$\leq \frac{1}{6}\|V-T\|^2\left[(1-\|V\|_1)^{-3}+(1-\|T\|_1)^{-3}\right]$$
$$\leq \frac{1}{3}\|V-T\|^2\max\left\{(1-\|V\|_1)^{-3},(1-\|T\|_1)^{-3}\right\},$$

for any for $T, V \in \mathcal{B}_1(H)$ with $\|V\|_1, \|T\|_1 < 1$.

Moreover, if $\|V\|_1, \|T\|_1 \le M < 1$, then

$$\left| \frac{\operatorname{tr}\left[(1_H - V)^{-1}\right] + \operatorname{tr}\left[(1_H - T)^{-1}\right]}{2} - \operatorname{tr}\left[\left(1_H - \frac{V+T}{2}\right)^{-1}\right] \right| \le \frac{1}{4} \|V - T\|^2 (1 - M)^{-3}. \tag{5.20}$$

The interested reader may choose other examples of power series to get similar results. However, the details are not presented here.

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