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Unitary skew-dilations of Hilbert space operators

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Abstract: The aim of this paper is to study, for a given sequence $(\rho_n)_{n\geq 1}$ of complex numbers, the class of Hilbert space operators possessing (ρ_n)-unitary dilations. This is the class of bounded linear operators T acting on a Hilbert space H, whose iterates T^n can be represented as $T^n = \rho_n P_H U^n|_H$, $n \geq 1$, for some unitary operator U acting on a larger Hilbert space, containing H as a closed subspace. Here P_H is the projection from this larger space onto H. The case when all ρ_n's are equal to a positive real number ρ leads to the class C_ρ introduced in the 1960s by Foias and Sz.-Nagy, while the case when all ρ_n's are positive real numbers has been previously considered by several authors. Some applications and examples of operators possessing (ρ_n)-unitary dilations, showing a behavior different from the classical case, are given in this paper.

Key words: Hilbert space operators, Dilations, Compressions of linear operators, Functional calculi, Numerical radius, ρ-radii, ρ-classes, (ρ_n)-classes If there are too many of them, you can remove ρ-radii and (ρ_n)-classes.

AMS *Subject Class.* (2010): 47A12, 47A20, 47A30, 47A60.

1. Introduction

Classes C_ρ have been introduced by B. Sz-Nagy and C. Foias [22] in 1966. For a complex Hilbert space H and a real number $\rho > 0$, a bounded linear operator $T \in \mathcal{L}(H)$ is said to be in the class $C_\rho(H)$ if all powers of T can be skew-dilated to powers of a unitary operator on a Hilbert space K, containing H as a closed subspace. This means that

$$T^n = \rho P_H U^n|_H, \quad \text{for all } n \geq 1,$$

where $U \in \mathcal{L}(K)$ is a suitable unitary operator, and $P_H \in \mathcal{L}(K)$ denotes the orthogonal projection onto H. Such an operator T is called a ρ-contraction, while the unitary operator U is called a *ρ-dilation*, or a ρ-unitary dilation, of T.

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The famous Sz.-Nagy dilation theorem (see [22]) shows that $C_1(H)$ is exactly the class of all Hilbert space contractions *i.e.*, operators of norm no greater than one. It is also known (see [6]) that the class $C_2(H)$ coincides with the class of all operators T with numerical range $W(T)$ included in the closed unit disk; equivalently, those T satisfying $w(T) \leq 1$. Here the *numerical range* $W(T)$ and the *numerical radius* $w(T)$ of T are defined by

$$W(T) = \big\{ \langle Tx, x\rangle : \|x\| = 1 \big\}\,; \quad w(T) = \sup \big\{ |\lambda| : \lambda \in W(T) \big\}.$$

Let T be an operator in the class C_ρ. Then

(i) T is power-bounded. More precisely, we have $\|T^n\| \leq \max(1, \rho)$, for all $n \geq 0$. In particular, the spectral radius $r(T)$ of T satisfies $r(T) \leq 1$;

(ii) T^k is in $C_\rho(H)$ for every $k \geq 1$;

(iii) For a closed subspace, F, of H which is stable by T (*i.e.*, $T(F) \subset F$), the restriction $T\,|_F$ is in $C_\rho(F)$;

(iv) The functional calculus map $f \mapsto f(T)$ that sends a polynomial f into $f(T)$ can be extended in a well-defined manner to the disk algebra $\mathbb{A}(\mathbb{D}) := C^0(\overline{\mathbb{D}}) \cap Hol(\mathbb{D})$. It is a morphism of Banach algebras, and satisfies

$$\|f(T)\| \leq \max(1, \rho)\|f\|_{L^\infty(\mathbb{D})};$$

(v) T is similar to a contraction: there is an invertible operator $L \in \mathcal{L}(H)$ such that $\|LTL^{-1}\| \leq 1$.

We refer the reader to [11, 12, 23, 17, 15] for proofs of these results, which mainly use several characterizations of classes $C_\rho(H)$. We record the principal ones in the following theorem.

Theorem. *Let T be an operator in $\mathcal{L}(H)$ and let $\rho > 0$. The following are equivalent:*

(i) $T \in C_\rho(H)$;

(ii) $r(T) \leq 1$ *and, for all* $z \in \mathbb{D}$, *we have* $\big(1 - \frac{2}{\rho}\big)I + \frac{2}{\rho}Re\big((I - zT)^{-1}\big) \geq 0$;

(iii) *For all* $z \in \mathbb{D}$ *and all* $h \in H$ *we have* $\big(\frac{2}{\rho} - 1\big)\|zTh\|^2 + \big(2 - \frac{2}{\rho}\big)\langle zTh, h\rangle \leq \|h\|^2$.

We remark that these characterization can be expressed in terms of classes of operator-valued holomorphic functions. For instance, (ii) says that the map $z \mapsto (1-\frac{2}{\rho})I + \frac{2}{\rho}(I-zT)^{-1}$ is in the Caratheodory class of operator-valued holomorphic functions on $\mathbb{D}$, having all real parts positive-definite operators. Item (iii) can be equivalently expressed by the membership of $z \mapsto zT((\rho-1)zT-\rho I)^{-1}$ to the Schur class of holomorphic maps $f:\mathbb{D}\to\mathcal{L}(H)$ having all norms no greater than one (i.e., $\|f(z)\|\leq 1$ for every $z\in\mathbb{D}$).

J.A.R. Holbrook [11] and J.P. Williams [24] introduced the notion of *ρ-radius* of an operator $T\in\mathcal{L}(H)$ as follows:

$$w_\rho(T) := \inf\Big\{u>0:\ \frac{1}{u}T\in C_\rho(H)\Big\}.$$

This ρ-radius is a quasi-norm on the Banach space $\mathcal{L}(H)$, equivalent to the operator norm, whose closed unit ball is exactly $C_\rho(H)$. Recall ([13]) that a *quasi-norm* satisfies all properties of a norm, except that the triangular inequality holds true up to a multiplicative constant. For $\rho>2$, the quasi-norm w_ρ satisfies ([23, 14])

$$w_\rho(T_1+T_2)\leq \rho\big(w_\rho(T_1)+w_\rho(T_2)\big).$$

Therefore the ρ-contractions are exactly the contractions for the ρ-radius, and many relationships between classes C_ρ can be expressed more easily using the associated ρ-radii. The ρ-radius is a usual Banach-space norm for $0<\rho\leq 2$.

Some generalizations of classes C_ρ have been studied, like classes $C_A(H)$ introduced by H. Langer (see [23, page 53] and its references, and [20]), or the classes $C_{(\rho_n)}(H)$ considered by several authors (see [17, 4, 18]). This latter generalization will be the main topic of study in this paper, with the novelty that we consider the general case when the ρ_n's are non-zero complex scalars. This will lead to classes of operators with several new features and different behavior.

2. Hilbert space operators with (ρ_n)-dilations

Definition and first properties. In light of the preceding discussion we introduce the following definition.

Definition 2.1. (Classes $C_{(\rho_n)}$) Let $(\rho_n)_{n\geq 1}$ be a sequence of complex numbers, with $\rho_n\neq 0$ for each n. We write $(\rho_n)_{n\geq1}\in(\mathbb{C}^*)^{\mathbb{N}^*}$. Let H be a complex Hilbert space. Define now

$$C_{(\rho_n)}(H) := \big\{T\in\mathcal{L}(H): \text{there exists a Hilbert space } K \text{ and a unitary operator } U\in\mathcal{L}(K) \text{ with } H\subset K \text{ and } T^n=\rho_n P_H U^n|_H,\ \forall n\geq 1\big\}.$$

Here $P_H \in \mathcal{L}(K)$ is the orthogonal projection from K onto its closed subspace H. We say in this case that T possesses (ρ_n)-*dilations*.

In other words, an operator T is in the class $C_{(\rho_n)}(H)$ if and only if all its powers admit dilations of the form $\rho_n U^n$ for a certain unitary operator U acting on a larger Hilbert space. For the rest of this paper, we will suppose that the Hilbert space H on which T acts is fixed. If there is no ambiguity, $C_{(\rho_n)}(H)$ will be abbreviated as $C_{(\rho_n)}$. Note also that the sequence $(\rho_n) = (\rho_n)_{n\geq 1}$ starts at $n = 1$: for $n = 0$ we have of course $T^0 = I_H = P_H U^0|_H$.

In the papers [17, 4, 18], the case when the ρ_n's are non-negative real numbers is considered. We went for a broader choice of sequences as the main ideas do not rely heavily on the fact that ρ_n are in $\mathbb{R}_+^*$ and as this eventually allows for some interesting new phenomena for the classes $C_{(\rho_n)}$. One first difference is recorded in the following remark.

Remark 2.2. The definition of $C_{(\rho_n)}$ easily gives that $T \in C_{(\rho_n)}$ if and only if $T^* \in C_{(\overline{\rho_n})}$. Therefore, when the ρ_n are real scalars, the class $C_{(\rho_n)}$ is stable under the adjoint map $T \mapsto T^*$. This is no longer true in the general case.

Remark 2.3. As another basic remark, we note that if T is in $C_{(\rho_n)}$, then we have $\|T^n\| \leq |\rho_n|$. Thus, $r(T) \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$. This relationship implies that two different cases appear in the study of the classes $C_{(\rho_n)}$:

(i) $0 < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) \leq +\infty$;

(ii) $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = 0$.

Although many of the proofs below work the same way in both cases, most of the results will be stated in the case (i). The study of the case (ii) is more problematic. Indeed, in case (ii), the class $C_{(\rho_n)}$ will only contain quasinilpotent operators, that is operators whose spectra reduce to $\{0\}$.

We also note that when $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = +\infty$, we trivially have $r(T) < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$ for every operator T. We also note that the condition $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = +\infty$ leads to small changes in the proofs below: the main difference between this condition and $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) < +\infty$ in case (i) is the fact that the quantity $\frac{1}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}$, which exists when $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) \in]0, +\infty[$, has to be replaced by 0 when $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = +\infty$. This motivates the following convention.

CONVENTION. For the rest of this paper, we assume that

$$\frac{1}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)} = 0 \quad \text{whenever} \quad \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = +\infty. \tag{2.1}$$

One of the main tools to characterize the classes $C_{(\rho_n)}$ is the following Herglotz-type theorem.

THEOREM 2.4. *Let H a Hilbert space. Let $F : \mathbb{D} \to H$ be an analytic function such that:*

(i) $F(0) = I$

(ii) $Re(F(z)) \geq 0, \ \forall z \in \mathbb{D}$.

Then, there exists a Hilbert space K containing H and $U \in \mathcal{L}(K)$ an unitary operator such that

$$F(z) = P_H(I + zU)(I - zU)^{-1}|_H, \quad \forall z \in \mathbb{D}$$

A proof of this theorem can be found in [8, pages 65−69].

DEFINITION 2.5. For $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ and for w in a complex Banach algebra, $f_{(\rho_n)}$ denotes the entire series given by $f_{(\rho_n)}(w) = \sum_{n\geq 1} \frac{2w^n}{\rho_n}$.

For $a \in \mathbb{R}$, we denote $Re_{\geq a}$ the half-plane $\{z \in \mathbb{C},\ Re(z) \geq a\}$, while $Re_{>a}$ is the half-plane $\{z \in \mathbb{C},\ Re(z) > a\}$.

PROPOSITION 2.6. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ and let $T \in \mathcal{L}(H)$. The following are equivalent:*

(i) $T \in C_{(\rho_n)}$;

(ii) *The series*

$$f_{(\rho_n)}(zT) = \sum_{n=1}^{\infty} \frac{2}{\rho_n} z^n T^n$$

is absolutely convergent in $\mathcal{L}(H)$ and

$$I + Re\big(f_{(\rho_n)}(zT)\big) \geq 0, \quad \forall z \in \mathbb{D}.$$

Proof. (i) ⇒ (ii) Let U be an unitary operator on a Hilbert space K, with K containing H as a closed subspace, such that

$$T^n = \rho_n P_H U^n|_H, \quad \forall n \geq 1.$$

For every polynomial $P(X) = a_0 + \cdots + a_n X^n$ and every $z \in \mathbb{D}$, we have

$$a_0 I + \frac{a_1}{\rho_1} zT + \cdots + \frac{a_n}{\rho_n}(zT)^n = P_H\big(a_0 I + a_1 zU + \cdots + a_n (zU)^n\big)|_H = P_H\, P(zU)|_H.$$

Since the series $1 + \sum_{n\geq 1} 2w^n$ converges absolutely to $f(w) = \frac{1+w}{1-w}$ for all $w \in \mathbb{D}$, and since U is unitary, the series $I + \sum_{n\geq 1} 2(zU)^n$ converges in norm to

$$f(zU) = (I + zU)(I - zU)^{-1}, \quad \forall z \in \mathbb{D}.$$

Thus, as

$$\left\| \frac{T^n}{\rho_n} \right\| = \big\| P_H U^n|_H \big\| \leq \big\| U^n \big\| \leq 1,$$

the series $I_H + \sum_{n\geq 1} \frac{2}{\rho_n}(zT)^n$ is absolutely convergent and converges to $P_H[(I + zU)(I - zU)^{-1}]|_H$ for all $z \in \mathbb{D}$. As U is unitary, $f(zU)$ is normal, so the closure of its numerical range $W(f(zU))$ is the convex hull of its spectrum. We have

$$\sigma\big(f(zU)\big) = f\big(\sigma(zU)\big) \subset f(\mathbb{D}) \subset Re_{>0}.$$

Thus,

$$W\big((I + zU)(I - zU)^{-1}\big) = W\big(f(zU)\big) \subset Hull\big(\sigma(f(zU))\big) \subset Re_{\geq 0}.$$

Furthermore, $W(P_H f(zU)|_H) \subset W(f(zU))$, so the numerical range of $I_H + f_{(\rho_n)}(zT)$ is included in $Re_{\geq 0}$. This is equivalent to $Re(I_H + f_{(\rho_n)}(zT)) \geq 0$, so (ii) is true.

(ii) ⇒ (i) We define $F(z) := I_H + f_{(\rho_n)}(zT)$. Thus, F is analytic on $\mathbb{D}$, $F(0) = I_H$, and $Re(F(z)) \geq 0$ for all $z \in \mathbb{D}$. By applying Theorem 2.4, we obtain a Hilbert space K and a unitary operator $U \in \mathcal{L}(K)$, such that $F(z) = P_H(I + zU)(I - zU)^{-1}|_H$, for all $z \in \mathbb{D}$. By developing both analytic expressions in entire series, and identifying their coefficients, we obtain $\frac{2}{\rho_n} T^n = 2P_H U^n|_H$ for all $n \geq 1$. Therefore $T \in C_{(\rho_n)}$. ∎

We will obtain most of the following results by applying Proposition 2.6. We can directly see by applying this proposition that any class $C_{(\rho_n)}$ contains 0, the null operator, so none of these classes is empty.

One remark is in order. We did not consider the case where $\rho_n = 0$ for some n in Definition 2.1. Indeed, this condition does not go well with computations similar to the ones in the proof of Proposition 2.6. Having $\rho_n = 0$

implies $T^n = 0$, but it does not give any information on $P_H U^n|_H$. This prevents us from showing that certain sums of powers of T and T^* are positive, which is a crucial tool when dealing with operators in the class $C_{(\rho_n)}$.

If we were to denote $m := \inf\{n : \rho_n = 0\}$, then any operator T in $C_{(\rho_n)}$ would need to be nilpotent of order at most m. The following corollary treats this nilpotent case and gives a characterization that was the one we expected in the case $\rho_m = 0$. See also [5, Proposition 6.1] for another use of the positivity condition (ii) below.

COROLLARY 2.7. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ and $m \geq 1$. Let $T \in \mathcal{L}(H)$ be such that $T^m = 0$. Then, the following are equivalent:*

(i) $T \in C_{(\rho_n)}$;

(ii) $I + Re\big(\sum_{n=1}^{m-1} z^n \frac{2}{\rho_n} T^n\big) \geq 0$ *for all* $z \in \mathbb{D}$.

Thus, for any sequence (τ_n) such that $\rho_k = \tau_k$, for all $1 \leq k < m$, we have $T \in C_{(\tau_n)}$ if and only if $T \in C_{(\rho_n)}$.

Proof. A direct application of Proposition 2.6 with the extra condition $T^m = 0$ gives the equivalence. ∎

Now we come back to Proposition 2.6. When $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$, we can see that the series $\sum_{n=1}^{\infty} \frac{2}{\rho_n} z^n T^n$ is absolutely convergent if and only if $|z| r(T) < \liminf_n(|\rho_n|^{\frac{1}{n}})$. We can thus reformulate Proposition 2.6 as follows.

THEOREM 2.8. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Let $T \in \mathcal{L}(H)$. Then, the following assertions are equivalent:*

(i) $T \in C_{(\rho_n)}$;

(ii) $r(T) \leq \liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)$ *and, for* $f_{(\rho_n)}(zT) := \sum_{n=1}^{\infty} \frac{2}{\rho_n} z^n T^n$, *we have*

$$I + Re\big(f_{(\rho_n)}(zT)\big) \geq 0, \quad \forall z \in \mathbb{D}.$$

Remark 2.9. Replacing the condition of absolute convergence of a series by a condition concerning the spectral radius of T is useful in several instances. We can first notice that if we take $v > 0$ small enough, then vT will satisfy the spectral radius condition. However, if $\liminf_n \big(|\rho_n|^{1/n}\big) = 0$, this condition must be replaced by $\limsup_n \big(\frac{\|T^n\|}{|\rho_n|}^{1/n}\big) \leq 1$, which can only be satisfied by certain quasinilpotent operators. Hence, aside from nilpotent operators and Corollary 2.7, knowing which operators can be "near" operators belonging to a class $C_{(\rho_n)}$ is a difficult problem. In this case, the map $f_{(\rho_n)}$ also has convergence radius 0, so we cannot use analytic or geometric properties related to the images of certain disks by $f_{(\rho_n)}$.

Many of the following results, related to specific operators or to $f_{(\rho_n)}$ will have no meaning in this case, but others will be true under the additional condition

$$\limsup_n \left(\frac{\|T^n\|}{|\rho_n|}\right)^{\frac{1}{n}} \leq 1.$$

We look now at the closure of the class $C_{(\rho_n)}$ for the operator norm.

Corollary 2.10. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. Then the class $C_{(\rho_n)}$ is closed for the operator norm: if $(T_m)_m$ a sequence of operators converging in $\mathcal{L}(H)$ to T, such that $T_m \in C_{(\rho_n)}$, then $T \in C_{(\rho_n)}$.*

Proof. Let $(T_m)_m$ a sequence of operators converging to T such that $T_m \in C_{(\rho_n)}$. We have

$$r(T) = \lim_m \left(r(T_m)\right) \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right).$$

Thus, for any $z \in \mathbb{D}$, the series $f_{(\rho_n)}(zT)$ converges absolutely and $f_{(\rho_n)}(zT) = \lim_m f_{(\rho_n)}(zT_m)$. Hence, for any $h \in H$, we have

$$Re\Big(\big\langle\big(I + f_{(\rho_n)}(zT)\big)h, h\big\rangle\Big) = Re\Big[\lim_m \big\langle\big(I + f_{(\rho_n)}(zT_m)\big)h, h\big\rangle\Big] \geq 0.$$

This implies that $I + Re\big(f_{(\rho_n)}(zT)\big) \geq 0$, and the proof is complete by using Theorem 2.8. ∎

Operator radii. The condition in Theorem 2.8 will be useful when studying the (ρ_n)-radius, which is introduced in the following definition.

Definition 2.11. Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$. Let $T \in \mathcal{L}(H)$. We define the *(ρ_n)-radius* of T as:

$$w_{(\rho_n)}(T) := \inf\left\{u > 0 : \tfrac{T}{u} \in C_{(\rho_n)}\right\} \in [0, +\infty].$$

The definition of the (ρ_n)-radius is similar to the definition of the ρ-radius that can be found in [11, 1, 3, 2]. As the classes $C_{(\rho_n)}$ and C_ρ share the same type of definition, the (ρ_n)-radius and the ρ-radius will share the same role with some slight different variations.

We will for now focus on properties of the (ρ_n)-radius.

LEMMA 2.12. *Let* $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ *with* $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. *Then, the map* $T \mapsto w_{(\rho_n)}(T)$ *takes values in* $[0, +\infty[$, *is a quasi-norm, is equivalent as a quasi-norm to the operator norm* $\|\cdot\|$, *and its closed unit ball is the class* $C_{(\rho_n)}$. *We also have*

$$w_{(\rho_n)}(T) \geq \left(\frac{\|T^m\|}{|\rho_m|}\right)^{\frac{1}{m}} \text{ and } w_{(\rho_n)}(T) \geq \frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}.$$

Proof. We start off by showing that the (ρ_n)-radius is finite while obtaining its equivalence with the operator norm $\|\cdot\|$. Let $T \in \mathcal{L}(H)$. Let $u > 0$ be such that $\frac{T}{u} \in C_{(\rho_n)}$. For any $m \geq 1$, we have $\frac{\|T^m\|}{u^m} \leq |\rho_m|$, that is

$$u \geq \left(\frac{\|T^m\|}{|\rho_m|}\right)^{\frac{1}{m}}.$$

Therefore, by taking the infimum over u such that $\frac{T}{u} \in C_{(\rho_n)}$, we get

$$w_{(\rho_n)}(T) \geq \left(\frac{\|T^m\|}{|\rho_m|}\right)^{\frac{1}{m}}.$$

For $m = 1$ we obtain $w_{(\rho_n)}(T) \geq \left(\frac{\|T\|}{|\rho_1|}\right)$. If we also take the lim sup of the right-hand side quantity, we get

$$w_{(\rho_n)}(T) \geq \frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}.$$

Now, let $r < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$. Therefore, the series $f_{(|\rho_n|)}(rz) := \sum_{n=1}^{\infty} \frac{2}{|\rho_n|} r^n z^n$ is absolutely convergent for all $z \in \mathbb{D}$, thus analytic on $\mathbb{D}$. Since $f_{(|\rho_n|)}(0) = 0$, there is a radius r_0, with $1 > r_0 > 0$, such that $|f_{(|\rho_n|)}(r_0 w)| \leq 1$ for all $|w| \leq r$. Let $u > 0$ be such that $\frac{\|T\|}{u} < r_0 r$. Thus, we have

$$r\left(\frac{T}{u}\right) < r_0 r < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right),$$

and for all $z \in \mathbb{D}$ we have

$$\left\| f_{(\rho_n)}\left(z\frac{T}{u}\right)\right\| \leq \sum_{n=1}^{\infty} \frac{2}{|\rho_n|}|z|^n \left(\frac{T}{u}\right)^n \leq \sum_{n=1}^{\infty} \frac{2}{\rho_n}|z|^n (r_0 r)^n = \left|f_{(|\rho_n|)}(r_0|z|r)\right| \leq 1.$$

We recall that for any $B \in \mathcal{L}(H)$ we have

$$Re(B) \geq -\|Re(B)\|I = -\left\|\frac{B+B^*}{2}\right\| I \geq -\|B\|I.$$

Thus, for any $z \in \mathbb{D}$, $f_{(\rho_n)}\big(z\frac{T}{u}\big)$ converges absolutely and we have

$$I + Re\left(f_{(\rho_n)}\left(z\frac{T}{u}\right)\right) \geq I - \left\|f_{(\rho_n)}\left(z\frac{T}{u}\right)\right\| I \geq 0.$$

This means that $\frac{T}{u} \in C_{(\rho_n)}$ according to Proposition 2.6, so $w_{(\rho_n)}(T) \leq u < +\infty$. Furthermore, since $\frac{T}{u} \in C_{(\rho_n)}$ for every u such that $u > \frac{\|T\|}{r_0 r}$, we get $w_{(\rho_n)}(T) \leq \frac{\|T\|}{r_0 r}$. Hence, we have

$$\frac{\|T\|}{|\rho_1|} \leq w_{(\rho_n)}(T) \leq \frac{\|T\|}{r_0 r}\,.$$

With these inequalities we immediately get

$$w_{(\rho_n)}(T) = 0 \quad \Leftrightarrow \quad T = 0.$$

These inequalities also imply that, for $S, T \in \mathcal{L}(H)$, we have

$$w_{(\rho_n)}(S+T) \leq \frac{\|S+T\|}{r_0 r} \leq \frac{\|S\|+\|T\|}{r_0 r} \leq \frac{|\rho_1|}{r_0 r}\big(w_{(\rho_n)}(S) + +w_{(\rho_n)}(T)\big).$$

In order to show that $w_{(\rho_n)}(\cdot)$ is a quasi-norm, we still have to show that it is homogeneous, that is $w_{(\rho_n)}(zT) = |z|w_{(\rho_n)}(T)$ for any $z \in \mathbb{C}$. Let $z \in \mathbb{C}$. The cases $z = 0$ and $T = 0$ have been treated, so we now consider $z = e^{it}|z| \neq 0$ and $T \neq 0$. Let $u \geq w_{(\rho_n)}(zT)$ be such that $\frac{zT}{u} \in C_{(\rho_n)}$. Denote $u' = \frac{u}{|z|}$. We can see that $r\big(\frac{zT}{u}\big) = r\big(\frac{T}{u'}\big)$ and that $f_{(\rho_n)}\big(w\frac{zT}{u}\big) = f_{(\rho_n)}\big(e^{it}w\frac{T}{u'}\big)$ for any $w \in \mathbb{D}$. Thus, the series $f_{(\rho_n)}\big(e^{it}w\frac{T}{u'}\big)$ converges absolutely and $I + Re\big(f_{(\rho_n)}(e^{it}w\frac{T}{u'})\big) \geq 0$, for any $w \in \mathbb{D}$. Hence $\frac{T}{u'} \in C_{(\rho_n)}$, so

$$u' = \frac{u}{|z|} \geq w_{(\rho_n)}(T).$$

Thus, by taking the infimum for $u \geq w_{(\rho_n)}(zT)$, we get

$$w_{(\rho_n)}(zT) \geq |z| w_{(\rho_n)}(T).$$

Applying the same result to $T' = zT$ and $z' = \frac{1}{z}$, we obtain

$$w_{(\rho_n)}(T) = w_{(\rho_n)}(z'T') \geq |z'| w_{(\rho_n)}(T') = \frac{1}{|z|} w_{(\rho_n)}(zT),$$

which proves the desired equality.

We will now prove that the closed unit ball for the (ρ_n)-radius is exactly $C_{(\rho_n)}$. Notice again that $w_{(\rho_n)}(T) = 0$ reduces to $T = 0$. If $T \in C_{(\rho_n)}$, then $w_{(\rho_n)}(T) \leq \frac{1}{1} = 1$. Conversely, suppose that $w_{(\rho_n)}(T) \leq 1$ and let $(u_m)_m$ be a sequence, with $u_m > 0$, converging to $w_{(\rho_n)}(T)$ such that $\frac{T}{u_m} \in C_{(\rho_n)}$. Using the fact that the class $C_{(\rho_n)}$ is closed for the operator norm, as proved in Corollary 2.10, we get $\frac{T}{w_{(\rho_n)}(T)} \in C_{(\rho_n)}$. Therefore, we have

$$r(T) \leq r\left(\frac{T}{w_{(\rho_n)}(T)}\right) \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$$

and

$$I + Re\big(f_{(\rho_n)}(zT)\big) \geq 0 \ \text{ for every } z \text{ with } |z| \leq \frac{1}{w_{(\rho_n)}(T)}\,.$$

Since $\frac{1}{w_{(\rho_n)}(T)} \geq 1$, we can conclude that $T \in C_{(\rho_n)}$. The proof is now complete. ∎

Remark 2.13. In the case when $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = 0$, we have $w_{(\rho_n)}(T) = +\infty$ unless T is quasinilpotent and the sequence of $\|T^n\|^{\frac{1}{n}}$ decreases to 0 fast enough.

Remark 2.14. Since the (ρ_n)-radius is homogeneous and

$$w_{(\rho_n)}(T) \leq 1 \quad \Leftrightarrow \quad T \in C_{(\rho_n)},$$

whenever $T \neq 0$, we have

$$\left\{u > 0\colon\ \tfrac{T}{u} \in C_{(\rho_n)}\right\} = \left[w_{(\rho_n)}(T), +\infty\right[.$$

COROLLARY 2.15. *Let* $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ *with* $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. *Let* $T \in \mathcal{L}(H)$. *We have*

(i) *For any* $z \neq 0$, $\frac{1}{|z|} w_{(\rho_n)}(T) = w_{(\rho_n)}(\frac{1}{z}T) = w_{(z^n\rho_n)}(T)$;

(ii) *If* $T \in C_{(\rho_n)}(H)$, *then* $T^k \in C_{(\rho_{kn})}(H)$, *for all* $k \geq 1$;

(iii) $w_{(\rho_{kn})_n}(T^k) \leq w_{(\rho_n)}(T)^k$, *for all* $k \geq 1$;

(iv) $w_{(\rho_n)}(T) = w_{(\overline{\rho_n})}(T^*)$.

Proof. (i) The left-hand equality is given by the homogeneity of $w_{(\rho_n)}(\cdot)$. For the right-hand one, we can see that

$$\left(\frac{T}{z}\right)^n = \rho_n P_H U^n|_H \text{ if and only if } T^n = z^n \rho_n P_H U^n|_H.$$

Thus $\frac{T}{z} \in C_{(\rho_n)}$ if and only if $T \in C_{(z^n\rho_n)}$. Lemma 2.12 implies that

$$w_{(\rho_n)}\left(\frac{1}{z}T\right) = w_{(z^n\rho_n)}(T).$$

(ii) By definition of the class $C_{(\rho_n)}$, if $T \in C_{(\rho_n)}$, then

$$(T^k)^m = \rho_{km} P_H (U^k)^m|_H,$$

so $T^k \in C_{(\rho_{kn})}(H)$.

(iii) The result is true when $T = 0$. When $T \neq 0$, consider $T' = \frac{T}{w_{(\rho_n)}(T)}$. By homogeneity of $w_{(\rho_n)}(\cdot)$, we have $w_{(\rho_n)}(T') = 1$, so $T' \in C_{(\rho_n)}$ according to Lemma 2.12. Thus, for any $k \geq 1$, $(T')^k \in C_{(\rho_{kn})}(H)$. Using again the homogeneity of the (ρ_n)-radius, we obtain

$$\frac{w_{(\rho_{kn})_n}(T^k)}{w_{(\rho_n)}(T)^k} = w_{(\rho_{kn})}\big((T')^k\big) \leq 1.$$

This completes the proof.

(iv) We use Remark 2.2 and Lemma 2.12 to obtain the equivalence

$$w_{(\rho_n)}(T) \leq 1 \quad \Leftrightarrow \quad w_{(\overline{\rho_n})}(T^*) \leq 1.$$

Since the (ρ_n)-radii are homogenous, these quantities must be equal. ∎

Corollary 2.16. *Let* $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ *with* $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) > 0$. *Let* $T \in \mathcal{L}(H)$. *The following assertions are true:*

(i) *Let F be an invariant closed subspace of T. Then* $w_{(\rho_n)}(T|_F) \leq w_{(\rho_n)}(T)$*;*

(ii) *For any isometry V we have* $w_{(\rho_n)}(VTV^*) \leq w_{(\rho_n)}(T)$*, with equality if V is unitary;*

(iii) *For a Hilbert space K we have* $w_{(\rho_n)}(T \otimes I_K) = w_{(\rho_n)}(T)$*;*

(iv) *For $T_m \in \mathcal{L}(H_m)$, $m \geq 1$ with $\sup_m(\|T_m\|) < +\infty$, we have*

$$w_{(\rho_n)}(\oplus_{m\geq 1} T_m) = \sup_{m\geq 1} \big(w_{(\rho_n)}(T_m)\big);$$

(v) *If $T^{(\infty)}$ denotes the countable orthogonal sum $T \oplus T \oplus \cdots$, then* $w_{(\rho_n)}\big(T^{(\infty)}\big) = w_{(\rho_n)}(T)$.

Proof. (i) We have $r(T|_F) \leq r(T)$. If $I + Re\big(f_{(\rho_n)}(zT)\big)$ is positive, then $I + Re\big(f_{(\rho_n)}(zT|_F)\big)$ is positive too. Thus, by using Lemma 2.12 we obtain

$$w_{(\rho_n)}(T) \leq 1 \quad \Rightarrow \quad w_{(\rho_n)}(T|_F) \leq 1.$$

The homogeneity of the (ρ_n)-radius gives the result.

(ii) We have $r(VTV^*) \leq r(T)$ and $(VTV^*)^n = VT^nV^*$. Thus, $f_{(\rho_n)}(zVTV^*) = Vf_{(\rho_n)}(zT)V^*$. Hence, for any $h \in H$ and any $z \in \mathbb{D}$, we have

$$Re\Big(\big\langle\big(I + f_{(\rho_n)}(zVTV^*)\big)h, h\big\rangle\Big) = Re\Big(\big\langle\big(I + f_{(\rho_n)}(zT)\big)V^*h, V^*h\big\rangle\Big).$$

By applying Theorem 2.8 and Lemma 2.12, we get

$$w_{(\rho_n)}(T) \leq 1 \quad \Rightarrow \quad w_{(\rho_n)}(VTV^*) \leq 1.$$

The homogeneity of the (ρ_n)-radii gives the desired inequality. When the isometry V is also invertible, the converse inequality is true, so both quantities are equal.

(iii) Since $\|T^n\| = \|(T \otimes I_K)^n\|$, we have $r(T) = r(T \otimes I_K)$. Let $u > 0$ be such that $u \geq \frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}$. Thus the series $f_{(\rho_n)}\big(z\frac{T\otimes I_K}{u}\big)$ is absolutely convergent for all $z \in \mathbb{D}$, and $f_{(\rho_n)}\big(z\frac{T\otimes I_K}{u}\big) = f_{(\rho_n)}\big(z\frac{T}{u}\big) \otimes I_K$. Since for any $h_1 \otimes k_1, h_2 \otimes k_2 \in H \otimes K$ we have

$$\langle h_1 \otimes k_1, h_2 \otimes k_2\rangle = \langle h_1, h_2\rangle\langle k_1, k_2\rangle,$$

we can see that the condition

$$\left\langle \left(I + Re\left(f_{(\rho_n)}\left(z\frac{T\otimes I_K}{u}\right)\right)\right)(h\otimes k), h\otimes k\right\rangle \geq 0, \quad \forall\, h\otimes k \in H\otimes K,$$

is equivalent to

$$\left\langle \left(I + Re \left(f_{(\rho_n)} \left(z\frac{T}{u} \right) \right) \right) (h), h \right\rangle \geq 0, \quad \forall\, h \in H.$$

Hence, $\frac{T \otimes I_K}{u} \in C_{(\rho_n)}(H \otimes K)$ is equivalent to $\frac{T}{u} \in C_{(\rho_n)}(H)$, which implies that $w_{(\rho_n)}(T) = w_{(\rho_n)}(T \otimes I_K)$.

(iv) Since $\sup_m(\|T_m\|) < +\infty$, the linear map $T = \oplus_{m\geq 1} T_m$ is bounded on the Hilbert space $H = \oplus_{m\geq 1} H_m$, and $\|T\| = \sup_m(\|T_m\|)$. Thus, $r(T) = \sup_m(r(T_m))$. Let $u > 0$ be such that $u \geq \frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}$. We have

$$r\left(\frac{T_m}{u}\right) \leq r\left(\frac{T}{u}\right) \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right).$$

Thus, the series $f_{(\rho_n)}\left(z\frac{T}{u}\right)$ and $f_{(\rho_n)}\left(z\frac{T_m}{u}\right)$ are absolutely convergent for all $z \in \mathbb{D}$, and

$$f_{(\rho_n)}\left(z\frac{T}{u}\right) = \oplus_{m\geq 1} f_{(\rho_n)}\left(z\frac{T_m}{u}\right).$$

Since for any $h = (h_m)_m \in H$, we have

$$\left[I + Re \left(f_{(\rho_n)} \left(z\frac{T}{u} \right) \right) \right] (h) = \left(\left(I + Re \left(f_{(\rho_n)} \left(z\frac{T_m}{u} \right) \right) \right) (h_m) \right)_m ,$$

this implies that

$$\left\langle \left(I + Re \left(f_{(\rho_n)} \left(z\frac{T}{u} \right) \right) \right) (h), h \right\rangle \geq 0, \quad \forall\, h \in H,$$

is equivalent to

$$\left\langle \left(I + Re \left(f_{(\rho_n)} \left(z\frac{T_m}{u} \right) \right) \right) (h_m), h_m \right\rangle \geq 0, \quad \forall\, h_m \in H_m,\ \forall\, m \geq 1.$$

Hence, the assertion $\frac{T}{u} \in C_{(\rho_n)}(H)$ is equivalent to $\frac{T_m}{u} \in C_{(\rho_n)}(H_m)$, $\forall n \geq 1$, which implies that $w_{(\rho_n)}(T) = \sup_m \left(w_{(\rho_n)}(T_m)\right)$.

(v) The proof is a consequence of item (iii) and [5, Remark 1.1]. ∎

The items (i) and (ii) of this corollary show that the classes $C_{(\rho_n)}$ are unitarily invariant, and stable under the restriction to an invariant closed subspace. The item (iv) is a generalization of a known property of direct sums of operators in the class $C_{(\rho)}$. Items (i), (ii) and (v) show that, under the condition of Corollary 2.16, the radius $w_{(\rho_n)}$ is an *admissible radius* in the terminology of [5, Definition 1.1]. Thus, all the results proved in [5] for admissible radii are valid for $w_{(\rho_n)}$ when $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) > 0$. In particular, the following result is true.

COROLLARY 2.17. *Let $T \in \mathcal{L}(H)$, with $\|T\| \le 1$ and $T^n = 0$ for some $n \ge 2$. Then, for each polynomial p with complex coefficients, we have*

$$w_{(\rho_n)}\big(p(T)\big) \le w_{(\rho_n)}\big(p(S_n^*)\big).$$

Here S_n^* is the nilpotent Jordan cell

$$S_n^* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

on the standard Euclidean space $\mathbb{C}^n$.

Some other consequences of the condition $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) > 0$ are proved in the next proposition.

PROPOSITION 2.18. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) > 0$. The following assertions are true:*

(i) *We have*

$$w_{(\rho_n)}(I) = \min\Big(\Big\{r \ge \liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)^{-1} : \ f_{(\rho_n)}\big(\mathbb{D}\big(0, \tfrac{1}{r}\big)\big) \subset Re_{\ge -1}\Big\}\Big);$$

(ii) *For any $T \in \mathcal{L}(H)$, we have $w_{(\rho_n)}(T) \ge r(T) w_{(\rho_n)}(I)$;*

(iii) *If T is normal, then $w_{(\rho_n)}(T) = \|T\| w_{(\rho_n)}(I)$.*

Proof. (i) Take $u = w_{(\rho_n)}(I)$ such that $\frac{I}{u} \in C_{(\rho_n)}$. We have $r(\frac{I}{u}) \le \liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)$, so $\frac{1}{u}$ is no greater than the convergence radius of $f_{(\rho_n)}$. For any $z \in \mathbb{D}$, we have $f_{(\rho_n)}\big(z\frac{I}{u}\big) = f_{(\rho_n)}(\frac{z}{u})I$. Thus, $I + Re\big(f_{(\rho_n)}\big(z\frac{I}{u}\big)\big) \ge 0$ for any $z \in \mathbb{D}$ if and only if $f_{(\rho_n)}\big(\mathbb{D}\big(0, \frac{1}{u}\big)\big) \in Re_{\ge -1}$.

(ii) Let $T \in \mathcal{L}(H)$. There is nothing to prove if $T = 0$ or $r(T) = 0$. Otherwise, let $u = w_{(\rho_n)}(T)$ be such that $\frac{T}{u} \in C_{(\rho_n)}$ (cf. Lemma 2.12). Since $I +$

$Re\big(f_{(\rho_n)}\big(z\frac{T}{u}\big)\big) \geq 0$, the spectrum of $I+f_{(\rho_n)}\big(z\frac{T}{u}\big)$ lies in $Re_{\geq 0}$. This spectrum is the set $\big\{1+f_{(\rho_n)}(zw),\ w \in \sigma\big(\frac{T}{u}\big)\big\}$. The union of these spectra, when z describes $\mathbb{D}$, is $\big\{1+f_{(\rho_n)}(w),\ |w| < \frac{r(T)}{u}\big\}$. Since $\frac{r(T)}{u} > 0$, we obtain from item (i) that $\frac{u}{r(T)} \geq w_{(\rho_n)}(I)$. Hence $w_{(\rho_n)}(T) \geq r(T) w_{(\rho_n)}(I)$.

(iii) Let T be a normal operator with $T \neq 0$. For $u = \|T\|.w_{(\rho_n)}(I)$, we have

$$r\left(\frac{T}{u}\right) = \frac{\|T\|}{u} = \frac{1}{w_{(\rho_n)}(I)} \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right).$$

Thus, we obtain that

$$\bigcup_{z\in\mathbb{D}} \sigma\left(I + f_{(\rho_n)}\left(z\frac{T}{u}\right)\right) = \left\{1 + f_{(\rho_n)}(w),\ |w| < \frac{1}{w_{(\rho_n)}(I)}\right\}.$$

Item (i) of this proposition tells us that this set is included in $Re_{\geq 0}$. As T is normal, $I + f_{(\rho_n)}\big(z\frac{T}{u}\big)$ is also normal, so

$$W\left(I + f_{(\rho_n)}\left(z\frac{T}{u}\right)\right) \subset Hull\left(\sigma\left(I + f_{(\rho_n)}\left(z\frac{T}{u}\right)\right)\right) \subset Re_{\geq 0}, \quad \forall z \in \mathbb{D}.$$

Hence, $I + Re\big(f_{(\rho_n)}\big(z\frac{T}{u}\big)\big) \geq 0$, and $\frac{T}{u} \in C_{(\rho_n)}$. By Lemma 2.12, we then have

$$w_{(\rho_n)}(T) \leq u = \|T\| w_{(\rho_n)}(I).$$

The inequality of item (ii) provides the desired equality. ∎

Remark 2.19. Since we also have

$$w_{(\rho_n)}(I) \geq \frac{1}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)},$$

the inequality in Proposition 2.18 is better than the last one of Lemma 2.12. Thus, if there is T such that $w_{(\rho_n)}(T) = \frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}$, the same must be true for the identity operator I. In the case when $\rho_n = \rho$, $\rho > 0$, this can only happen when $\rho \geq 1$.

3. Classes $C_{(\rho)}$ for $\rho \neq 0$

In this section, we will focus on the case where $\rho_n = \rho$, for some $\rho \in \mathbb{C}^*$. This is an intermediate class between the classical case considered by Sz.-Nagy and Foias (classes C_τ for $\tau > 0$) and the the general $C_{(\rho_n)}$-classes. Thus the obtained results are already known when $\rho > 0$, but the generalization to the case $\rho \in \mathbb{C}^*$ seems to be new. Nevertheless, we acknowledge the influence of [23, 1, 2, 14] for the results of this section. The results obtained here will turn out to be useful when we will look again at $C_{(\rho_n)}$-classes in the next section.

Some characterizations.

Lemma 3.1. *Let $\rho \neq 0$ and $\rho_n = \rho$, $\forall n \geq 1$. Let $T \in \mathcal{L}(H)$. The following are equivalent:*

(i) $T \in C_{(\rho)}(H)$;

(ii) $r(T) \leq 1$ *and* $Re\big(\big(1 - \frac{2}{\rho}\big)I + \frac{2}{\rho}(I - zT)^{-1}\big) \geq 0$, $\forall z \in \mathbb{D}$;

(iii) $r(T) \leq 1$ *and* $Re\big(\frac{2}{\rho}(I - zT)\big) + Re\big(1 - \frac{2}{\rho}\big)(I - zT)^*(I - zT) \geq 0$, $\forall z \in \mathbb{D}$;

(iv) $Re\big(\frac{2}{\rho}(I - zT)\big) + Re\big(1 - \frac{2}{\rho}\big)(I - zT)^*(I - zT) \geq 0$, $\forall z \in \mathbb{D}$.

Proof. (i) $\Leftrightarrow$ (ii) We have $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) = 1$. When $r(T) \leq 1$, for $z \in \mathbb{D}$, we have

$$I + \sum_{n \geq 1} \frac{2}{\rho}(zT)^n = \left(1 - \frac{2}{\rho}\right)I + \frac{2}{\rho}(I - zT)^{-1}.$$

Apply now Proposition 2.6.

(ii) $\Leftrightarrow$ (iii) We will use several times the known fact that for $A, B \in \mathcal{L}(H)$, with A invertible,

$$Re(B) \geq 0 \quad \Leftrightarrow \quad Re(A^*BA) \geq 0.$$

We obtain the equivalence (ii) $\Leftrightarrow$ (iii) by choosing

$$A = (I - zT),\ B = \left(1 - \frac{2}{\rho}\right)I + \frac{2}{\rho}(I - zT)^{-1}$$

and by rearranging the expression, using that $(I - zT)^*(I - zT)$ is a positive self-adjoint operator and $Re(A^*) = Re(A)$.

(iii) $\Rightarrow$ (iv) It is immediate.

(iv) $\Rightarrow$ (iii) Suppose that $r(T) > 1$. Thus, there exists $\gamma \in \mathbb{C}$ such that $|\gamma| = r(T) > 1$, and there is a sequence (h_n) of vectors $h_n \in H$ such that $\|h_n\| = 1$ and $\|(T - \gamma I)h_n\| \to 0$ as $n \to \infty$. Let $0 < \epsilon < |\gamma| - 1$ and set $g_n := (T - \gamma I)h_n$. Let also $\eta = \epsilon e^{it}$, for some t that will be chosen later on. Let $z := \frac{1+\eta}{\gamma}$. Then, $|z| < \frac{1+(|\gamma|-1)}{|\gamma|} = 1$. Furthermore, we have

$$(I - zT)h_n = \left(I - \frac{1}{\gamma}T\right) h_n - \frac{\eta}{\gamma}Th_n + \eta h_n - \eta h_n = -zg_n - \eta h_n.$$

Thus, we obtain

$$Re\Bigg(\Bigg\langle \left[\frac{2}{\rho}(I - zT) + \left(1 - \frac{2}{\rho}\right)(I - zT)^*(I - zT)\right] h_n,\, h_n\Bigg\rangle\Bigg) \geq 0$$

$$\Rightarrow Re\Bigg(\frac{2}{\rho}\left[-\eta\|h_n\|^2 - \langle zg_n, h_n\rangle\right] + \left(1 - \frac{2}{\rho}\right)\|(I - zT)h_n\|^2\Bigg) \geq 0$$

$$\Rightarrow Re\Bigg(\frac{2}{\rho}\left[-\eta - \langle zg_n, h_n\rangle\right] + \left(1 - \frac{2}{\rho}\right)\left[|\eta|^2 + 2Re(\langle zg_n, h_n\rangle) + |z|^2\|g_n\|^2\right]\Bigg) \geq 0.$$

Hence, by taking the limit as $n \to +\infty$, we obtain

$$Re\left(\frac{2}{\rho}(-\eta) + \left(1 - \frac{2}{\rho}\right)|\eta|^2\right) = Re\left(\frac{-2}{\rho}e^{it}\right)\epsilon + Re\left(1 - \frac{2}{\rho}\right)\epsilon^2 \geq 0.$$

We can then choose $t \in \mathbb{R}$ depending on $arg(\rho)$ and $sgn\big(Re\big(1 - \frac{2}{\rho}\big)\big)$ to obtain

$$\text{either } \frac{-2}{|\rho|}\epsilon + \left|Re\left(1 - \frac{2}{\rho}\right)\right|\epsilon^2 \geq 0 \text{ or } \frac{2}{|\rho|}\epsilon - \left|Re\left(1 - \frac{2}{\rho}\right)\right|\epsilon^2 \leq 0.$$

But since $\frac{-2}{|\rho|} < 0$, there is some $\epsilon > 0$ such that $\frac{-2}{|\rho|} + \big|Re\big(1 - \frac{2}{\rho}\big)\big|\epsilon$ is strictly negative, which is impossible. This contradiction shows that $r(T) \leq 1$, which concludes the proof. ∎

LEMMA 3.2. *Let $\rho \neq 0$ and $\alpha > 0$ be two scalars. Let $T \in \mathcal{L}(H)$. The following assertions are equivalent:*

(i) $w_{(\rho)}(T) \leq \alpha$;

(ii) $r(T) \leq \alpha$, $\big((\rho - 1)zT - \rho\alpha I\big)$ *is invertible and* $\big\|(zT)\big((\rho - 1)zT - \rho\alpha I\big)^{-1}\big\| \leq 1$, $\forall z \in \mathbb{D}$;

(iii) $r(T) \leq \alpha$, $\big((\rho-1)T - \rho wI\big)$ *is invertible and* $\big\|T\big((\rho-1)T - \rho wI\big)^{-1}\big\| \leq 1$, $\forall |w| > \alpha$.

Proof. (i) $\Rightarrow$ (ii) When replacing T with $\frac{T}{\alpha}$, all expressions in (i) and (ii) are reduced to the case $\alpha = 1$. Now, as $w_{(\rho_n)}(T) \leq \alpha = 1$, we use Lemma 3.1 to have $r(T) \leq 1$ and

$$Re\left(\left(1 - \frac{2}{\rho}\right) I + \frac{2}{\rho}(I - zT)^{-1}\right) \geq 0, \quad \forall z \in \mathbb{D}.$$

We denote $C_z := \left(1 - \frac{2}{\rho}\right)I + \frac{2}{\rho}(I - zT)^{-1}$, for $z \in \mathbb{D}$. We recall that since $Re(C_z) \geq 0$, we have $(C_z + I)$ invertible and

$$\left\|(C_z - I)(C_z + I)^{-1}\right\| \leq 1.$$

A computation gives

$$C_z - I = \frac{2}{\rho} zT(I - zT)^{-1} \quad \text{and} \quad C_z + I = \left[2I + \left(\frac{2}{\rho} - 2\right) zT\right] (I - zT)^{-1}.$$

Thus,

$$\begin{aligned}(C_z - I)(C_z + I)^{-1} &= \frac{1}{\rho} zT \left[I + \left(\frac{1}{\rho} - 1\right) zT\right]^{-1} \\ &= zT\big[\rho I + (1 - \rho)zT\big]^{-1} = -zT\big[-\rho I + (\rho - 1)zT\big]^{-1}.\end{aligned}$$

This means that all the conditions of (ii) are fulfilled.

(ii) $\Rightarrow$ (i) We again reduce to the case $\alpha = 1$. We denote

$$D_z = zT\big[\rho I - (\rho - 1)zT\big]^{-1}, \quad \text{for } z \in \mathbb{D}.$$

Since $\|D_z\| \leq 1$, we have $D_z \in C_{(1)}$, so $r(D_z) \leq 1$ and

$$Re\big((I + wD_z)(I - wD_z)^{-1}\big) \geq 0, \quad \text{for all } w \in \mathbb{D}.$$

We obtain:

$$I + wD_z = \big[\rho I + (w + 1 - \rho)zT\big]\big[\rho I - (\rho - 1)zT\big]^{-1}$$

and

$$I - wD_z = \big[\rho I + (-w + 1 - \rho)zT\big]\big[\rho I - (\rho - 1)zT\big]^{-1}.$$

Thus,

$$(I + wD_z)(I - wD_z)^{-1} = \big[\rho I + (w + 1 - \rho)zT\big]\big[\rho I + (-w + 1 - \rho)zT\big]^{-1}.$$

Since $r(T) \le 1$, $(I - zT)$ is invertible so $[\rho I + (-w + 1 - \rho)zT]^{-1}$ converges to $\frac{1}{\rho}(I - zT)^{-1}$ when w tends to 1, by continuity of the inverse map. Thus,

$$\lim_{w\to 1,\ w\in\mathbb{D}} (I + wD_z)(I - wD_z)^{-1} = \frac{1}{\rho}\big(\rho I + (2-\rho)zT\big)(I - zT)^{-1} = C_z.$$

Hence, $Re(C_z) \ge 0$ for all $z \in \mathbb{D}$ and $r(T) \le 1$, so $T \in C_{(\rho)}$.

(ii) $\Leftrightarrow$ (iii) For $z \neq 0$, we take $w = \frac{\alpha}{z}$ to obtain the result. The converse gives the result for all $z \in \mathbb{D}$, $z \neq 0$, which extends to $\mathbb{D}$ by continuity. ∎

Reducing to the case $\rho > 0$. With this characterization of $C_{(\rho)}$ classes, we are now able to obtain the main relationship between (ρ)-radii and (τ)-radii, $\rho \in \mathbb{C}^*$, $\tau > 0$. This relationship extends the "symmetric" relationship

$$\tau w_{(\tau)}(T) = (2-\tau) w_{(\tau)}(T), \quad 0 < \tau < 2,$$

that was already known (see [2, Theorem 3]).

Proposition 3.3. *Let $\rho \neq 0$ and $\alpha > 0$ be two scalars. Let $T \in \mathcal{L}(H)$. The following assertions are equivalent:*

(i) $w_{(\rho)}(T) \le \alpha$;

(ii) $\big((\rho-1)zT - \rho\alpha I\big)$ *is invertible and* $\big\|(zT)\big((\rho-1)zT - \rho\alpha I\big)^{-1}\big\| \le 1$, $\forall z \in \mathbb{D}$;

(iii) $\big((\rho-1)T - \rho w I\big)$ *is invertible and* $\big\|T\big((\rho-1)T - \rho w I\big)^{-1}\big\| \le 1$, $\forall |w| > \alpha$.

Furthermore, we have:

$$|\rho| w_{(\rho)}(T) = \big(1 + |\rho - 1|\big) w_{1+|\rho-1|}(T). \tag{3.1}$$

Hence, the map $\rho \in \mathbb{C}^* \mapsto |\rho| w_{(\rho)}(T)$ is constant on circles of center 1, is continuous on $\mathbb{C}^*$ and can be extended continuously to $2w_{(2)}(T)$ at 0.

Proof. Using the results of Lemma 3.2, we can see that items (ii) and (iii) are equivalent and that item (i) implies item (ii). We only need to show that item (ii) implies $r(T) \le \alpha$. We can reduce the proof to the case $\alpha = 1$ by considering $\frac{T}{\alpha}$ instead of T. We also recall that if $\rho > 0$, the result is valid (see [19, Thm. 1] or [7] for a proof). Let $\rho \neq 0$. We denote $S = \frac{1+|\rho-1|}{|\rho|} T$. Suppose that $[(\rho-1)zT-\rho I]^{-1}$ exists and that $\|zT[(\rho-1)zT-\rho I]^{-1}\| \leq 1$, for all $z \in \mathbb{D}$. With $\rho - 1 = |\rho-1|e^{it}$, $\rho = |\rho|e^{is}$ and $w = z.e^{-is+it}$ we then have

$$
\begin{aligned}
& \left\| zT\big[(\rho-1)zT-\rho I\big]^{-1}\right\| \leq 1 \\
\Leftrightarrow\ & \left\| zT\big[|\rho-1|e^{it}zT-|\rho|e^{is}I\big]^{-1}\right\| \leq 1 \\
\Leftrightarrow\ & \left\| ze^{-is}e^{it}e^{-it}T\big[|\rho-1|ze^{-is}e^{it}T-|\rho|I\big]^{-1}\right\| \leq 1. \\
\Leftrightarrow\ & \left|e^{-it}\right| \left\| wT\big[|\rho-1|wT-|\rho|I\big]^{-1}\right\| \leq 1 \\
\Leftrightarrow\ & \left\| wT\big[\big(1+|\rho-1|-1\big)wT-|\rho|I\big]^{-1}\right\| \leq 1 \\
\Leftrightarrow\ & \left\| w\frac{1+|\rho-1|}{|\rho|}T\left[\big(1+|\rho-1|-1\big)w\frac{1+|\rho-1|}{|\rho|}T-\big(1+|\rho-1|\big)I\right]^{-1}\right\| \leq 1 \\
\Leftrightarrow\ & \left\| wS\big[\big(1+|\rho-1|-1\big)wS-\big(1+|\rho-1|\big)I\big]^{-1}\right\| \leq 1.
\end{aligned}
$$

Since w describes $\mathbb{D}$ when z does, this is true for all $w \in \mathbb{D}$. Therefore $w_{(1+|\rho-1|)}(S) \leq 1$ as $1+|\rho-1| > 0$ (see the beginning of the proof and Lemma 3.2). Thus, $r(S) \leq 1$, which implies $r(T) \leq \frac{|\rho|}{1+|\rho-1|} \leq 1$.

Now that we have showed that the condition about the spectral radius of T is not necessary, we can see that the equivalences in the previous computations give

$$w_{(\rho)}(T) \leq 1 \quad \Leftrightarrow \quad w_{(1+|\rho-1|)}\left(\frac{1+|\rho-1|}{|\rho|}T\right) \leq 1.$$

By homogeneity of the (ρ_n)-radii, this is equivalent to

$$|\rho| w_{(\rho)}(T) = \big(1+|\rho-1|\big) w_{(1+|\rho-1|)}(T).$$

The properties of the map $\rho \in \mathbb{C}^* \mapsto |\rho| w_{(\rho)}(T)$ can now be obtained from its restriction to $[1, +\infty[$, which is known to be continuous (see [2, Corollary 2] for example). ∎

Equation (3.1) gives a simple geometric understanding of a formula that was previously known only for real numbers ρ between 0 and 2. It also implies the following relationship between C_ρ classes.

COROLLARY 3.4. *We have*

$$C_{(\rho)} = \frac{1+|\rho-1|}{|\rho|} C_{(1+|\rho-1|)}.$$

We conclude that complex (ρ)-radii of an operator T can be expressed in terms of the real positive ones.

Corollary 3.5. *Let $\rho \neq 0$ and let $T \in \mathcal{L}(H)$. We have:*

(i) $w_{(\rho)}(I) = \frac{1+|\rho-1|}{|\rho|}$, $\forall \rho \neq 0$;

(ii) *If T is normal, then* $w_{(\rho)}(T) = \|T\| \frac{1+|\rho-1|}{|\rho|}$;

(iii) *If $T^2 = 0$, then* $w_{(\rho_n)}(T) = w_{(\rho_1)}(T) = \frac{2w(T)}{|\rho_1|} = \frac{\|T\|}{|\rho_1|}$;

(iv) *It $T^2 = bI$, $b \in \mathbb{C}$, then* $|\rho| w_{(\rho)}(T) = w(T) + \sqrt{w_2(T)^2 + |b|(|\rho-1|^2-1)}$;

(v) *It $T^2 = aT$, $a \in \mathbb{C}$, then* $|\rho| w_\rho(T) = 2w(T) + |a||\rho - 1|$.

Proof. (i) It is known that $w_{(\rho)}(I) = 1$ when $1 \leq \rho$. The relationship of Proposition 3.3 gives the result.
(ii) When T is normal, we have $w_{(\rho_n)}(T) = \|T\| w_{(\rho_n)}(I)$.
(iii) If $T^2 = 0$, then $T \in C_{(\rho_n)}$ if and only if $I + Re\big(\frac{2}{\rho_1} zT\big) \geq 0$ for all $z \in \mathbb{D}$. By Corollary 2.7, this is equivalent to $\frac{T}{|\rho_1|} \in C_{(1)}$, to $\frac{2T}{|\rho_1|} \in C_{(2)}$ and to $T \in C_{(\rho_1)}$. Thus, Lemma 2.12 and the following facts

$$w_{(2)}(T) = w(T) \quad \text{and} \quad w_{(1)}(T) = \|T\|$$

imply that

$$w_{(\rho_n)}(T) = w_{(\rho_1)}(T) = \frac{2w(T)}{|\rho_1|} = \frac{\|T\|}{|\rho_1|}.$$

(iv), (v) We can reduce these cases to $T^2 = I$ (respectively $T^2 = T$) by taking δ to be a square root of b (respectively a) and looking at $\frac{T}{\delta}$ (respectively $\frac{T}{\delta^2}$). Then, [2, Theorem 6] gives the result when $\rho > 0$, and we extend it to $\rho \in \mathbb{C}^*$ by using Proposition 3.3. ∎

Computations and some applications. For the next auxiliary result we need some notation. For an operator T acting on H and for $h \in H$, define

$$V_h := \overline{Span\big(T^n(h),\, n \geq 0\big)} \quad \text{and} \quad T_h := T|_{V_h} \in \mathcal{L}(V_h).$$

Lemma 3.6. *Let $T \in \mathcal{L}(H)$. Let $\rho \neq 0$. Then, with the previous notation, we have*

$$w_{(\rho)}(T) = \sup_{h \in H} \big(w_{(\rho)}(T_h)\big).$$

If we also have $P(T) = 0$ for some $P \in C[X]$ with $deg(P) = n$, then T_h can be identified as some matrix $S \in M_n(\mathbb{C})$ such that $P(S) = 0$, and the computation of $w_{(\rho)}(T_h)$ can be obtained from the computation of $w_{(\rho)}(S)$.

Proof. Let $h \in H$. We already proved in Corollary 2.16 that $w_{(\rho)}(T_h) \leq w_{(\rho)}(T)$. Conversely, for $\frac{1}{u} = \sup_{h\in H}\big(w_{(\rho)}(T_h)\big)$, $\big(I - z\frac{T}{u}\big)$ is invertible as $\big(I - z\frac{T_h}{u}\big)$ is invertible for all $h \in H$ and we have $Re\big(\big\langle\big(I + f_{(\rho)}\big(z\frac{T}{u}\big)\big)g, g\big\rangle\big) \geq 0$ for all $g \in H$. Thus $\frac{T}{u} \in C_{(\rho)}$, which implies $\sup_{h\in H}\big(w_{(\rho)}(T_h)\big) \geq w_{(\rho)}(T)$ and concludes the proof. ∎

Remark 3.7. Here is an attempt to compute $w_{(\rho)}(T)$, $\rho > 1$, when T satisfies the quadratic equation $T^2 + aT + bI = 0$. We use some ideas from [2], which allows one to obtain an expression of $w_{(\rho)}(T)$ depending on $w_{(2)}(T)$ when $a = 0$ or $b = 0$.

Up to considering $re^{it}T$, we can assume that $|b| = 1$ and $Re(\bar{a}b) = 0$. With α, β the roots of $X^2 + aX + b$ and $\eta \in \mathbb{C}$, we want to compute $w_{(\rho)}(M)$, for $M = \begin{pmatrix} \alpha & \eta \\ 0 & \beta \end{pmatrix}$. Using Lemma 3.2 (ii) and Proposition 3.3, we obtain that $w_{(\rho)}(M)$ is the largest (in modulus) z that is solution of

$$2(\rho-1)^2 + \rho^2|z|^2\left(\frac{|a|^2 + |a^2 - 4b|}{2}\right) + |\eta|^2\rho^2|z|^2 = \left|(\rho-1)^2\frac{ia}{|a|} + a(\rho-1)\rho z + \rho^2 z^2\right|^2 + 1.$$

In the case $\rho = 2$, the equation simplifies into

$$2 + 2|z|^2\left(|a|^2 + |a^2 - 4b|\right) + 4|\eta|^2|z|^2 = \left|4z^2 + 2az + \frac{ia}{|a|}\right|^2 + 1.$$

However, unlike the case where $a = 0$ or $b = 0$ in [2], we couldn't find a way to have an algebraic expression of $w_{(\rho)}(T)$ in terms of $w_{(2)}(T)$.

Using Proposition 3.3, we can also generalize some results of [1] about characterizing unitary operators through their ρ-radii.

Proposition 3.8. *Let $T \in \mathcal{L}(H)$ be invertible. Then*

(i) *T is unitary if and only if $\sigma(T) \subset \partial\mathbb{D}$ and there exists $\rho \in \mathbb{C}^*$ such that*

$$w_{(\rho)}(T) \leq w_{(\rho)}(I).$$

(ii) *$T = \|T\|U$ for U unitary if and only if there exists $\rho \in \mathbb{C}^*$ and $m > 0$ such that*

$$\frac{w_{(\rho)}(T^{-m})}{w_{(\rho)}(I)} = \left(\frac{w_{(\rho)}(T)}{w_{(\rho)}(I)}\right)^{-m}.$$

Proof. (i) The formula of Proposition 3.3 can be rewritten as $w_{(\rho)}(S) = w_{(\rho)}(I)w_{(1+|\rho-1|)}(S)$. It allows us to obtain the same relationship between T and I for $w_{(1+|\rho-1|)}$, and we can then apply [1, Theorem 2.1] to get the result.

(ii) The formula of Proposition 3.3 allows us to obtain the same relationship for $w_{(1+|\rho-1|)}$, which simplifies into:

$$w_{1+|\rho-1|}(T^{-m}) = w_{1+|\rho-1|}(T)^{-m}.$$

We can now apply [1, Theorem 1.1], and the proof is complete. ∎

Proposition 3.9. *Let $\rho \neq 0$ be a complex number. Then*

(i) *The ρ-radius $w_{(\rho)}(\cdot)$ is a norm on $\mathcal{L}(H)$ if and only if $|\rho - 1| \leq 1$;*

(ii) *If $|\rho - 1| > 1$, then, for all operators T_1 and T_2 in $\mathcal{L}(H)$, we have*

$$w_\rho(T_1 + T_2) \leq \big(1 + |\rho - 1|\big)\big(w_\rho(T_1) + w_\rho(T_2)\big).$$

Proof. For two operators T_1, T_2, we have

$$w_{(\rho)}(T_1 + T_2) \leq C\left(w_{(\rho)}(T_1) + w_{(\rho)}(T_2)\right)$$

if and only if the same is true for $w_{(1+|\rho-1|)}$. It is known [23, 14] that for $\tau > 0$, $w_{(\tau)}$ is a norm if and only if $0 < \tau \leq 2$. We conclude that $w_{(\rho)}(\cdot)$ is a norm if and only if ρ lies in the closed circle of center 1 and radius 1. Moreover, when $\tau > 2$, $w_{(\tau)}$ is a quasi-norm with multiplicative constant (also called the modulus of concavity of the quasi-norm [13]) lower or equal to τ. We thus obtain (ii). ∎

For the next proposition we recall that for $r > 0$ the disc algebra over the disc $\mathbb{D}(0, r)$, $\mathbb{A}(\overline{\mathbb{D}(0, r)})$, is the set of holomorphic functions on $\mathbb{D}(0, r)$ that are continuous on $\overline{\mathbb{D}(0, r)}$.

Proposition 3.10. *Let $\rho \neq 0$ be a complex number. Let $T \in C_{(\rho)}$. Then the functional calculus map $f \mapsto f(T)$ that sends a polynomial f into $f(T)$*

can be extended continuously to the disk algebra $\mathbb{A}\big(\mathbb{D}\big(0,\frac{1}{w_{(\rho)}(I)}\big)\big)$*. It is a morphism of Banach algebras, and satisfies*

$$\|f(T)\| \le \big(1+|\rho-1|\big)\|f\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)}.$$

Furthermore, for $f \in \mathbb{A}\big(\mathbb{D}\big(0,\frac{1}{w_{(\rho)}(I)}\big)\big)$ *such that* $f(0)=0$*, we have*

$$w_{(\rho)}\big(f(T)\big) \le w_{(\rho)}(I)\|f\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)}.$$

If $f \in \mathbb{A}(\mathbb{D})$ *with* $f(0)=0$*, we also have*

$$w_{(\rho)}\big(f(T)\big) \le \|f\|_{L^\infty(\mathbb{D})}.$$

The constants in these three inequalities are optimal.

Proof. We notice first that $T \in C_{(\rho)}$ is equivalent to $w_{(\rho)}(T) \le 1$, which is equivalent to

$$w_{(1+|\rho-1|)}(T) \le \frac{|\rho|}{1+|\rho-1|} = \frac{1}{w_{(\rho)}(I)} \le 1.$$

Hence, $w_{(\rho)}(I)T$ lies in $C_{(1+|\rho-1|)}$, so there exists a Hilbert space K and an unitary operator U over K such that

$$\big(w_{(\rho)}(I)T\big)^n = \big(1+|\rho-1|\big)P_H U^n|_H, \quad \forall n \ge 1.$$

Therefore, if we denote $V := \frac{U}{w_{(\rho)}(I)}$, for any polynomial P we get

$$P(T) = P_H\big[\big(1+|\rho-1|\big)P(V) - |\rho-1|P(0)I\big]|_H.$$

Since V is a normal operator with spectral radius $\frac{1}{w_{(\rho)}(I)}$, we then have

$$\begin{aligned}\|P(T)\| &\le \big\|\big(1+|\rho-1|\big)P(V) - |\rho-1|P(0)I\big\| \\ &\le \big\|\big(1+|\rho-1|\big)P - |\rho-1|P(0)\big\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)}.\end{aligned}$$

As the polynomials are dense in the algebra $\mathbb{A}\big(\mathbb{D}\big(0,\frac{1}{w_{(\rho)}(I)}\big)\big)$, the morphism of algebras $P \mapsto P(T)$ extends continuously on $\mathbb{A}\big(\mathbb{D}\big(0,\frac{1}{w_{(\rho)}(I)}\big)\big)$.

Let us estimate the norm of this map. For f in the algebra we denote $g(z) := f\big(\frac{z}{w_{(\rho)}(I)}\big)$. Hence, $g \in \mathbb{A}(\mathbb{D})$, and we have $f(T) = g\big(w_{(\rho)}(I)T\big)$. Applying a reformulation of Theorem 2 in [15] by Ando and Okubo, we obtain

$$\begin{aligned}\|f(T)\| = \big\|g\big(w_{(\rho)}(I)T\big)\big\| &\leq \max\big(1, 1+|\rho-1|\big)\|g\|_{L^\infty(\mathbb{D})}\\ &= \big(1+|\rho-1|\big)\|f\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)}.\end{aligned}$$

We will now prove the two remaining inequalities. The fact that V is normal implies that $f \mapsto f(V)$ is well defined and bounded on $\mathbb{A}\big(\mathbb{D}\big(0, \frac{1}{w_{(\rho)}(I)}\big)\big)$. Therefore

$$f(T) = P_H\big[\big(1+|\rho-1|\big)f(V) - |\rho-1|f(0)I\big]|_H, \ \forall f \in \mathbb{A}\left(\mathbb{D}\left(0, \frac{1}{w_{(\rho)}(I)}\right)\right).$$

We now suppose that f satisfies $f(0) = 0$. If $f \equiv 0$, then $f(T) = 0$ and the statements are true. Otherwise, up to dividing f by its norm, we may assume that $\|f\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)} = 1$. For a fixed $n \geq 1$, we get

$$f(T)^n = f^n(T) = \big(1+|\rho-1|\big)P_H f^n(V)|_H = \big(1+|\rho-1|\big)P_H f(V)^n|_H.$$

As we have $\|f(V)\| \leq \|f\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)} = 1$, the operator $f(V)$ lies in $C_{(1)}$ which in turns implies that $f(V)$ can be dilated on a larger Hilbert space as follows

$$f(V)^m = P_K W^m|_K, \ \forall m \geq 1,$$

with W a suitable unitary operator. Combining the two dilations, we obtain

$$f(T)^n = \big(1+|\rho-1|\big)P_H W^n|_H, \ \forall n \geq 1.$$

Therefore $f(T)$ lies in $C_{(1+|\rho-1|)}$, which is equivalent to $w_{(1+|\rho-1|)}(f(T)) \leq 1$. This inequality is in turn equivalent to $w_{(\rho)}(f(T)) \leq w_{(\rho)}(I)$, which proves the second inequality of this Proposition. Lastly, if $f \in \mathbb{A}(\mathbb{D})$ with $f(0) = 0$, we can use Schwarz's lemma to obtain

$$\|f\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)} \leq \frac{1}{w_{(\rho)}(I)}\|f\|_{L^\infty(\mathbb{D})},$$

which in turn gives $w_{(\rho)}(f(T)) \leq \|f\|_{L^\infty(\mathbb{D})}$.

For the optimality of these inequalities, let us take T such that $T^2 = 0$ and $\|T\| = |\rho|$, and $f(z) = z$. We then have

$$w_{(\rho)}(T) = \frac{\|T\|}{|\rho|} = 1 = w_{(\rho)}(I)\|f\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)} = \|f\|_{L^\infty(\mathbb{D})}$$

and

$$\|f(T)\| = |\rho| = \big(1 + |\rho - 1|\big)\|f\|_{L^\infty\left(\mathbb{D}\left(0,\frac{1}{w_{(\rho)}(I)}\right)\right)}.$$

The proof is complete. ∎

When ρ does not lie in $[1, +\infty[$, the algebra where the functional calculus is defined strictly contains the disc algebra $\mathbb{A}(\mathbb{D})$. For $0 < \rho < 1$, the norm of this map is then $2 - \rho$. This result differs from [15, Theorem 2] as Ando and Okubo looked in [15] at the map $f \mapsto f(T)$ on $\mathbb{A}(\mathbb{D})$ and not on a larger algebra.

4. Inequalities and parametrizations for (ρ_n)-radii

Operator radii of products and tensor products. A useful tool, used to study the behavior of a product or sum of double-commuting operators, is the following result, proved in [11, Theorem.4.2].

Proposition 4.1. *Let $T_n, S_n \in \mathcal{L}(H)$, $n \in \mathbb{Z}$, be such that for all $0 \le r < 1$, $t \in \mathbb{R}$, the series $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} T_n$ and $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} S_n$ converge absolutely and have self-adjoint non-negative sums. If, moreover, we have $T_n \cdot S_m = S_m \cdot T_n$, $\forall\, m, n \in \mathbb{Z}$, then the series $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} T_n \cdot S_n$ converges absolutely and has a self-adjoint non-negative sum, for all $0 \le r < 1$, $t \in \mathbb{R}$.*

Using Proposition 4.1 we can easily obtain the following auxiliary result.

Lemma 4.2. *Let $T, S \in \mathcal{L}(H)$ be two operators that are double-commuting (i.e., $TS = ST$, $TS^* = S^*T$). Let $(\rho_n)_n, (\tau_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) > 0$ and $\liminf_n \big(|\tau_n|^{\frac{1}{n}}\big) > 0$. Then, we have*

$$w_{(\rho_n\tau_n)}(ST) \le w_{(\rho_n)}(S)\, w_{(\tau_n)}(T).$$

Proof. If $S = 0$ or $T = 0$, then $ST = 0$ and both sides of the inequality are equal to zero. If $S \ne 0$ and $T \ne 0$, then, up to dividing S and T by their respective radius, we can consider that $w_{(\rho_n)}(S) = w_{(\tau_n)}(T) = 1$. Thus, we need to prove that $w_{(\rho_n\tau_n)}(S.T) \leq 1$. We define

$$T_m := \begin{cases} \frac{1}{\rho_m}T^m & \text{if } m \geq 1, \\ I & \text{if } m = 0, \\ \frac{1}{\rho_{|m|}}(T^*)^{|m|} & \text{if } m \leq -1, \end{cases} \qquad S_m := \begin{cases} \frac{1}{\tau_m}S^m & \text{if } m \geq 1, \\ I & \text{if } m = 0, \\ \frac{1}{\tau_{|m|}}(S^*)^{|m|} & \text{if } m \leq -1. \end{cases}$$

The condition $w_{(\rho_n)}(S) = w_{(\tau_n)}(T) = 1$, together with Lemma 2.12 and Proposition 2.6, ensure us that the conditions of Proposition 4.1 are fulfilled, since $I + Re\big(f_{(\rho_n)}(re^{it}S)\big) = \sum_{m\in\mathbb{Z}} r^{|m|}e^{imt}S_m$, for all $0 \leq r < 1$, $t \in \mathbb{R}$. Thus, $\sum_{m\in\mathbb{Z}} r^{|m|}e^{imt}S_mT_m$ converges absolutely, is self-adjoint, and has a positive sum, for all $0 \leq r < 1$, $t \in \mathbb{R}$. This implies that the series

$$\sum_{n\geq 1} \frac{2}{\rho_n\tau_n}\big(re^{it}ST\big)^n = f_{(\rho_n\tau_n)}\big(re^{it}ST\big)$$

is absolutely convergent and that $I + Re\big(f_{(\rho_n\tau_n)}(re^{it}ST)\big) \geq 0$ for all $0 \leq r < 1$, $t \in \mathbb{R}$. Thus $ST \in C_{(\rho_n\tau_n)}$ and $w_{(\rho_n\tau_n)}(ST) \leq 1$, which concludes the proof. ∎

Corollary 4.3. *Let $T, S \in \mathcal{L}(H)$ and let $(\rho_n)_n, (\tau_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) > 0$ and $\liminf_n \big(|\tau_n|^{\frac{1}{n}}\big) > 0$.*

(i) *If T and S double-commute, then*

$$w_{(\rho_n)}(ST) \leq w_{(1)}(S)w_{(\rho_n)}(T) \leq |\tau_1|w_{(\tau_n)}(S)w_{(\rho_n)}(T).$$

This inequality is optimal when $\dim(H) \geq 4$.

(ii) *We have*

$$w_{(1)}(ST) \leq w_{(1)}(S)w_{(1)}(T) \leq |\tau_1||\rho_1|w_{(\tau_n)}(S)w_{(\rho_n)}(T).$$

This inequality is optimal when $\dim(H) \geq 2$.

(iii) *For $R \in \mathcal{L}(H')$, we have*

$$w_{(\rho_n\tau_n)}(T \otimes R) \leq w_{(\rho_n)}(T)w_{(\tau_n)}(R).$$

Proof. (i) We use Lemma 4.2 for S, T and $(1)_n, (\rho_n)_n$ to get the left-hand side inequality. The right-hand side inequality comes from the fact $w_{(\tau_n)}(S) \geq \frac{\|S\|}{|\tau_1|}$ (cf. Lemma 2.12). By taking

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ such that } ST = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

some computation show that S and T double-commute, and that

$$\|S\| = \|T\| = \|ST\|, \quad S^2 = T^2 = (ST)^2 = 0.$$

Corollary 3.5(iii) shows that all three quantities are equal to $\frac{\|ST\|}{|\rho_1|}$.

(ii) The inequality on the right-hand side follows again from Lemma 2.12. By taking

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ such that } ST = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\|S\| = \|T\| = \|ST\| = 1, \ S^2 = T^2 = 0, \text{ and } ST \text{ is self-adjoint.}$$

Thus, $w_{(\tau_n)}(S) = w_{(\rho_n)}(T) = \frac{1}{\rho}$ and $w_{(1)}(ST) = 1$, so all quantities are equal to 1.

(iii) As I_H, T double-commute and $I_{H'}, R$ double-commute too, we can apply Lemma 4.2 to $(T \otimes I_{H'})(I_H \otimes R) = T \otimes R$. We then apply Corollary 2.16(iii). ∎

Although these inequalities are optimal for some operators, they tend to lose a good part of the information in the general case. For example, we have $w_{(3)}(I) = 1 \leq w_{(-1)}(I)w_{(-3)}(I) = 5$. Such a loss of information on the radius of the identity operator I also impacts almost every estimate of radii for other operators in $\mathcal{L}(H)$.

Corollary 4.4. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. Then,*

$$\frac{\|T\|}{|\rho_1|} \leq w_{(\rho_n)}(T) \leq \|T\| w_{(\rho_n)}(I).$$

Furthermore, the coefficients in this equivalence of quasi-norms are optimal.

Proof. The left-hand side inequality $\frac{\|T\|}{|\rho_1|} \leq w_{(\rho_n)}(T)$ has been obtained in Lemma 2.12. The equality case is obtained for T such that $T^2 = 0$, as seen in Corollary 3.5. The right-hand side inequality comes from Lemma 4.2, with $S = I$ and $\tau_n = 1$. It is an improvement of the one that was obtained in Lemma 2.12. The equality case is obtained for any T normal of norm 1. ∎

Operator radii as 1-parameter families. To better understand the behavior of the associated radii associated with classes of operators, it is useful to look at (ρ_n)-radii as 1-parameter families. This is obtained by studying the map $z \mapsto w_{(z\rho_n)}$. We will present results for the real parameter case ($r \in]0, +\infty[$) and for the complex one ($z \in \mathbb{C}^*$).

The two main ingredients we are using are the double-commuting inequality of Lemma 4.2 for T, I and $(\rho_n)_n, (1)_1$, and the fact that $f_{(z\rho_n)} = \frac{1}{z} f_{(\rho_n)}$.

Proposition 4.5. *Let $T \in \mathcal{L}(H)$ and consider $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$.*

(i) *For all $z \neq 0$, we have:*

$$\frac{|z|}{1 + |z-1|} w_{(z\rho_n)}(T) \leq w_{(\rho_n)}(T) \leq w_{(z\rho_n)}(T)\big(|z| + |z-1|\big).$$

(ii) *The map $z \mapsto w_{(z\rho_n)}(T)$ is continuous on $\mathbb{C}^*$, and $r \mapsto w_{(re^{it}\rho_n)}(T)$ is continuous and decreasing on $]0, +\infty[$, for all $t \in]-\pi, \pi]$.*

(iii) *We have*

$$\frac{1}{3} w_{(z\rho_n)}(T) \leq w_{(|z|\rho_n)}(T) \leq 3 w_{(z\rho_n)}(T),$$

and these inequalities are optimal.

Proof. (i) We use Lemma 4.2 to obtain

$$w_{(z\rho_n)}(T) \leq w_{(z)}(I) w_{(\rho_n)}(T) \text{ and } w_{(\rho_n)}(T) \leq w_{(z^{-1})}(I) w_{(z\rho_n)}(T).$$

As $w_{(z)}(I) = \frac{1+|z-1|}{|z|}$ and $w_{(z^{-1})}(I) = |z| + |z-1|$, we obtain the desired inequalities.

(ii) Up to changing $(\rho_n)_n$ by $(w\rho_n)_n$, the continuity must only be shown at the point $w = 1$, that is when $z \to 1$. As we have

$$w_{(\rho_n)}(T) \leq w_{(z\rho_n)}(T)\big(|z| + |z-1|\big) \leq w_{(\rho_n)}(T)\big(|z| + |z-1|\big)\frac{1 + |z-1|}{|z|}$$

and as $(|z|+|z-1|)$, $\frac{1+|z-1|}{|z|}$ both tend to 1 from above as $z \to 1$, we obtain

$$\lim_{z\to 1} w_{(z\rho_n)}(T) = w_{(\rho_n)}(T).$$

For any $t \in \mathbb{R}$ and $0 < r < R$, we have

$$w_{(Re^{it}\rho_n)}(T) \le w_{(Rr^{-1})}(I)w_{(re^{it}\rho_n)}(T) = w_{(re^{it}\rho_n)}(T).$$

Thus, $r \mapsto w_{(re^{it}\rho_n)}(T)$ is decreasing on $]0,+\infty[$.

(iii) We use the fact that $w_{(e^{it})}(I) = 1 + |e^{it}-1|$ has a maximum of 3 when $e^{it} = -1$. The equality case for the inequality on the left-hand side is attained at $T = I$, $\rho_n = 1$ and $z = -1$, whereas the equality case for the one on the right-hand side is attained at $T = I$, $\rho_n = -1$, $z = -1$. ∎

Since $r \mapsto w_{(r\rho_n)}(T)$ is decreasing, the classes $C_{(r\rho_n)}$ are increasing (for the usual order of inclusion of sets), for $r \in]0,+\infty[$. By using nilpotent operators of order 2, and item (iii) of Corollary 3.5, we can also immediately show that these inclusions are always strict.

For the following propositions, we recall that $\frac{1}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)} = 0$ if $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = +\infty$.

Proposition 4.6. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ and $T \in \mathcal{L}(H)$ be such that $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > r(T) \ge 0$. Then, there is $r > 0$ such that for all z with $|z| = r$,*

$$\frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)} \le w_{(z\rho_n)}(T) \le 1.$$

Proof. Let $s > 1$ be such that $r(sT) < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$. As

$$\limsup_{n\to\infty} \left(\frac{2s^n\|T^n\|}{|\rho_n|}\right)^{\frac{1}{n}} = \frac{r(sT)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)} < 1,$$

there is $B > 0$ such that $\frac{2s^n\|T^n\|}{|\rho_n|} \le B$. Thus, for all $w \in \mathbb{D}$, we have

$$\left\|f_{(z\rho_n)}(wT)\right\| \le \sum_{n\ge 1} \frac{2\|T^n\|}{|z||\rho_n|} \le \sum_{n\ge 1} \frac{B}{|z|s^n} = \frac{1}{|z|}\frac{sB}{1-s} < +\infty.$$

By taking $|z|$ large enough, we have $\|f_{(z\rho_n)}(wT)\| < 1$, which implies that

$$I + Re\big(f_{(z\rho_n)}(wT)\big) \ge 0, \ \forall w \in \mathbb{D}.$$

Thus $w_{(z\rho_n)}(T) \leq 1$. The left-hand side inequality comes from items (i) and (ii) of Proposition 2.18: we have

$$w_{(z\rho_n)}(T) \geq r(T) w_{(z\rho_n)}(I) \quad \text{and} \quad w_{(z\rho_n)}(I) \geq \frac{1}{\liminf_n \left(|z\rho_n|^{\frac{1}{n}}\right)}.$$

∎

Proposition 4.7. *Let $T \in \mathcal{L}(H)$ and let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. Then*

$$\lim_{|z| \to +\infty} \left(w_{(z\rho_n)}(T)\right) = \frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}.$$

Proof. According to Proposition 4.5 and Proposition 2.18, the map $r \mapsto w_{(re^{it}\rho_n)}(T)$ is decreasing on $]0, +\infty[$ and

$$w_{(\rho_n)}(T) \geq r(T) w_{(\rho_n)}(I) \geq \frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}.$$

We will show that $w_{(z\rho_n)}(T)$ is as close to this lower bound as we want when z is large enough. Let $\epsilon > 0$. If $r(T) = 0$, then $r\left(\frac{1}{\epsilon}T\right) = 0$, so Proposition 4.6 implies the existence of $r > 0$ such that $w_{(z\rho_n)}\left(\frac{T}{\epsilon}\right) \leq 1$ for all z with $|z| = r$. Thus, $w_{(z\rho_n)}(T) \leq \epsilon$. If $r(T) \neq 0$, for $0 < R < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$ we have

$$r\left(\frac{RT}{(1+\epsilon) r(T)}\right) \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right).$$

Thus, by Proposition 4.6, there exists $r > 0$ such that $w_{(z\rho_n)}\left(\frac{RT}{(1+\epsilon)r(T)}\right) \leq 1$ for all z with $|z| = r$. Hence,

$$\frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)} \leq w_{(z\rho_n)}(T) \leq \frac{(1+\epsilon) r(T)}{R}.$$

We then obtain the result by taking $R = \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)(1-\epsilon)$ if $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$ is finite, or $R = \frac{1}{\epsilon}$ if $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = +\infty$. ∎

Proposition 4.8. *Let $T \in \mathcal{L}(H)$. Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. We have:*

(i) *$z \mapsto w_{(z\rho_n)}(T)$ is uniformly continuous on $\mathbb{C} \setminus \mathbb{D}(0,\epsilon)$, for all $\epsilon > 0$. This maps tends to $+\infty$ as $|z| \to 0$, and to $\frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}$ as $|z| \to +\infty$;*

(ii) *For any $t \in \mathbb{R}$, the map $r \mapsto w_{(re^{it}\rho_n)}(T)$ is log-convex on $]0,+\infty[$.*

Proof. (i) On the closed set $\mathbb{C} \setminus \mathbb{D}(0,\epsilon)$, the function

$$z \mapsto w_{(z\rho_n)}(T)$$

is continuous, decreasing on every half-line of the form $e^{it}[\epsilon,+\infty[$, and converges to

$$\frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)}$$

as $|z| \to +\infty$. Thus, a standard argument (considering two cases, $\epsilon \leq |z| \leq R$ and $|z| \geq R$) shows that this map is uniformly continuous. One can also use the double-commuting inequality of Lemma 4.2 for T and I_H, as well as the uniform continuity of the map $z \mapsto w_{(z)}(I)$ on $\mathbb{C} \setminus \mathbb{D}(0,\eta)$, in order to prove the uniform continuity of $z \mapsto w_{(z\rho_n)}(T)$. The limit as $|z| \to +\infty$ has been obtained in Proposition 4.7, while the limit as $|z| \to 0$ comes from the fact that $w_{(z\rho_n)}(T) \geq \frac{\|T\|}{|z||\rho_1|}$, as remarked in Lemma 2.12.

(ii) Let $t \in \mathbb{R}$. Denote $G'(z) := -e^{-it} f_{(\rho_n)}(zT)$. For any $\alpha > 0$, we have $w_{(re^{it}\rho_n)}(T) \leq \alpha$ if and only if $f_{(e^{it}\rho_n)}\big(z\frac{T}{\alpha}\big)$ is analytic on $\mathbb{D}$ and $I + Re\big(\frac{1}{r} f_{(e^{it}\rho_n)}\big(z\frac{T}{\alpha}\big)\big) \geq 0$, for all $z \in \mathbb{D}$. By taking $w = \frac{z}{\alpha}$, this is equivalent to $G'(w)$ being analytic on $\mathbb{D}\big(0,\frac{1}{\alpha}\big)$ and $Re(G'(w)) \leq rI$, for all $w \in \mathbb{D}\big(0,\frac{1}{\alpha}\big)$. The result is then obtained by mimicking the proof of [2, Theorem 1] by Ando and Nishio and replacing G with G'. ∎

Even though the expression of $f_{(z\rho_n)}$ is more complex than $f_{(z)}(w) = \frac{2}{z}\frac{w}{1-w}$, the main regularity properties remain valid due to its analyticity.

Proposition 4.9. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. If one of the following assertions is true*

(i) $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) < 1$;

(ii) $|\rho_n| < 1$ *for some* $n \geq 1$;

(iii) $w_{(\rho_n)}(I) > 1$;

(iv) $\rho_n = M + x_n$, $(x_n)_n \in \ell_2(\mathbb{C})$, *then all operators in* $C_{(\rho_n)}(H)$ *are similar to contractions. If, on the contrary, we have:*

(i') $w_{(\rho_n)}(I) < 1$,

then $C_{(\rho_n)}(H)$ *contains operators that are not similar to contractions.*

Both statements remain true if the conditions are only fulfilled for the subsequence $(\rho_{kn})_n$*, for some fixed* $k \geq 1$.

Proof. (i), (ii), (iii) We can see that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If (iii) is true, then for $T \in C_{(\rho_n)}$, we have

$$r(T) \leq \frac{w_{(\rho_n)}(T)}{w_{(\rho_n)}(I)} < 1,$$

so T is similar to a contraction.

(iv) It has been shown in [17, Chapter 2] (see also [4, Corollary 5.2.1]) that when $\rho_n = M + x_n$, $(x_n)_n \in \ell_2(\mathbb{C})$, all operators in $C_{(\rho_n)}$ are similar to contractions.

(i') On the contrary, when $w_{(\rho_n)}(I) < 1$, $\frac{1}{w_{(\rho_n)}(I)} I \in C_{(\rho_n)}$ and this operator is not similar to a contraction.

The last assertion of the theorem follows from two facts. The first one is that $T \in C_{(\rho_n)}$ implies $T^k \in C_{(\rho_{kn})}$. The second one is that T^k is similar to a contraction if and only if T is similar to a contraction: see [10, Problem 6 (ii)] for a proof when $k = 2$ that extends to any k by taking $((f, g)) := \sum_{j=1}^{k-1} \langle A^j f, A^j g \rangle$. ∎

Proposition 4.10. *Let* $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ *be such that* $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$.

(i) *If* $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = +\infty$*, then* $\bigcup_{r>0} C_{(r\rho_n)}(H) = \mathcal{L}(H)$.

(ii) *If* $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) < +\infty$*, then we have*

$$\left\{T : r(T) < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)\right\} \subset \bigcup_{r>0} C_{(r\rho_n)}(H) \subset \left\{T : r(T) \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)\right\}.$$

(iii) *Moreover, we have*

$$\left\{T : r(T) < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)\right\} = \bigcup_{r>0} C_{(r\rho_n)}(H)$$

if and only if

$$w_{(r\rho_n)}\Big(\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)I\Big) > 1, \ \ \forall r > 0.$$

Proof. (i) By using Proposition 4.6, for any T there exists $r > 0$ such that $w_{(r\rho_n)}(T) \leq 1$.

(ii) We use again Proposition 4.6 in order to obtain the left-hand side inclusion. The other inclusion follows from Proposition 2.6.

(iii) Suppose that there is a number $r > 0$ and an operator T with $r(T) = \liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)$ such that $w_{(r\rho_n)}(T) \leq 1$. Then

$$1 \geq w_{(r\rho_n)}(T) \geq r(T) w_{(r\rho_n)}(I) \geq \frac{r(T)}{\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)} = 1.$$

Thus all inequalities are equalities, and $w_{(r\rho_n)}\big(\liminf_n (|\rho_n|^{\frac{1}{n}}).I\big) = 1$. Hence, the union of all $C_{(r\rho_n)}$ strictly contains $\big\{T : r(T) < \liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)\big\}$ if and only if it contains $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)I$. The proof is complete. ∎

Remark 4.11. Replacing $(\rho_n)_n$ by $(e^{it}\rho_n)_n$ leaves unchanged the quantity $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big)$. However, the union of all classes $C_{(r\rho_n)}$ can become a different set.

With $\rho_n = \rho$, we have $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) = 1$ and $w_{(\rho)}(I) = 1$ if and only if $\rho \in [1, +\infty[$. Thus,

$$\bigcup_{r>0} C_{(re^{it})_n}(H) = \big\{T : r(T) < 1\big\} \ \text{ if } \ t \neq 0 \ [2\pi].$$

This is not an equality if $t = 0$ (look at the identity operator I). However, the set $\bigcup_{r>0} C_{(r)}(H)$ does not contain all operators with spectral radius one. Indeed, it has been proven in [17, Chapter 2] (see also [4, Corollary 5.2.1]) that all operators contained in this union are all similar to contractions. Furthermore, all operators similar to a contraction are not in this union. For a counterexample, any non-orthogonal projection T (that is $T^2 = T$ and $\|T\| > 1$) is not in this union since Corollary 4.3,(v), says that $w_{(\rho)}(T) = \frac{\|T\|+|\rho-1|}{|\rho|} > 1$.

- For a sequence $(\rho_n)_n$ that satisfies $\alpha = \liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) \in]0, +\infty[$, we can go back to the case $\liminf_n \big(|\rho_n|^{\frac{1}{n}}\big) = 1$ by considering the sequence $\big(\frac{\rho_n}{\alpha^n}\big)_n$. As this normalization is equivalent to a dilation by a factor $\frac{1}{\alpha}$ on the class

$C_{(\rho_n)}$, we can then try to see if in this case the class $C_{(\rho_n)}$ is always included in the set of operators that are similar to contractions. This question is motivated by Properties 4.10 and 4.9. The answer is true when $w_{(\rho_n)}(I) > 1$, but Corollary 4.18 will give a negative answer in many remaining cases, even if we consider the inclusion in the set of power-bounded operators.

At this point we would like to mention that, for every $k \geq 2$, there is ([9]) a Hilbert space operator $T \notin \cup_{\rho>0} C_\rho$ but with T^k belonging to $C_{(\tau)}$ for every $\tau \geq 1$. Related results are given in the next proposition.

Proposition 4.12. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that* $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. *Let H a Hilbert space of dimension at least 2.*

(i) *For $T \in \mathcal{L}(H)$ with $T^2 = 0$ and $\|T\| > |\rho_1|$, T^k is in the class $C_{(\rho_n)}$ for every $k \geq 2$, but T is not.*

(ii) *If $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 1$, then $T^k \in \bigcup_{r>0} C_{(r\rho_n)}$ for some $k \geq 2$ implies that $T \in \bigcup_{r>0} C_{(r\rho_n)}$.*

(iii) *If $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) < 1$, then there exists $T \in \mathcal{L}(H)$ such that T^k lies in $\bigcup_{r>0} C_{(r\rho_n)}$ for every $k \geq 2$ whereas T does not.*

(iv) *If $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = 1$ and $I \notin \bigcup_{r>0} C_{(r\rho_n)}$, then $T^k \in \bigcup_{r>0} C_{(r\rho_n)}$ for some $k \geq 2$ implies that $T \in \bigcup_{r>0} C_{(r\rho_n)}$.*

Proof. (i) As we have $\|T\| > |\rho_1|$, T cannot lie in $C_{(\rho_n)}$, whereas $T^k = 0$ does.

(ii) Let T be such that $T^k \in \bigcup_{r>0} C_{(r\rho_n)}$ for some $k \geq 2$. Then, $r(T^k) \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$. Hence,

$$r(T) \leq \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)^{\frac{1}{k}} < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right),$$

that is $T \in \bigcup_{r>0} C_{(r\rho_n)}$ according to Proposition 4.10, (i) and (ii).

(iii) Take $r > 0$ such that

$$\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) < r < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)^{\frac{1}{2}},$$

and denote $T = rI$. Thus, using item (ii) of Proposition 4.10, we can see that since for every $k \geq 2$ we have

$$r(T^k) \leq r(T^2) < \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) < r(T),$$

T doesn't lie in $\bigcup_{r>0} C_{(r\rho_n)}$ whereas T^k does.

(iv) If $T^k \in \bigcup_{r>0} C_{(r\rho_n)}$, then $r(T^k) < 1$ according to item (iii) of Proposition 4.10. This implies that $r(T) < 1$, which implies in turn that $T \in \bigcup_{r>0} C_{(r\rho_n)}$. ∎

Remark 4.13. As the classes $C_{(r\rho_n)}$ are increasing for the inclusion of sets, the assertion $T \in \bigcup_{r>0} C_{(r\rho_n)}$ is equivalent to the existence of $R > 0$ such that $T \in C_{(r\rho_n)}$ for every $r \geq R$.

When $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = 1$ and $I \in \bigcup_{r>0} C_{(r\rho_n)}$, which is the case when $\rho_n = \rho > 0$, we do not know if the result of Găvruţa [9] stays true, as the type of operators he used in his proof is not suited in this setting: since there are sequences (ρ_n) such that $\bigcup_{r>0} C_{(r\rho_n)}$ contains all power-bounded operators (see Corollary 4.18), taking a T such that $T^k = I$ will not work.

EXAMPLE 4.14. For $\rho_n = 2(n!)$, we have $I + f_{(\rho_n)}(zT) = exp(zT)$, and a quick computation gives $w_{(2(n!))}(I) = \frac{2}{\pi} < 1$ (see item (iv) of Example 4.20 for another proof). Therefore $\frac{\pi}{2} I \in C_{(2(n!))}(H)$ and this class contains an operator not similar to a contraction.

We can also try to obtain some relationships between the $(\gamma_n\rho_n)$-radii of an operator, for sequences $(\gamma_n)_n \in \partial\mathbb{D}^{\mathbb{N}^*}$, in order to see for which sequences $(\gamma_n)_n$ the maximal or minimal radii are attained. The following lemma answers the question for the maximal radii when $T = I$.

LEMMA 4.15. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\alpha = \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$. If $\lim_{x\to\alpha^-} f_{(|\rho_n|)}(x) > 1$, then $f_{(|\rho_n|)}(x) = 1$ has a unique solution, r_1, on $]0, \alpha[$. Otherwise, denote $r_1 = \alpha$. We then have:*

(i) $w_{(-|\rho_n|)}(I) = \frac{1}{r_1}$;

(ii) $w_{(-|\rho_n|)}(I) \geq w_{(\gamma_n\rho_n)}(I) \geq \frac{1}{\alpha}$, *for any $(\gamma_n)_n \in \partial\mathbb{D}^{\mathbb{N}^*}$;*

(iii) *The condition*

$$w_{(r\gamma_n\rho_n)}(I) = \frac{1}{\alpha}, \ \forall r \geq 1, \ \forall \gamma_n \in \partial\mathbb{D}$$

is equivalent to

$$\lim_{x\to\alpha^-} f_{(|\rho_n|)}(x) \leq 1$$

and to

$$w_{(-|\rho_n|)}(I) = \frac{1}{\alpha}.$$

Proof. (i), (ii) The right-hand side inequality of (ii) is the last inequality of Lemma 2.12.

For any $z \in \mathbb{D}(0,\alpha)$ and $\gamma_n \in \partial\mathbb{D}$, we have

$$\left|f_{(\gamma_n\rho_n)}(z)\right| \leq \sum_{n\geq 1} \frac{2|z|^n}{|\rho_n|} = f_{(|\rho_n|)}(|z|).$$

Also, the map $x \mapsto f_{(|\rho_n|)}(x)$ is strictly increasing on $]0,\alpha[$, as $f_{(|\rho_n|)}$ is non-constant with positive Taylor coefficients, so if $\lim_{x\to\alpha^-} f_{(|\rho_n|)}(x) > 1$ the real number r_1 is indeed unique. Let $u > 0$ be such that $u \geq \frac{1}{r_1} \geq \frac{1}{\alpha}$. Then $\frac{1}{u} \leq r_1$ and

$$\lim_{x\to\frac{1}{u}^-} f_{|\rho_n|}(x) \leq 1.$$

Since we have

$$f_{(\gamma_n\rho_n)}\Big(\mathbb{D}\Big(0,\frac{1}{u}\Big)\Big) \subset \mathbb{D}\Big(0, \lim_{x\to\frac{1}{u}^-} f_{(|\rho_n|)}(x)\Big) \subset \mathbb{D},$$

Proposition 2.18 implies that $w_{(\gamma_n\rho_n)}(I) \leq \frac{1}{r_1}$. When $\gamma_n = -\frac{\overline{\rho_n}}{|\rho_n|}$, we have

$$f_{(\gamma_n\rho_n)}(x) = f_{(-|\rho_n|)}(x) = -f_{(|\rho_n|)}(x).$$

Thus, the negative number $\lim_{x\to\frac{1}{u}^-}\big(-f_{(|\rho_n|)}(x)\big)$ lies in the adherence of $f_{(-|\rho_n|)}\big(\mathbb{D}\big(0,\frac{1}{u}\big)\big)$, and the smallest $u \geq \frac{1}{\alpha}$ such that

$$f_{(-|\rho_n|)}\Big(\mathbb{D}\Big(0,\frac{1}{u}\Big)\Big) \subset Re_{\geq -1}$$

is $\frac{1}{r_1}$. Hence,

$$w_{(-|\rho_n|)}(I) = \frac{1}{r_1} \geq w_{(\gamma_n\rho_n)}(I).$$

(iii) By (ii) and using that $r \mapsto w_{(r\gamma_n\rho_n)}(I)$ is decreasing, we have

$$w_{(r\gamma_n\rho_n)}(I) = \frac{1}{\alpha},\ \ \forall r \geq 1,\ \ \forall(\gamma_n)_n \in \partial\mathbb{D}^{\mathbb{N}^*}$$

if and only if

$$w_{(-|\rho_n|)}(I) = \frac{1}{\alpha}.$$

This equation is equivalent to $r_1 = \alpha$, that is $\lim_{x\to\alpha^-}\big(f_{(|\rho_n|)}(x)\big) \leq 1$. ∎

We do not know if the $(|\rho_n|)$-radius of I is always the minimal one. The idea of the proof of Lemma 4.15 can be transported to any operator T if we add a summability condition to the sequence $(\rho_n)_n$.

PROPOSITION 4.16. *Let $a = (a_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\sum_{n\geq 1} \frac{1}{|a_n|} \leq 1$. Let $T \in \mathcal{L}(H)$ and define*

$$\rho_n := \begin{cases} 2a_n\|T^n\| & \text{if } T^n \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

(i) *If $r(T) > 0$ or if T is nilpotent, then $T \in C_{(\rho_n)}$.*

(ii) *If $r(T) > 0$ and $\liminf_n \left(|a_n|^{\frac{1}{n}}\right) = 1$, then $w_{(z_n\rho_n)}(T) = 1$, for all z_n such that $|z_n| \geq 1$ and $\lim_n \left(|z_n|^{\frac{1}{n}}\right) = 1$.*

Proof. (i) Suppose first that $r(T) > 0$. Since $\sum_n \frac{1}{|a_n|} < +\infty$, we have $\liminf_n(|a_n|^{\frac{1}{n}}) \geq 1$, thus $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) \geq r(T) > 0$. We also have:

$$\left\|f_{(\rho_n)}(zT)\right\| \leq \sum_{n\geq 1} \frac{2|z|^n\|T^n\|}{2|a_n|\|T^n\|} \leq \sum_{n\geq 1} \frac{1}{|a_n|} \leq 1.$$

Thus, $I + Re\big(f_{(\rho_n)}(zT)\big) \geq \big(1 - \left\|f_{(\rho_n)}(zT)\right\|\big)I \geq 0$ for all $z \in \mathbb{D}$, so $T \in C_{(\rho_n)}$. If T is nilpotent then $f_{(\rho_n)}(zT)$ becomes a finite sum and the same computation gives the result, as $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) > 0$.

(ii) When $r(T) > 0$ and $\liminf_n \left(|a_n|^{\frac{1}{n}}\right) = 1$, we have $r(T) = \liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)$, so

$$1 \geq w_{(\rho_n)}(T) \geq \frac{r(T)}{\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right)} = 1.$$

Thus $w_{(\rho_n)}(T) = 1$. If we multiply each a_n by a complex number z_n with $|z_n| \geq 1$ and $\lim_n \left(|z_n|^{\frac{1}{n}}\right) = 1$, the sum $\sum_{n\geq 1} \frac{1}{|z_n a_n|}$ decreases, while $\liminf_n \left(|z_n a_n|^{\frac{1}{n}}\right) = 1$. Thus, we can apply the previous result to $(z_n\rho_n)_n$ and obtain $w_{(z_n\rho_n)}(T) = 1$. ∎

Remark 4.17. For any T with $r(T) > 0$, if we take a sequence $(\rho_n)_n$ as in item (ii) of the previous Proposition, then the result says that $z \mapsto w_{(z\rho_n)}(T)$ is constant and equal to 1 on $\mathbb{C} \setminus \mathbb{D}$.

- The choice of $(\rho_n)_n$ only depends on $\|T^n\|$. For example, with any T normal with $\|T\| = 1$, by taking $a_n = \frac{\pi^2}{6} n^2$, we have $w_{(2a_n z_n)}(T) = 1$ for any sequence $(z_n)_n$ such that $1 \leq |z_n|$ and $\sup |z_n| < +\infty$.

- If T is quasinilpotent but not nilpotent, we have $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = 0$. However, the statement of item (i) holds true for such a T, with a very similar proof.

Using the ideas in the proof of Proposition 4.16, we can show that some sets $\bigcup_{r>0} C_{(r\rho_n)}$ largely differ from $\bigcup_{\rho>0} C_{(\rho)}$ or $\{T : r(T) < 1\}$ even if $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = 1$.

Corollary 4.18. *Let $(\rho_n)_n$ be such that $\liminf_n \left(|\rho_n|^{\frac{1}{n}}\right) = 1$. The following assertions are true:*

(i) *If $\left(\frac{1}{\rho_n}\right) \in \ell^1$, then $\bigcup_{r>0} C_{(r\rho_n)}$ contains all power-bounded operators.*

(ii) *If $f_{(\rho_n)} \in H^\infty(\mathbb{D})$ and $f'_{(\rho_n)} \in H^\infty(\mathbb{D})$, then $\bigcup_{r>0} C_{(r\rho_n)}$ contains an operator that is not power-bounded.*

(iii) *If $n^{k+1+\epsilon} = O(|\rho_n|)$ for $k \in \mathbb{N}^*$ and some $\epsilon > 0$, then $\bigcup_{r>0} C_{(r\rho_n)}$ contains all operators T such that $\|T^n\| = O(n^k)$.*

Proof. (i) Let T be a power-bounded operator with $\|T^n\| \leq C$. Let $r > 0$ and $z \in \mathbb{D}$. We have

$$\left\| f_{(r\rho_n)}(zT) \right\| \leq \sum_{n\geq 1} \frac{2}{r|\rho_n|} |z|^n \|T^n\| \leq \frac{2C}{r} \sum_{n\geq 1} \frac{1}{|\rho_n|} < +\infty.$$

Hence, for r large enough, we have $\left\| f_{(r\rho_n)}(zT) \right\| \leq 1$ for every $z \in \mathbb{D}$. This implies that

$$I + Re\big(f_{(r\rho_n)}(zT)\big) \geq 0, \ \forall z \in \mathbb{D}.$$

This in turn implies that $T \in C_{(r\rho_n)}$ since we also know that $r(T) \leq 1 = \liminf_n \left(|r\rho_n|^{\frac{1}{n}}\right)$.

(ii) We first note that both the entire series $f_{(\frac{\rho_n}{n})}(z) = \sum_{n\geq 1} \frac{2n}{\rho_n} z^n$ and $f_{(\rho_n)}$ have radii of convergence 1, so their sum is analytic on $\mathbb{D}$. We also have $f_{(\frac{\rho_n}{n})}(z) = z f'_{(\rho_n)}(z)$. Let N be a nilpotent operator of order 2 and set $T = I + N$. Since $T^n = I + nN$, we have $\|T^n\| \simeq n\|N\|$ so T is not power-bounded. We will show that T belongs to a class $C_{(r\rho_n)}$ for large enough $r > 0$. Let

$r > 0$ and $z \in \mathbb{D}$. We have:

$$\begin{aligned}\left\|f_{(r\rho_n)}(zT)\right\| &= \left\|\sum_{n\geq 1} \frac{2}{r\rho_n} z^n (I + nN)\right\| = \left\|\frac{1}{r} f_{(\rho_n)}(z) I + \frac{1}{r} z f'_{(\rho_n)}(z) N\right\| \\ &\leq \frac{1}{r}\Big(\left\|f_{(\rho_n)}\right\|_{H^\infty} + \left\|f'_{(\rho_n)}\right\|_{H^\infty} \|N\|\Big) < +\infty.\end{aligned}$$

Hence, for r large enough, we have $\left\|f_{(r\rho_n)}(zT)\right\| \leq 1$ for every $z \in \mathbb{D}$, which implies that

$$I + Re\big(f_{(r\rho_n)}(zT)\big) \geq 0, \ \forall z \in \mathbb{D}.$$

This in turn implies that $T \in C_{(r\rho_n)}$ since we also know that $r(T) = 1 = \liminf_n \big(|r\rho_n|^{\frac{1}{n}}\big)$.

(iii) Let T be such that $\|T^n\| = O(n^k)$ and let $z \in \mathbb{D}$. We have $\frac{\|T^n\|}{|\rho_n|} = O\big(\frac{1}{n^{1+\epsilon}}\big)$, so this sequence is in ℓ^1. If T is nilpotent, then T is power-bounded and we can apply (i) to get a positive $r > 0$ such that $T \in C_{(r\rho_n)}$. Otherwise, we can consider the complex numbers

$$a_n := \frac{\rho_n}{\|T^n\|} \left\|\left(\frac{\|T^n\|}{|\rho_n|}\right)_n\right\|_{\ell^1}.$$

We have

$$\sum_{n\geq 1} \frac{1}{|a_n|} = \left\|\left(\frac{\|T^n\|}{|\rho_n|}\right)_n\right\|_{\ell^1}^{-1} \sum_{n\geq 1} \frac{\|T^n\|}{|\rho_n|} = 1.$$

Thus, for $\tau_n := 2a_n\|T^n\|$, we can use Proposition 4.16 to obtain $T \in C_{(\tau_n)}$. Since $\tau_n = 2\rho_n \left\|\left(\frac{\|T^n\|}{|\rho_n|}\right)_n\right\|_{\ell^1}$, we have $\tau_n = r\rho_n$ for some $r > 0$, which concludes the proof. ∎

The condition $f'_{(\rho_n)} \in H^\infty(\mathbb{D})$ implies that the sequence $\big(\frac{n}{\rho_n}\big)_n$ is bounded, but it does not imply the condition $\big(\frac{1}{\rho_n}\big) \in \ell^1$ from (i). Thus, for a sequence (ρ_n) satisfying the conditions of item (ii), the set $\bigcup_{r>0} C_{(r\rho_n)}$ may not contain every power-bounded operator.

Some examples. We conclude this paper by providing a computation of $w_{(z\rho_n)}(I)$ in two examples, where sequences $(\rho_n)_n$ were chosen to match some common analytic maps. The difficulty lies in the computation of the boundary of $f_{(z\rho_n)}\big(\mathbb{D}\big(1, \frac{1}{u}\big)\big)$, as some specific points on the boundary do not always have an explicit expression.

EXAMPLE 4.19. Let $R > 0$ and $-\pi < t \leq \pi$. We have:

(i) $I + f_{(Re^{it}n)}(zI) = I - \frac{2}{Re^{it}} \log(1 - zI)$;
(ii) $w_{(Re^{it}n)}(I) = 1$ if $t = 0$ and $R \geq 2\log(2)$;
(iii) $w_{(Re^{it}n)}(I) = \frac{1}{\exp(\frac{R}{2})-1} > 1$ if $t = 0$ and $0 < R < 2\log(2)$;
(iv) $w_{(Re^{it}n)}(I) = \frac{1}{1-\exp(\frac{-R}{2})} > 1$ if $t = \pi$;
(v) $w_{(Re^{it}n)}(I) = 1$ if $t = \pm\frac{\pi}{2}$ and $R \geq \pi$;
(vi) $w_{(Re^{it}n)}(I) = \frac{1}{\sin(\frac{R}{2})}$ if $t = \pm\frac{\pi}{2}$ and $0 < R < \pi$;
(vii) $w_{(Re^{it}n)}(I) = 1$ if $0 < |t| < \frac{\pi}{2}$ and $R \geq 2\cos(t)\log(2\cos(t)) + 2\sin(t)t$;
(viii) If we have $0 < |t| < \frac{\pi}{2}$ and $0 < R < 2\cos(t)\log(2\cos(t)) + 2\sin(t)t$, then

$$w_{(Re^{it}n)}(I) = \inf\left\{u > 1 : 1 - \frac{2}{R}\, g_t(u) \geq 0\right\} > 1$$

with

$$g_t(u) := \cos(t)\log\left(\frac{\sqrt{u^2 - \sin(t)^2} + \cos(t)}{u}\right) + \arcsin\left(\frac{\sin(t)}{u}\right)\sin(t).$$

The same holds if $\frac{\pi}{2} < |t| < \pi$.

Proof. Let $R > 0$, $t \in]-\pi, \pi]$. As $n \in \mathbb{R}$, we have $w_{(Re^{-it}n)}(I) = w_{(Re^{it}n)}(I)$, so we can restrict the study to $t \in [0, \pi]$. A direct computation gives:

$$f_{(Re^{it}n)}(zT) = -\frac{2}{Re^{it}} \log(1 - zT).$$

As $\liminf_n \left(|n|^{\frac{1}{n}}\right) = 1$, we have $w_{(Re^{it}n)}(I) \geq 1$. Thus, we consider those $u > 1$ such that $I + Re\big(f_{(Re^{it}n)}\big(\frac{zI}{u}\big)\big)$ is positive for every $z \in \mathbb{D}$. It is equivalent to look at the positivity of

$$1 + Re\Big(f_{(Re^{it}n)}\Big(\frac{z}{u}\Big)\Big) = 1 - \frac{2}{R} Re\Big(e^{-it}\log\Big(1 - \frac{z}{u}\Big)\Big).$$

We start off by studying the boundary of $\log\left(\mathbb{D}\big(1, \frac{1}{u}\big)\right)$. By analyticity, we have $\partial\log\left(\mathbb{D}\big(1, \frac{1}{u}\big)\right) \subset \log\left(\partial\mathbb{D}\big(1, \frac{1}{u}\big)\right)$. As $\log\left(e^{is}\mathbb{R} \cap \mathbb{D}\big(1, \frac{1}{u}\big)\right)$ is a horizontal interval that is non-empty if and only if $|s| \leq \arcsin\left(\frac{1}{u}\right)$, the previous sets are equal and $\log\big(\mathbb{D}(1,\frac{1}{u})\big)$ is convex. Thus, the set $\log\big(\partial\mathbb{D}(1,\frac{1}{u})\big)$ can be parameterized by two arcs depending on the imaginary part of its elements:

$$s \mapsto \log\left(\cos(s) \pm \frac{1}{u}\sqrt{1-\sin(s)^2u^2}\right) + is,\;\; s \in \Big[-\arcsin\Big(\frac{1}{u}\Big); \arcsin\Big(\frac{1}{u}\Big)\Big].$$

We want to compute the minimum of $1 - \frac{2}{R}Re\big(e^{-it}\log\big(1-\frac{e^{is}}{u}\big)\big)$ in order to find for which $u > 1$ this minimum is non-negative. For the cases $t = 0$, $t = \pi$, and $t = \frac{\pi}{2}$, computing this minimum amounts to finding the extrema of the real or imaginary part of the elements in $\log\big(\partial\mathbb{D}(1,\frac{1}{u})\big)$. As these extrema are $\log\big(1 \pm \frac{1}{u}\big)$ for the real part and $\pm\arcsin\big(\frac{1}{u}\big)$ for the imaginary part, an easy computation gives all the $u > 1$ such that

$$\inf_{w\in\mathbb{R}}\Big(1 - \frac{2}{R}Re\Big(e^{-it}\log\Big(1-\frac{e^{iw}}{u}\Big)\Big)\Big) \geq 0$$

in all three cases, which proves the items (ii), (iii), (iv), (v), (vi).

For $0 < t < \frac{\pi}{2}$, computing this minimum leads to searching the lower bound of

$$f_1(s) := \cos(\pi-t)\log\left(\cos(s) - \frac{1}{u}\sqrt{1-\sin(s)^2u^2}\right) - s\sin(\pi-t).$$

For $\frac{\pi}{2} < t < \pi$, computing this minimum leads to searching the lower bound of

$$f_2(s) := \cos(\pi-t)\log\left(\cos(s) + \frac{1}{u}\sqrt{1-\sin(s)^2u^2}\right) - s\sin(\pi-t).$$

The derivatives of these maps are:

$$f_1'(s) = \frac{\sin(s)u\cos(\pi-t)}{\sqrt{1-\sin(s)^2u^2}} - \sin(\pi-t),\; f_2'(s) = -\frac{\sin(s)u\cos(\pi-t)}{\sqrt{1-\sin(s)^2u^2}} - \sin(\pi-t).$$

Both of them only have one zero, at $s = -\arcsin\big(\frac{\sin(t)}{u}\big)$. And in both cases the searched minimum for $1 - \frac{2}{R}Re\big(e^{-it}\log\big(1-\frac{e^{is}}{u}\big)\big)$ is:

$$1 - \frac{2}{R}\left[\cos(t)\log\left(\frac{\sqrt{u^2-\sin(t)^2}+\cos(t)}{u}\right) + \arcsin\left(\frac{\sin(t)}{u}\right)\sin(t)\right] = 1 - \frac{2}{R}g_t(u).$$

If $0 < t < \frac{\pi}{2}$, this minimum decreases towards

$$1 - \frac{2}{R}g_t(1) := 1 - \frac{2}{R}\big[\cos(t)\log\big(2\cos(t)\big) + t\sin(t)\big]$$

when $u \to 1^+$. So $\frac{I}{u} \in C_{(Re^{it}n)}$ for every $u > 1$ if and only if $1 - \frac{2}{R} g_t(1) \geq 0$, that is $R \geq 2g_t(1)$. This proves item (vii) and half of item (viii).

If $\frac{\pi}{2} < t < \pi$, this minimum decreases towards $-\infty$ as $u \to 1^+$, so the smallest u for which this minimum is non-negative verifies $u > 1$ and $w_{(Re^{it}n)}(I) = u$. This gives the other half of item (viii) and concludes the proof. ∎

Example 4.20. Let $R > 0$ and $-\pi < t \leq \pi$. We have:

(i) $I + f_{(Re^{it}n!)}(zI) = I + \frac{2}{R.e^{it}}\big(\exp(zI) - I\big)$;

(ii) $w_{(Re^{it}n!)}(I) = \frac{1}{\log(\frac{R}{2}+1)}$ if $t = \pi$;

(iii) $w_{(Re^{it}n!)}(I) = \frac{1}{\log(\frac{2}{2-R})}$ if $t = 0$ and $0 < R \leq 2 - \frac{2}{e}$;

(iv) $w_{(Re^{it}n!)}(I) = \frac{1}{\frac{\pi}{2}-t}$ if $0 \leq |t| < \frac{\pi}{2}$ and $R = 2\cos(t)$;

(v) $w_{(Re^{it}n!)}(I) \leq \frac{1}{\log(\frac{R}{2}-\cos(t))}$ for $R > 2 + 2\cos(t)$;

(vi) $w_{(Re^{it}n!)}(I) \geq \frac{1}{\sqrt{\pi^2+\log\left(\frac{R}{2\cos(t)}-1\right)^2}}$ if $0 \leq |t| < \frac{\pi}{2}$ and $R > 4\cos(t)$;

(vii) In general, we have

$$w_{(Re^{it}.n!)_n}(I) = \inf\Bigg\{u > 0 : \forall\theta \in [-\pi, \pi] \text{ with } \theta + \frac{\sin(\theta)}{u} = t + k\pi,$$
$$k \in \mathbb{Z}, \text{ we have } (-1)^k e^{\frac{\cos(\theta)}{u}} \cos(\theta) \geq \cos(t) - \frac{R}{2}\Bigg\}.$$

For $R \geq 2e^{\pi/2} - 2$, we can restrict the infimum after u in $]0, \frac{2}{\pi}]$ and to the smallest $\theta \in]\frac{-\pi}{2}, 0]$ such that $\theta + \frac{\sin(\theta)}{u} = t + k\pi$.

Proof. Let $R > 0$, $t \in [-\pi, \pi]$ and $u > 0$. As $n \in \mathbb{R}$, we have $w_{(Re^{-it}n!)}(I) = w_{(Re^{it}n!)}(I)$, so we restrict the study to $t \in [0, \pi]$. A computation gives

$$I + f_{(Re^{it}n!)}(zI) = I + \frac{2}{R.e^{it}}\big(\exp(zI) - I\big).$$

We will first use Lemma 4.15 to compute $w_{(-Rn!)}(I)$ and rule out the case $t = \pi$. As $f_{(Rn!)}(x) = \frac{2}{R}\big(\exp(x) - 1\big)$, we get

$$f_{(Rn!)}(x) = 1 \quad \Leftrightarrow \quad x = \log\left(\frac{R}{2} + 1\right).$$

Hence, $w_{(-Rn!)}(I) = \frac{1}{\log(\frac{R}{2}+1)}$ and item (ii) is proved.

As $\liminf_n \big(|n!|^{\frac{1}{n}}\big) = +\infty$, we have $\frac{I}{u} \in C_{(Re^{it}n!)}$ if and only if $u \geq w_{(Re^{it}n!)}(I)$, if and only if $I + Re\big(f_{(Re^{it}n!)}\big(z\frac{I}{u}\big)\big)$ for every $z \in \mathbb{D}$. Thus, we need to study the positivity of

$$1 + Re\Big(f_{(Re^{it}n!)}\Big(\frac{z}{u}\Big)\Big) = 1 + \frac{2}{R}Re\Big(e^{-it}\Big(\exp\Big(\frac{z}{u}\Big) - 1\Big)\Big),$$

for every $z \in \mathbb{D}$ and for any $u > 0$. By analyticity, we only need to make the computations for $z \in \partial\mathbb{D}$. We have

$$\begin{aligned} & 1 + \frac{2}{R}Re\Big(\exp\Big(\frac{z}{u} - it\Big) - e^{-it}\Big) \geq 0 \\ \Leftrightarrow \quad & \exp\Big(Re\Big(\frac{z}{u}\Big)\Big)\cos\Big(\frac{Im(z)}{u} - t\Big) \geq -\frac{R}{2} + \cos(t). \end{aligned}$$

Denote, for $s \in [-\pi, \pi]$,

$$g_u(s) := e^{\frac{\cos(s)}{u}}\cos\Big(t - \frac{\sin(s)}{u}\Big).$$

Thus, $\frac{I}{u} \in C_{(Re^{it}n!)}$ is equivalent to

$$\min_{s\in[-\pi,\pi]}(g_u(s)) \geq -\frac{R}{2} + \cos(t).$$

Therefore, this inequality will be verified if and only if $u \geq w_{(Re^{it}n!)}(I)$. Also, since

$$\min_s \big(g_u(s)\big) = \min_{|w|=\frac{1}{u}} \big(Re\big(\exp(w - it)\big)\big) = \min_{|w|<\frac{1}{u}} \big(Re\big(\exp(w - it)\big)\big),$$

we can see that $\min_s \big(g_u(s)\big)$ is the minimum of a harmonic non-constant map over the disc $\mathbb{D}\big(0, \frac{1}{u}\big)$. The maximum principle implies that the map $u \mapsto \min_s \big(g_u(s)\big)$ is strictly increasing. Hence, $w_{(Re^{it}n!)}(I)$ is the only number $u > 0$ such that $\min_{s\in[-\pi,\pi]} \big(g_u(s)\big) = -\frac{R}{2} + \cos(t)$.

Let us focus now on the minimum of g_u. The derivative of g_u is

$$g_u'(s) = \frac{1}{u}e^{\frac{\cos(s)}{u}}\sin\Big(t - \frac{\sin(s)}{u} - s\Big).$$

Hence, the minimum of g_u will be reached for a s_0 such that $h_u(s_0) := t - s_0 - \frac{\sin(s_0)}{u} = k\pi$, for some $k \in \mathbb{Z}$. For such a s_0, we will also have

$$g_u(s_0) = (-1)^k e^{\frac{\cos(s_0)}{u}} \cos(s_0).$$

If $u \geq 1$, the map h_u is strictly decreasing, with range $[t-\pi, t+\pi]$. Hence, there will only be 2 (resp. 3) values of s such that $h_u(s) = k\pi$ if $t \in]0,\pi[$ (resp. $t = 0$). If $t = 0$ and $u \geq 1$, these values of s will be $-\pi$, 0, π, and the minimum of g_u will be $g_u(\pi) = \exp\left(\frac{-1}{u}\right)$. Thus, if $t = 0$ and $w_{(Rn!)}(I) \geq 1$, we will have

$$\exp\left(\frac{-1}{w_{(Rn!)}(I)}\right) = -\frac{R}{2} + 1,$$

which is equivalent to $0 < R \leq 2 - \frac{2}{e}$. Thus $w_{(Rn!)}(I) = \frac{1}{\log(\frac{2}{2-R})}$, proving item (iii).

When $t \in]0,\pi[$ and $u \geq 1$, we have however no explicit formula for the two values of s mentioned above.

For $t \in [0, \frac{\pi}{2}[$ and $R = 2\cos(t)$, we will have $\min_s \left(g_{w_{(Re^{it}n!)}(I)}(s)\right) = 0$. As $e^{\frac{\cos(s)}{u}} \cos(s) = 0$ if and only if $s = \pm\frac{\pi}{2}$, this minimum will be attained at $\frac{\pi}{2}$ or $\frac{-\pi}{2}$, and $w_{(Re^{it}n!)}(I)$ will be the largest $u > 0$ such that $g_u(\frac{\pi}{2}) = 0$ or $g_u(-\frac{\pi}{2}) = 0$. The latter condition is equivalent to $\frac{1}{u} \pm t = \frac{\pi}{2} + k\pi$, that is $\frac{1}{u} = \frac{\pi}{2} \pm t + k\pi$. Since we have $0 \leq t < \frac{\pi}{2}$, the integer k needs to be positive. By looking at the smallest possible value for $\frac{1}{u}$ we get $w_{(Re^{it}n!)}(I) = \frac{1}{\frac{\pi}{2} - t}$, proving item (iv).

In general, we can see that $\min_s \left(g_u(s)\right) \geq -e^{\frac{1}{u}}$. When $R > 2 + 2\cos(t)$, the inequality $-e^{\frac{1}{u}} \geq \cos(t) - \frac{R}{2}$ is equivalent to $u \geq \frac{1}{\log(\frac{R}{2} - \cos(t))}$, which proves item (v).

If $u\pi < 1$, we have $u\pi = \sin(\alpha)$ for some $\alpha > 0$, and

$$g_u(\alpha) = -\cos(t) e^{\frac{\sqrt{1-u^2\pi^2}}{u}}.$$

When $R > 4\cos(t)$, the inequality $g_u(\alpha) \leq \cos(t) - \frac{R}{2}$ is equivalent to

$$u \leq \frac{1}{\sqrt{\pi^2 + \log\left(\frac{R}{2\cos(t)} - 1\right)^2}}.$$

Item (vi) is now proved.

Taking $R \geq 2e^{\pi/2} - 2 = R_0$, we get $-\frac{R}{2} + 1 \leq 2 - e^{\pi/2} < -1$ and

$$w_{(Re^{it}n!)}(I) \leq w_{(-Rn!)}(I) \leq w_{(-R_0 n!)}(I) = \frac{2}{\pi},$$

for every $0 \leq t \leq \pi$, according to Lemma 4.15. We can then see that for $u = w_{(Re^{it}n!)}(I)$ and for a number s_0 such that $g_u(s_0) = \min_s \big(g_u(s)\big)$ and $h_u(s_0) = k\pi$, the relationship

$$(-1)^k e^{\frac{\cos(s_0)}{u}} \cos(s_0) = g_u(s_0) = -\frac{R}{2} + \cos(t) < -1$$

implies that $\cos(s_0) > 0$ and that k is odd. In this case, $|s_0|$ will be the smallest real s in $[0, \frac{\pi}{2}[$ such that $h_u(s)$ or $h_u(-s)$ is equal to $k\pi$ with k odd. As we also have $h_u(\frac{-\pi}{2}) = t + \frac{\pi}{2} + \frac{1}{u} \geq \pi \geq t = h_u(0) \geq 0$, we can see that s_0 lies in $]\frac{-\pi}{2}, 0]$. This gives all the announced results. ∎

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