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Radon-Nikodýmification of arbitrary measure spaces

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Abstract: We study measurable spaces equipped with a σ-ideal of negligible sets. We find conditions under which they admit a localizable locally determined version – a kind of fiber space that locally describes their directions – defined by a universal property in an appropriate category that we introduce. These methods allow to promote each measure space $(X, \mathscr{A}, \mu)$ to a strictly localizable version $(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$, so that the dual of $\mathbf{L}_1(X, \mathscr{A}, \mu)$ is $\mathbf{L}_\infty(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$. Corresponding to this duality is a generalized Radon-Nikodým theorem. We also provide a characterization of the strictly localizable version in special cases that include integral geometric measures, when the negligibles are the purely unrectifiable sets in a given dimension.

Key words: Measurable space with negligibles; Radon-Nikodým Theorem; strictly localizable measure space; integral geometric measure; purely unrectifiable.

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Contents

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1. Foreword

The Radon-Nikodým Theorem does not hold for every measure space $(X, \mathscr{A}, \mu)$. One way to phrase this precisely is to consider the canonical embedding

$$\Upsilon : \mathbf{L}_\infty(X, \mathscr{A}, \mu) \to \mathbf{L}_1(X, \mathscr{A}, \mu)^*.$$

The following hold.

(A) Υ is injective (this corresponds to the uniqueness almost everywhere of Radon-Nikodým derivatives) if and only if $(X, \mathscr{A}, \mu)$ is semi-finite.

(B) Υ is surjective (this corresponds to the existence of Radon-Nikodým derivatives) if and only if the Boolean algebra $\mathscr{A}/\mathscr{N}_{\mu,\mathrm{loc}}$ is Dedekind complete (i.e. order complete as a lattice).

While (A) is classical, see e.g. [6, 243G(a)], (B) is recent and due to the second author, see [2, 4.6]. Let us recall the relevant definitions. Given a measure space $(X, \mathscr{A}, \mu)$, we abbreviate $\mathscr{A}^f := \mathscr{A} \cap \{A : \mu(A) < \infty\}$ and $\mathscr{N}_\mu := \mathscr{A} \cap \{N : \mu(N) = 0\}$. We say that $(X, \mathscr{A}, \mu)$ is *semi-finite* if every $A \in \mathscr{A}$ of infinite measure contains some $F \in \mathscr{A}^f \setminus \mathscr{N}_\mu$. Equivalently, $\mu(A) = \sup\{\mu(F) : A \supseteq F \in \mathscr{A}^f\}$. We further define the σ-ideal of *locally μ-null* sets as follows: $\mathscr{N}_{\mu,\mathrm{loc}} := \mathscr{A} \cap \left\{A : A \cap F \in \mathscr{N}_\mu \text{ for all } F \in \mathscr{A}^f\right\}$. It is easy to see [2, 4.4] that $(X, \mathscr{A}, \mu)$ is semi-finite if and only if $\mathscr{N}_{\mu,\mathrm{loc}} = \mathscr{N}_\mu$. Thus we obtain the following classical criterion, [6, 243G(b)].

(C) Υ is an isometric isomorphism if and only if $(X, \mathscr{A}, \mu)$ is semi-finite and the Boolean algebra $\mathscr{A}/\mathscr{N}_\mu$ is Dedekind complete.

Though semi-finiteness is a natural property, Caratheodory's method does not always provide it. For instance, the measure spaces $(\mathbb{R}^2, \mathscr{A}_{\mathscr{H}^1}, \mathscr{H}^1)$ and $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathscr{I}^1_\infty)$ are not semi-finite – see [8, 439H] and [4, 3.3.20]. Here, $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure in the Euclidean plane [4, 2.10.2], $\mathscr{A}_{\mathscr{H}^1}$ is the σ-algebra consisting of $\mathscr{H}^1$-measurable sets in Caratheodory's sense, $\mathscr{I}^1_\infty$ is a 1-dimensional integral geometric measure [4, 2.10.5(1)], and $\mathscr{B}(\mathbb{R}^2)$ is the σ-algebra whose members are the Borel subsets of $\mathbb{R}^2$. Both $\mathscr{H}^1(\Gamma)$ and $\mathscr{I}^1_\infty(\Gamma)$ coincide with the usual Euclidean length of Γ when this is a Lipschitz curve.

It is natural to want to associate, with an arbitrary $(X, \mathscr{A}, \mu)$, an improved version of itself – in a universal way – ideally one for which the Radon-Nikodým Theorem holds. This is one of our several achievements in this paper. It is not difficult to modify slightly the measure μ, keeping the underlying measurable space $(X, \mathscr{A})$ untouched, in order to make it semi-finite. Specifically, letting $\mu_{\mathrm{sf}}(A) = \sup\left\{\mu(A \cap F) : F \in \mathscr{A}^f\right\}$, for $A \in \mathscr{A}$, one checks that $(X, \mathscr{A}, \mu_{\mathrm{sf}})$ is semi-finite and that $\mathscr{N}_{\mu_{\mathrm{sf}}} = \mathscr{N}_{\mu,\mathrm{loc}}$. However, it appears to be a more delicate task to modify $(X, \mathscr{A}, \mu)$ in a canonical way in order for Υ to become surjective.

An idea for testing if Υ is surjective is as follows. Given $\alpha \in \mathbf{L}_1(X, \mathscr{A}, \mu)^*$ we apply the Radon-Nikodým Theorem "locally", as it is valid on each finite measure subspace $(F, \mathscr{A}_F, \mu_F)$, $F \in \mathscr{A}^f$, i.e. we represent by integration the functional $\alpha \circ \iota_F \in \mathbf{L}_1(F, \mathscr{A}_F, \mu_F)^*$, where $\iota_F \colon \mathbf{L}_1(F, \mathscr{A}_F, \mu_F) \to \mathbf{L}_1(X, \mathscr{A}, \mu)$ is the obvious map. This produces a family of Radon-Nikodým derivatives $\langle f_F \rangle_{F \in \mathscr{A}^f}$. By the almost everywhere uniqueness of Radon-Nikodým derivatives in finite measure spaces, this is a *compatible family* in the sense that $F \cap F' \cap \{f_F \neq f_{F'}\} \in \mathscr{N}_\mu$, for every $F, F' \in \mathscr{A}^f$. In order to obtain a globally defined Radon-Nikodým derivative, one ought to be able to "glue" together the functions of this family. A *gluing* of $\langle f_F \rangle_{F \in \mathscr{A}^f}$ is, by definition, an $\mathscr{A}$-measurable function $f \colon X \to \mathbb{R}$ such that $F \cap \{f \neq f_F\} \in \mathscr{N}_\mu$ for every $F \in \mathscr{A}^f$.

The question whether such a gluing exists takes us away from the realm of measure spaces, as it rather pertains to *measurable spaces with negligibles*, abbreviated MSNs, i.e. triples $(X, \mathscr{A}, \mathscr{N})$ where $(X, \mathscr{A})$ is a measurable space and $\mathscr{N} \subseteq \mathscr{A}$ is a σ-ideal. The notion of compatible family $\langle f_E \rangle_{E \in \mathscr{E}}$ of $\mathscr{A}_E$-measurable functions $E \to \mathbb{R}$ subordinated to an arbitrary collection $\mathscr{E} \subseteq \mathscr{A}$ readily makes sense in this more general setting, as does the notion of gluing of such a compatible family. That each compatible family of partially defined measurable functions admits a gluing is equivalent to the Boolean algebra $\mathscr{A}/\mathscr{N}$ being Dedekind complete. In this case we say that $(X, \mathscr{A}, \mathscr{N})$ is *localizable*. Equivalently, $(X, \mathscr{A}, \mathscr{N})$ is localizable if and only if every collection $\mathscr{E} \subseteq \mathscr{A}$ admits an $\mathscr{N}$-essential supremum (see 3.3 for a definition), which corresponds to taking an actual supremum in the Boolean algebra $\mathscr{A}/\mathscr{N}$. For a proof of these classical equivalences, see e.g. [2, 3.13]. It will be convenient to call *$\mathscr{N}$-generating* a collection $\mathscr{E} \subseteq \mathscr{A}$ that admits X as an $\mathscr{N}$-essential supremum. For instance, one easily checks that if $(X, \mathscr{A}, \mu)$ is a semi-finite measure space, then $\mathscr{A}^f$ is $\mathscr{N}_\mu$-generating.

Let $(X, \mathscr{A}, \mathscr{N})$ be a localizable MSN, $\mathscr{E} \subseteq \mathscr{A}$ be $\mathscr{N}$-generating, and $\langle f_E \rangle_{E \in \mathscr{E}}$ be a compatible family of partially defined measurable functions. The problem of gluing this compatible family in our setting is reminiscent of the fact that, for a topological space X, the functor of continuous functions on open sets is a sheaf. However, unlike in the case of continuous functions, in order to define f globally, we ought to make choices on the domains $E \cap E'$, for $E, E' \in \mathscr{E}$, because f_E and $f_{E'}$ do not coincide everywhere there, but merely *almost everywhere*. In an attempt to avoid the issue, one can replace $\mathscr{E}$ with an almost disjointed refinement of itself, say $\mathscr{F}$. By this we mean that each member of $\mathscr{F}$ is contained in a member of $\mathscr{E}$, that $\mathscr{F}$ is $\mathscr{N}$-generating, and that $F \cap F' \in \mathscr{N}$ whenever $F, F' \in \mathscr{F}$ are distinct. The existence of $\mathscr{F}$ follows from Zorn's Lemma, see 4.9. Still, $F \cap F'$ may not be empty whenever $F, F' \in \mathscr{F}$ are distinct and we are again in a position to make choices. A step further along the road would be to produce from $\mathscr{F}$ a disjointed family $\mathscr{G}$ whose union is conegligible. From classical measure theory, we learn of two situations when this is doable. First, in the presence of a lower density of $(X, \mathscr{A}, \mathscr{N})$ (see 10.1 for a definition), and second when $\operatorname{card} \mathscr{F} \leqslant \mathfrak{c}$ (see the proof of 7.6). In those cases, a gluing exists. In fact, in the context of measure spaces, the existence of a lower density yields a somewhat stronger structure than localizability. In order to state this, we need one more definition. We say that a measure space $(X, \mathscr{A}, \mu)$ is *locally determined* if it is semi-finite and if the following holds:

$$\forall A \subseteq X : \left[\forall F \in \mathscr{A}^f : A \cap F \in \mathscr{A}\right] \Rightarrow A \in \mathscr{A}.$$

A complete locally determined measure space $(X, \mathscr{A}, \mu)$ admits a lower density if and only if it is strictly localizable, which means, by definition, that there exists a partition $\mathscr{G} \subseteq \mathscr{A}^f$ of X such that

$$\mathscr{A} = \mathscr{P}(X) \cap \{A : A \cap G \in \mathscr{A} \text{ for all } G \in \mathscr{G}\}$$

and $\mu(A) = \sum_{G \in \mathscr{G}} \mu(A \cap G)$, for $A \in \mathscr{A}$. See [7, 341M] for a proof. The existence of a lower density for a strictly localizable measure space follows from the case of finite measure spaces by gluing, and the case of finite measure spaces is a consequence of a martingale convergence theorem. Even though the notion of a lower density makes sense for MSNs, their existence does not hold for even the most natural generalization of finite measure spaces, namely ccc MSNs (satisfying the countable chain condition, 4.3 and 4.5), see [16].

Both notions of localizability (of an MSN) and local determination (of a measure space) seem to express in different ways the fact that "there are enough measurable sets".

For instance, one easily checks that an MSN $(X, \mathscr{A}, \{\emptyset\})$, such that $\mathscr{A}$ contains all singletons, is localizable if and only if $\mathscr{A} = \mathscr{P}(X)$. Thus, given an arbitrary MSN $(X, \mathscr{A}, \mathscr{N})$, one may naively attempt to "add measurable sets" in a smart way in order to obtain a localizable MSN $\left(X, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right)$, just as many as needed, and that would be a "localizable version" of $(X, \mathscr{A}, \mathscr{N})$. Unfortunately, within ZFC this cannot always be done while "sticking in the base space X", as shown by the following, quoted from [2].

Theorem. *Assume that:*

(1) $C \subseteq [0,1]$ *is some Cantor set of Hausdorff dimension 0;*

(2) $X = C \times [0,1]$*;*

(3) $\mathscr{A}$ *is a* σ*-algebra such that* $\mathscr{B}(X) \subseteq \mathscr{A} \subseteq \mathscr{P}(X)$*;*

(4) $\mathscr{N} = \mathscr{N}_{\mathscr{H}^1}$ or $\mathscr{N} = \mathscr{N}_{\mathrm{pu}}$.

Then $(X, \mathscr{A}, \mathscr{N})$ *is consistently not localizable.*

Here, $\mathscr{N}_{\mathrm{pu}}$ consists of those subsets S of X that are *purely 1-unrectifiable*, i.e. $\mathscr{H}^1(S \cap \Gamma) = 0$ for every Lipschitz (or, for that matter, C^1) curve $\Gamma \subseteq \mathbb{R}^2$. Thus, one may need to also add points to the base space X and, in particular cases such as the one above, we give a very specific way of doing so, in the last section of this paper. In the case of a general measure space $(X, \mathscr{A}, \mu)$, we can get a feeling of what needs to be done, when trying to define the gluing of a compatible family $\langle f_F \rangle_{F \in \mathscr{A}^f}$. Indeed, each $x \in X$ may belong to several $F \in \mathscr{A}^f$ and this calls for considering an appropriate quotient of the fiber bundle $\big\{(x, F) : x \in F \in \mathscr{A}^f\big\}$.

One of the tasks that we assign ourselves in this paper is to define a general notion of "localization" of an MSN and to prove existence results in some cases. Since a definition of "localization" will involve a universal property, it is critical to determine which category is appropriate for our purposes. As this offers unexpected surprises, we describe the several steps in some detail. The objects of our first category MSN are the *saturated* MSN $(X, \mathscr{A}, \mathscr{N})$, by what we mean that for every $N, N' \subseteq X$, if $N \subseteq N'$ and $N' \in \mathscr{N}$, then $N \in \mathscr{N}$. This is in analogy with the notion of a complete measure space. In order to define the morphisms between two objects $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$, we say that a map $f \colon X \to Y$ is $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurable if $f^{-1}(B) \in \mathscr{A}$ for every $B \in \mathscr{B}$ and $f^{-1}(M) \in \mathscr{N}$ for every $M \in \mathscr{M}$. For instance, if X is a Polish space and μ is a diffuse probability measure on X, there exists [17, 3.4.23] a Borel isomorphism $f \colon X \to [0,1]$ such that $f_{\#}\mu = \mathscr{L}^1$, where $\mathscr{L}^1$ is the Lebesgue measure, thus f is $[(\mathscr{B}(X), \mathscr{N}_\mu), (\mathscr{B}([0,1]), \mathscr{N}_{\mathscr{L}^1})]$-measurable. We define an equivalence relation for such measurable maps $f, f' \colon X \to Y$ by saying that $f \sim f'$ if and only if $\{f \neq f'\} \in \mathscr{N}$. The morphisms in the category MSN between the objects $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$ are the equivalence classes of

$[(\mathscr{A},\mathscr{N}),(\mathscr{B},\mathscr{M})]$-measurable maps. At this stage, we need to suppose that $(X,\mathscr{A},\mathscr{N})$ is saturated for the relation of equality almost everywhere to be transitive 2.7. With this assumption, the composition of measurable maps is also compatible with $\sim$, see 2.8.

We let LOC be the full subcategory of MSN whose objects are the localizable MSNs. We may be tempted to define the localization of a saturated MSN $(X,\mathscr{A},\mathscr{N})$ as its coreflection (if it exists) along the forgetful functor $\mathsf{Forget}\colon \mathsf{LOC}\to\mathsf{MSN}$, and the question of existence in general becomes that of the existence of a right adjoint to Forget. Specifically, we may want to say that a pair $\big[\big(\hat{X},\hat{\mathscr{A}},\hat{\mathscr{N}}\big),\mathbf{p}\big]$, where $\big(\hat{X},\hat{\mathscr{A}},\hat{\mathscr{N}}\big)$ is saturated localizable MSN and $\mathbf{p}$ is a morphism $\hat{X}\to X$, is a localization of $(X,\mathscr{A},\mathscr{N})$ whenever the following universal property holds. For every pair $[(Y,\mathscr{B},\mathscr{M}),\mathbf{q}]$, where $(Y,\mathscr{B},\mathscr{M})$ is a saturated localizable MSN and $\mathbf{q}$ is a morphism $Y\to X$, there exists a unique morphism $\mathbf{r}\colon Y\to\hat{X}$ such that $\mathbf{q}=\mathbf{p}\circ\mathbf{r}$.

$$\begin{tikzcd} (Y,\mathscr{B},\mathscr{M}) \arrow[rr, dotted, "\exists!\mathbf{r}"] \arrow[dr, "\mathbf{q}"'] & & \big(\hat{X},\hat{\mathscr{A}},\hat{\mathscr{N}}\big) \arrow[dl, "\mathbf{p}"] \\ & (X,\mathscr{A},\mathscr{N}) & \end{tikzcd} \tag{$\bigtriangledown$}$$

However, we now illustrate that the notion of morphism defined so far is not yet the appropriate one that we are after. We consider the MSN $(X,\mathscr{A},\{\emptyset\})$ where $X=\mathbb{R}$ and $\mathscr{A}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. We recall that we want the localization of $(X,\mathscr{A},\{\emptyset\})$ to be $[(X,\mathscr{P}(X),\{\emptyset\}),\mathbf{p}]$ with $\mathbf{p}$ induced by the identity id_X. Assume if possible that this is the case. In the diagram above we consider $(Y,\mathscr{B},\mathscr{M})=(X,\mathscr{A},\mathscr{N}_{\mathscr{L}^1})$ and $\mathbf{q}$ induced by the identity. Note that this is, indeed, a localizable MSN since it is associated with a σ-finite measure space (see 4.5 and 4.4). Thus, there would exist a morphism $\mathbf{r}$ in MSN such that $\mathbf{p}\circ\mathbf{r}=\mathbf{q}$. Picking $r\in\mathbf{r}$, this implies that $X\cap\{x : r(x)\neq x\}$ is Lebesgue negligible. The measurability of r would then imply that $r^{-1}(S)\in\mathscr{A}$ for every $S\in\mathscr{P}(X)$, contradicting the existence of non Lebesgue measurable subsets of $\mathbb{R}$.

The problem with the example above is that the objects $(X,\mathscr{A},\mathscr{N}_{\mathscr{L}^1})$ and $(X,\mathscr{A},\{\emptyset\})$ should not be compared, in other words that $\mathbf{q}$ should not be a legitimate morphism. We say that a morphism $\mathbf{f}\colon(X,\mathscr{A},\mathscr{N})\to(Y,\mathscr{B},\mathscr{M})$ of the category MSN is *supremum preserving* if the following holds for (one, and therefore every) $f\in\mathbf{f}$. If $\mathscr{F}\subseteq\mathscr{B}$ admits an $\mathscr{M}$-essential supremum $S\in\mathscr{B}$, then $f^{-1}(S)$ is an $\mathscr{N}$-essential supremum of $f^{-1}(\mathscr{F})$. It is easy to see that adding this condition to the definition of morphism rules out the $\mathbf{q}$

considered in the preceding paragraph. We define the category $\mathsf{MSN}_{\mathsf{sp}}$ to be that whose objects are the saturated MSNs and whose morphisms are those morphisms of MSN that are supremum preserving. We define similarly $\mathsf{LOC}_{\mathsf{sp}}$. We now define the *localizable version* (if it exists) of a saturated MSN with the similar universal property illustrated in ($\bigtriangledown$), except for we now require all morphisms to be in $\mathsf{MSN}_{\mathsf{sp}}$, i.e. supremum preserving. In other words, it is a coreflection of an object of $\mathsf{MSN}_{\mathsf{sp}}$ along $\mathsf{Forget}\colon \mathsf{LOC}_{\mathsf{sp}} \to \mathsf{MSN}_{\mathsf{sp}}$. Unfortunately, this is not quite yet the right setting. Indeed, we show in 4.13 that if X is uncountable and $\mathscr{C}(X)$ is the countable-cocountable σ-algebra of X, then $[(X, \mathscr{P}(X), \{\emptyset\}), \boldsymbol{\iota}]$ (with $\boldsymbol{\iota}$ induced by id_X) is not the localizable version of $(X, \mathscr{C}(X), \{\emptyset\})$. This prompts us to introduce a new category.

We say that an object $(X, \mathscr{A}, \mathscr{N})$ of MSN is *locally determined* if for every $\mathscr{N}$-generating collection $\mathscr{E} \subseteq \mathscr{A}$ the following holds:

$$\forall A \subseteq X : \left[\forall E \in \mathscr{E} : A \cap E \in \mathscr{A}\right] \Rightarrow A \in \mathscr{A}.$$

In case $(X, \mathscr{A}, \mathscr{N})$ is the MSN associated with some complete semi-finite measure space $(X, \mathscr{A}, \mu)$, then it is locally determined (in the sense of MSNs) if and only if $(X, \mathscr{A}, \mu)$ is locally determined (in the sense of measure spaces) – see 5.3(F) – even though the latter sounds stronger because we test with any generating family $\mathscr{E}$. We say that an object of MSN is *lld* if it is both localizable and locally determined, and we let $\mathsf{LLD}_{\mathsf{sp}}$ be the corresponding full subcategory of $\mathsf{LOC}_{\mathsf{sp}}$. We now define the *lld version* of an object of $\mathsf{MSN}_{\mathsf{sp}}$ to be its coreflection (if it exists) along $\mathsf{Forget}\colon \mathsf{LLD}_{\mathsf{sp}} \to \mathsf{MSN}_{\mathsf{sp}}$, i.e. it satisfies the corresponding universal property illustrated in ($\bigtriangledown$) with Y and $\hat{X}$ being lld, and the morphisms being supremum preserving. This definition is satisfactory in at least the simplest case, 5.4 : If $(X, \mathscr{A}, \{\emptyset\})$ is so that $\mathscr{A}$ contains all singletons, then it admits $[(X, \mathscr{P}(X), \{\emptyset\}), \boldsymbol{\iota}]$ as its lld version.

Our general question has now become whether $\mathsf{Forget} : \mathsf{LLD}_{\mathsf{sp}} \to \mathsf{MSN}_{\mathsf{sp}}$ admits a right adjoint. Freyd's Adjoint Functor Theorem [1, 3.3.3] could prove useful, however do not know whether it applies, mostly because we do not know whether coequalizers exist in $\mathsf{MSN}_{\mathsf{sp}}$. We gather in Table 1 the information that we know about limits and colimits in the three categories we introduced.

In view of proving some partial existence result for lld versions, we introduce the intermediary notion of a *4c* (or *cccc*) saturated MSN, short for coproduct (in $\mathsf{MSN}_{\mathsf{sp}}$) of ccc saturated MSNs. It is easy to see that 4c MSNs are lld, 4.6 and 4.4. The *4c version* of an object of $\mathsf{MSN}_{\mathsf{sp}}$ is likewise defined by its universal property in diagram ($\bigtriangledown$), using supremum preserving morphisms. Our main results are about *locally ccc* MSNs, i.e. those saturated MSNs $(X, \mathscr{A}, \mathscr{N})$ such that $\mathscr{E}_{\mathrm{ccc}} = \mathscr{A} \cap \{Z : \text{ the subMSN } (Z, \mathscr{A}_Z, \mathscr{N}_Z) \text{ is ccc}\}$ is $\mathscr{N}$-generating. A complete semi-finite measure space $(X, \mathscr{A}, \mu)$ is clearly locally ccc, since $\mathscr{A}^f$ is $\mathscr{N}_\mu$-generating. Similarly, one can define the more general *locally localizable* objects in $\mathsf{MSN_{sp}}$. In 4.14, we give an example of an MSN which is not even locally localizable.

	$\mathsf{MSN_{sp}}$	$\mathsf{LOC_{sp}}$	$\mathsf{LLD_{sp}}$
equalizers	exist if $\{f = g\}$ is meas. 3.7(C)	**?**	exist 5.10
products	(countable) exist 2.13	**?**	**?**
coequalizers	**?**	**?** see 5.7	**?** see 5.7
coproducts	exist 3.7(D)	exist 4.6	exist 4.6 and 5.3(D)

Table 1: Limits and colimits in the three categories of MSNs.

THEOREM. *Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated locally ccc MSN. The following hold:*

(1) *$(X, \mathscr{A}, \mathscr{N})$ admits a 4c version,* 7.4.

(2) *If furthermore $\mathscr{E}_{\mathrm{ccc}}$ contains an $\mathscr{N}$-generating subcollection $\mathscr{E}$ such that $\operatorname{card} \mathscr{E} \leqslant \mathfrak{c}$ and each $(Z, \mathscr{A}_Z)$ is countably separated, for $Z \in \mathscr{E}$, then $(X, \mathscr{A}, \mathscr{N})$ admits an lld version which is also its 4c version.*

By saying that a measurable space $(Z, \mathscr{A}_Z)$ is countably separated we mean that $\mathscr{A}_Z$ contains a countable subcollection that separates points in Z. The 4c version $\left(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right)$ is obtained as a coproduct $\coprod_{Z \in \mathscr{E}} (Z, \mathscr{A}_Z, \mathscr{N}_Z)$ where $\mathscr{E}$ is an $\mathscr{N}$-generating almost disjointed refinement of $\mathscr{E}_{\mathrm{ccc}}$, whose existence ensues from Zorn's Lemma. In order to establish that this, in fact, is also the lld version under the extra assumptions in (2), we need to build an appropriate morphism $\mathbf{r}$ in diagram $(\triangledown)$, associated with an lld pair $[(Y, \mathscr{B}, \mathscr{M}), \mathbf{q}]$. It is obtained as a gluing of $\langle q_Z \rangle_{Z \in \mathscr{E}}$ where $q_Z \colon q^{-1}(Z) \to \hat{X}$ is the obvious map. Since $\mathscr{E}$ is almost disjointed, $\langle q_Z \rangle_{Z \in \mathscr{E}}$ is compatible and, since $(Y, \mathscr{B}, \mathscr{M})$ in diagram $(\triangledown)$ is localizable, the only obstruction to gluing is that $\hat{X}$ is not $\mathbb{R}$. Notwithstanding, $\left(\hat{X}, \hat{\mathscr{A}}\right) = \coprod_{Z \in \mathscr{E}} (Z, \mathscr{A}_Z)$ is itself countably separated because $\operatorname{card} \mathscr{E} \leqslant \mathfrak{c}$, 6.8 so that the local determinacy of $(Y, \mathscr{B}, \mathscr{M})$ and the fact that $q^{-1}(\mathscr{E})$ is $\mathscr{M}$-generating (because $\mathscr{E}$ is $\mathscr{N}$-generating and q is supremum preserving) provides a gluing r, 6.10.

We now explain how this applies to associating, in a canonical way, a strictly localizable measure space with *any* measure space $(X, \mathscr{A}, \mu)$. First, we recall that without changing the base space X we can render the measure space complete and semi-finite. In that case, $\mathscr{A}^f$ is $\mathscr{N}_\mu$-generating and witnesses the fact that the saturated MSN $(X, \mathscr{A}, \mathscr{N}_\mu)$ is locally ccc. By the theorem above, it admits a 4c version $\big[\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big), \mathbf{p}\big]$.

Theorem. *Let $(X, \mathscr{A}, \mu)$ be a complete semi-finite measure space and $\big[\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big), \mathbf{p}\big]$ its corresponding 4c version. Let $p \in \mathbf{p}$. There exists a unique (and independent of the choice of p) measure $\hat{\mu}$ defined on $\hat{\mathscr{A}}$ such that $p_\# \hat{\mu} = \mu$ and $\mathscr{N}_{\hat{\mu}} = \hat{\mathscr{N}}$. Furthermore $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mu}\big)$ is a strictly localizable measure space, and the Banach spaces $\mathbf{L}_1(X, \mathscr{A}, \mu)$ and $\mathbf{L}_1\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mu}\big)$ are isometrically isomorphic.*

Of course, the general process for constructing $\hat{X}$ is non constructive, as it involves the axiom of choice to turn $\mathscr{A}^f$ into an almost disjointed generating family. This is why, in the last two sections of this paper, we explore a particular case where we are able to describe explicitly $\hat{X}$ as a quotient of a fiber bundle, all "hands on". We start with the measure space $\big(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m), \mathscr{I}^k_\infty\big)$ where $1 \leqslant k \leqslant m-1$ are integers, $\mathscr{B}(\mathbb{R}^m)$ is the σ-algebra of Borel subsets of $\mathbb{R}^m$, and $\mathscr{I}^k_\infty$ is the integral geometric measure described in [4, 2.10.5(1)] and [11, 5.14]. Note that it is not semi-finite, [4, 3.3.20]. Thus, we replace it with its complete semi-finite version $\Big(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \tilde{\mathscr{I}}^k_\infty\Big)$. We let $\mathscr{E}$ be the collection of k-dimensional submanifolds $M \subseteq \mathbb{R}^m$ of class C^1 such that $\phi_M = \mathscr{H}^k \llcorner M$ is locally finite. It follows from the Besicovitch Structure Theorem [4, 3.3.14] that $\mathscr{E}$ is $\mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}$-generating, 11.2(ii). Now, for each $x \in \mathbb{R}^m$ we define $\mathscr{E}_x = \mathscr{E} \cap \{M : x \in M\}$ and we define on $\mathscr{E}_x$ an equivalence relation as follows. We declare that $M \sim_x M'$ if and only if

$$\lim_{r \to 0^+} \frac{\mathscr{H}^k(M \cap M' \cap \mathbf{B}(x,r))}{\boldsymbol{\alpha}(k) r^k} = 1.$$

Letting $[M]_x$ denote the equivalence class of $M \in \mathscr{E}_x$, we prove 11.2 that underlying set of the 4c, lld, and strictly localizable version of the MSN $\Big(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}\Big)$ can be taken to be

$$\hat{X} = \big\{(x, [M]_x) : x \in \mathbb{R}^m \text{ and } M \in \mathscr{E}_x\big\}.$$

This leads to an explicit description of the dual of $\mathbf{L}_1\Big(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \tilde{\mathscr{I}}^k_\infty\Big)$ as $\mathbf{L}_\infty\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\Big)$.

We close this foreword with a comment about the "Baire category" counterpart of our work on measure spaces.

Consider a nonempty topological space X which is either completely metrizable or locally compact Hausdorff. We recall that a subset of X is termed *meager* if it is a countable union of nowhere dense subsets of X (i.e. subsets whose closure has empty interior). Meager sets in X clearly form a σ-ideal which we denote by $\mathscr{M}$. Furthermore, $A \subseteq X$ is called *Baire measurable* if it is the symmetric difference of an open set and a meager set. Letting $\mathscr{B}$ be the collection of Baire measurable subsets of X, we note that $(X, \mathscr{B}, \mathscr{M})$ is a saturated MSN and that, under our assumption on X, $X \notin \mathscr{M}$. In case X is Polish (i.e. completely metrizable and *separable*), everything turns out perfect from the point of view of this paper, reflecting the situation of σ-finite measure spaces: This is because $(X, \mathscr{B}, \mathscr{M})$ then satisfies the countable chain condition. It is therefore localizable, by 4.4, and locally determined, by 5.3(C). In the general case, however, we do not know whether our results in Section 7 apply to showing that $(X, \mathscr{B}, \mathscr{M})$ admits a localizable version.

We are indebted to David Fremlin whose point of view on measure theory – generously shared in his immense treatise [5, 6, 7, 8, 9, 10] – influenced our work in this paper. It is the second author's pleasure to record useful conversations with Francis Borceux.

2. Measurable spaces with negligibles

Definition 2.1. (σ-algebra) Let X be a set. A *σ-algebra* on X is a set $\mathscr{A} \subseteq \mathscr{P}(X)$ such that

(1) $\emptyset \in \mathscr{A}$;

(2) If $A \in \mathscr{A}$ then $X \setminus A \in \mathscr{A}$;

(3) If $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathscr{A}$ then $\bigcup_{n \in \mathbb{N}} A_n \in \mathscr{A}$.

If $\mathscr{A}$ is a σ-algebra on X then $X \in \mathscr{A}$ and $\bigcap_{n \in \mathbb{N}} A_n \in \mathscr{A}$ whenever $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathscr{A}$. Clearly $\{\emptyset, X\}$ and $\mathscr{P}(X)$ are σ-algebras on X, respectively the coarsest and the finest. If $\langle \mathscr{A}_i \rangle_{i \in I}$ is a nonempty family of σ-algebras on X then $\bigcap_{i \in I} \mathscr{A}_i$ is a σ-algebra on X. Thus each $\mathscr{E} \subseteq \mathscr{P}(X)$ is contained in a coarsest σ-algebra on X which we will denote by $\sigma(\mathscr{E})$. If $\mathscr{E}, \mathscr{A} \subseteq \mathscr{P}(X)$ and $\mathscr{A} = \sigma(\mathscr{E})$ we say that the σ-algebra $\mathscr{A}$ *is generated* by $\mathscr{E}$. Clearly, if

$\mathscr{E}_1 \subseteq \mathscr{E}_2 \subseteq \mathscr{P}(X)$ then $\sigma(\mathscr{E}_1) \subseteq \sigma(\mathscr{E}_2)$. A *measurable space* is a couple $(X, \mathscr{A})$ where X is a set and $\mathscr{A}$ is a σ-algebra on X. In this case, if no confusion is possible we call *measurable* the members of $\mathscr{A}$.

Definition 2.2. (Measurable maps) Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces and $f\colon X \to Y$. We say that f is $(\mathscr{A}, \mathscr{B})$*-measurable* (or simply *measurable* if no confusion can occur) if $f^{-1}(B) \in \mathscr{A}$ whenever $B \in \mathscr{B}$. Measurable spaces, together with measurable maps, form a well-defined category, as one can check that the composition of two measurable maps is measurable.

Definition 2.3. (σ-ideal) Let $(X, \mathscr{A})$ be a measurable space. A σ*-ideal* $\mathscr{N}$ of $\mathscr{A}$ is a subset of $\mathscr{A}$ that satisfies the following requirements:

(1) $\emptyset \in \mathscr{N}$;

(2) If $A \in \mathscr{A}$, $N \in \mathscr{N}$ and $A \subseteq N$ then $A \in \mathscr{N}$;

(3) If $\langle N_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathscr{N}$, then $\bigcup_{n\in\mathbb{N}} N_n \in \mathscr{N}$.

Definition 2.4. (Measurable space with negligibles) A *measurable space with negligibles* (abbreviated *MSN*) is a triple $(X, \mathscr{A}, \mathscr{N})$ where $(X, \mathscr{A})$ is a measurable space and $\mathscr{N}$ is a σ-ideal of $\mathscr{A}$. Given an MSN $(X, \mathscr{A}, \mathscr{N})$, elements belonging to $\mathscr{N}$ are referred to as $\mathscr{N}$*-negligible* sets (simply negligible sets if no confusion can occur). Complements of $\mathscr{N}$-negligible sets are called $\mathscr{N}$*-conegligible* sets (or simply conegligible sets).

We can associate to any measure space $(X, \mathscr{A}, \mu)$ the MSN $(X, \mathscr{A}, \mathscr{N}_\mu)$ where $\mathscr{N}_\mu$ is the σ-ideal $\mathscr{N}_\mu = \mathscr{A} \cap \{N : \mu(N) = 0\}$. Conversely, any MSN $(X, \mathscr{A}, \mathscr{N})$ derives from a measure space: it suffices to consider the measure $\mu\colon \mathscr{A} \to [0, \infty]$ that sends negligible sets to 0 and the remaining sets to ∞.

Definition 2.5. (Saturated MSNs) An MSN $(X, \mathscr{A}, \mathscr{N})$ is called *saturated* whenever the following property holds: For all $N \in \mathscr{N}$, any subset $N' \subseteq N$ is $\mathscr{A}$-measurable – therefore, in fact, $N' \in \mathscr{N}$. This property is of purely technical nature, as an MSN $(X, \mathscr{A}, \mathscr{N})$ that does not have it can be turned into a saturated MSN $(X, \bar{\mathscr{A}}, \bar{\mathscr{N}})$, by setting:

$$\begin{aligned}
\bar{\mathscr{N}} &= \mathscr{P}(X) \cap \{\bar{N} : \bar{N} \subseteq N \text{ for some } N \in \mathscr{N}\}, \\
\bar{\mathscr{A}} &= \mathscr{P}(X) \cap \{\bar{A} : A \ominus \bar{A} \in \bar{\mathscr{N}} \text{ for some } A \in \mathscr{A}\} \\
&= \mathscr{P}(X) \cap \{A \ominus \bar{N} : A \in \mathscr{A} \text{ and } \bar{N} \in \bar{\mathscr{N}}\}.
\end{aligned}$$

Here, $\ominus$ denotes the symmetric difference of sets. We call $\left(X, \bar{\mathscr{A}}, \bar{\mathscr{N}}\right)$ the *saturation* of $(X, \mathscr{A}, \mathscr{N})$. In case the original MSN corresponds to a measure space

$(X, \mathscr{A}, \mu)$, its saturation corresponds to the measure space usually referred to as the completion of $(X, \mathscr{A}, \mu)$. We will denote the latter by $\big(X, \bar{\mathscr{A}}, \bar{\mu}\big)$.

Definition 2.6. Let $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$ be two MSNs. We say that a map $f \colon X \to Y$ is $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-*measurable* (or simply *measurable*) if

(1) f is $(\mathscr{A}, \mathscr{B})$-measurable;
(2) $f^{-1}(M) \in \mathscr{N}$ for every $M \in \mathscr{M}$.

It is easy to check that measurability in the above sense is preserved by composition.

Definition 2.7. (Morphisms of saturated MSNs) Let $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$ be two saturated MSNs. A *morphism* from $(X, \mathscr{A}, \mathscr{N})$ to $(Y, \mathscr{B}, \mathscr{M})$ is an equivalence class of $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurable maps under the relation $\sim$ of equality almost everywhere: $f \sim f'$ whenever $\{f \neq f'\} \in \mathscr{N}$. In order to check that this relation is, indeed, transitive, it is important to assume that $(X, \mathscr{A}, \mathscr{N})$ is saturated for otherwise we would not know that $\{f \neq f''\} \in \mathscr{A}$ when $f, f'' : X \to Y$ are both $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurable. Also, in the special case where X is $\mathscr{N}$-negligible and $Y = \emptyset$, we follow the convention that there is unique morphism from $(X, \mathscr{A}, \mathscr{N})$ to $(\emptyset, \{\emptyset\}, \{\emptyset\})$.

Lemma 2.8. *Let $(X, \mathscr{A}, \mathscr{N})$, $(Y, \mathscr{B}, \mathscr{M})$ and $(Z, \mathscr{C}, \mathscr{P})$ be MSNs and let $f, f' \colon X \to Y$ and $g, g' \colon Y \to Z$ be maps. If*

(A) *f, f' are $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurable;*
(B) *g, g' are $[(\mathscr{B}, \mathscr{M}), (\mathscr{C}, \mathscr{P})]$-measurable;*
(C) *$(X, \mathscr{A}, \mathscr{N})$ is saturated;*
(D) *$f \sim f'$, $g \sim g'$,*

then $g \circ f$, $g' \circ f'$ are $[(\mathscr{A}, \mathscr{N}), (\mathscr{C}, \mathscr{P})]$-measurable and $g \circ f \sim g' \circ f'$.

Proof. The first conclusion follows from hypotheses (A) and (B) and Paragraph 2.6. The second conclusion is a consequence of

$$\{g \circ f \neq g' \circ f'\} \subseteq \{f \neq f'\} \cup f^{-1}(\{g \neq g'\})$$

and hypotheses (A), (C) and (D). ∎

Definition 2.9. (Category MSN) Thanks to the preceding result, there is a notion of composition for morphisms between saturated MSNs: If $\mathbf{f}\colon (X,\mathscr{A},\mathscr{N}) \to (Y,\mathscr{B},\mathscr{M})$ and $\mathbf{g}\colon (Y,\mathscr{B},\mathscr{M}) \to (Z,\mathscr{C},\mathscr{P})$ are morphisms, we let $\mathbf{g}\circ\mathbf{f}\colon (X,\mathscr{A},\mathscr{N}) \to (Z,\mathscr{C},\mathscr{P})$ be the equivalence class of $g\circ f$ where $f\in\mathbf{f}$ and $g\in\mathbf{g}$.

This allows to define the category MSN whose objects are saturated MSNs and whose morphisms are described in the paragraph 2.7. Additionally, we add the convention that, for a negligible saturated MSN, i.e. an MSN of the form $\big(X,\mathscr{P}(X),\mathscr{P}(X)\big)$, there is a unique morphism from $\big(X,\mathscr{P}(X),\mathscr{P}(X)\big)$ to $\big(\emptyset,\{\emptyset\},\{\emptyset\}\big)$. This way, negligible saturated MSNs are isomorphic to one another in the category MSN.

The categorical point of view is rarely considered in measure theory, mainly due to the lack of a well-behaved notion of morphism between measure spaces. The category MSN also appears in the work [14] under the name $\mathsf{StrictEMS}$. We start to investigate the existence of limits and colimits in this category.

Definition 2.10. (subMSN) Let $(X,\mathscr{A},\mathscr{N})$ be an MSN and $Z\in\mathscr{P}(X)$. We define the *subMSN* $(Z,\mathscr{A}_Z,\mathscr{N}_Z)$, where $\mathscr{A}_Z := \{A\cap Z : A\in\mathscr{A}\}$ and $\mathscr{N}_Z := \{N\cap Z : N\in\mathscr{N}\}$. Note that in the special case where Z is $\mathscr{A}$-measurable, we have $\mathscr{A}_Z = \mathscr{A}\cap\{A : A\subseteq Z\}$ and $\mathscr{N}_Z = \mathscr{N}\cap\{N : N\subseteq Z\}$.

The inclusion map $\iota_Z\colon Z\to X$ is $[(\mathscr{A}_Z,\mathscr{N}_Z),(\mathscr{A},\mathscr{N})]$-measurable and induces a morphism $\boldsymbol{\iota}_Z$ between $(Z,\mathscr{A}_Z,\mathscr{N}_Z)$ and $(X,\mathscr{A},\mathscr{N})$.

Proposition 2.11. *Let $\mathbf{f},\mathbf{g} : (X,\mathscr{A},\mathscr{N}) \to (Y,\mathscr{B},\mathscr{M})$ be a pair of morphisms in the category MSN, represented by the maps $f\in\mathbf{f}$ and $g\in\mathbf{g}$, and set $Z := \{f=g\}$. Then the equalizer of $\mathbf{f},\mathbf{g}$ is $[(Z,\mathscr{A}_Z,\mathscr{N}_Z),\boldsymbol{\iota}_Z]$.*

Proof. As $f\circ\iota_Z = g\circ\iota_Z$, we have clearly $\mathbf{f}\circ\boldsymbol{\iota}_Z = \mathbf{g}\circ\boldsymbol{\iota}_Z$. Let $\mathbf{h}$ be any other morphism $(T,\mathscr{C},\mathscr{P}) \to (X,\mathscr{A},\mathscr{N})$ that satisfies the relation $\mathbf{f}\circ\mathbf{h} = \mathbf{g}\circ\mathbf{h}$ and let $h\in\mathbf{h}$. Then $h^{-1}(Z)$ is conegligible in T. Up to modifying h, we can suppose that it has values in Z. The restriction $h'\colon T\to Z$ of h is $[(\mathscr{C},\mathscr{P}),(\mathscr{A}_Z,\mathscr{N}_Z)]$-measurable and we have $h = \iota_Z\circ h'$, leading to a factorization $\mathbf{h} = \boldsymbol{\iota}_Z\circ\mathbf{h}'$. This factorization is unique, as any morphism $\mathbf{h}'$ satisfying $\mathbf{h} = \boldsymbol{\iota}_Z\circ\mathbf{h}'$ must derive from a map $h'\colon T\to Z$ that coincides almost everywhere with h. ∎

Proposition 2.12. *The category MSN has coproducts. Consider a family of saturated MSNs $\langle(X_i,\mathscr{A}_i,\mathscr{N}_i)\rangle_{i\in I}$. Its coproduct is the MSN $(X,\mathscr{A},\mathscr{N})$ whose underlying set is $X = \coprod_{i\in I} X_i$, and whose σ-algebra and σ-ideal are* *defined by*

$$\mathscr{A} = \mathscr{P}(X) \cap \{A : A \cap X_i \in \mathscr{A}_i \text{ for all } i \in I\},$$
$$\mathscr{N} = \mathscr{P}(X) \cap \{N : N \cap X_i \in \mathscr{N}_i \text{ for all } i \in I\}.$$

For $i \in I$, the canonical morphism $\boldsymbol{\iota}_i \colon (X_i, \mathscr{A}_i, \mathscr{N}_i) \to (X, \mathscr{A}, \mathscr{N})$ is the morphism induced by the inclusion map $\iota_i \colon X_i \to X$.

Proof. Notice that, indeed, $(X, \mathscr{A}, \mathscr{N})$ is a saturated MSN. Let $(Y, \mathscr{B}, \mathscr{M})$ be a saturated MSN and $\langle \mathbf{f}_i \rangle_{i \in I}$ be a collection of morphisms from $(X_i, \mathscr{A}_i, \mathscr{N}_i)$ to $(Y, \mathscr{B}, \mathscr{M})$, each $\mathbf{f}_i$ being represented by a measurable map f_i. We set $f = \coprod_{i \in I} f_i$, the map such that $f \circ \iota_i = f_i$ for any $i \in I$. It is clear that f is $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurable and $\mathbf{f} \circ \boldsymbol{\iota}_i = \mathbf{f}_i$ holds for all $i \in I$. We need to show that $\mathbf{f}$ is the unique morphism $(X, \mathscr{A}, \mathscr{N}) \to (Y, \mathscr{B}, \mathscr{M})$ with this property.

Suppose $\mathbf{g} \colon (X, \mathscr{A}, \mathscr{N}) \to (Y, \mathscr{B}, \mathscr{M})$ is another morphism, represented by a measurable map $g \colon X \to Y$, for which $\mathbf{g} \circ \boldsymbol{\iota}_i = \mathbf{f}_i$ for all $i \in I$. Then f and g coincide almost everywhere on each X_i, which implies, due to the choice of $\mathscr{N}$, that f and g are equal almost everywhere and $\mathbf{f} = \mathbf{g}$. ∎

PROPOSITION 2.13. *The category* MSN *has countable products. Let $\big\langle (X_i, \mathscr{A}_i, \mathscr{N}_i) \big\rangle_{i \in I}$ be a countable family of saturated MSNs. Its product is the MSN $(X, \mathscr{A}, \mathscr{N})$, whose underlying set is the product $X = \prod_{i \in I} X_i$, whose σ-ideal is*

$$\mathscr{N} = \mathscr{P}(X) \cap \left\{ N : \exists \langle N_i \rangle_{i \in I} \in \prod_{i \in I} \mathscr{N}_i, \quad N \subseteq \bigcup_{i \in I} \pi_i^{-1}(N_i) \right\},$$

where $\pi_i \colon X \to X_i$ denotes the projection map, and whose σ-algebra is the saturation of $\bigotimes_{i \in I} \mathscr{A}_i$:

$$\mathscr{A} = \left\{ A \ominus N : A \in \bigotimes_{i \in I} \mathscr{A}_i \text{ and } N \in \mathscr{N} \right\}.$$

For $i \in I$, the projection morphism $\boldsymbol{\pi}_i \colon (X, \mathscr{A}, \mathscr{N}) \to (X_i, \mathscr{A}_i, \mathscr{N}_i)$ is the map induced by the projection map π_i.

Proof. First note that, by construction of $\mathscr{A}$ and $\mathscr{N}$, the projection maps π_i are $[(\mathscr{A}, \mathscr{N}), (\mathscr{A}_i, \mathscr{N}_i)]$-measurable. Let $(Y, \mathscr{B}, \mathscr{M})$ be a saturated MSN and $\langle \mathbf{f}_i \rangle_{i\in I}$ be a collection of morphisms from $(Y, \mathscr{B}, \mathscr{M})$ to $(X_i, \mathscr{A}_i, \mathscr{N}_i)$, each $\mathbf{f}_i$ being represented by a measurable map $f_i \colon Y \to X_i$. We define $f = \prod_{i\in I} f_i \colon Y \to \prod_{i\in I} X_i$ that assigns $y \in Y$ to $\langle f_i(y) \rangle_{i\in I}$. Clearly, f is $(\mathscr{B}, \bigotimes_{i\in I} \mathscr{A}_i)$-measurable. Moreover, for any negligible set $N \in \mathscr{N}$, we can find a sequence $\langle N_i \rangle_{i\in I}$ such that $N \subseteq \bigcup_{i\in I} \pi_i^{-1}(N_i)$. Thus

$$f^{-1}(N) \subseteq \bigcup_{i\in I} (\pi_i \circ f)^{-1}(N_i) = \bigcup_{i\in I} f_i^{-1}(N_i).$$

As I is countable and $f_i^{-1}(N_i) \in \mathscr{M}$ for all $i \in I$, we find that $f^{-1}(N) \in \mathscr{M}$, which entails that the map f is $[(\mathscr{B}, \mathscr{M}), (\mathscr{A}, \mathscr{N})]$-measurable.

Let $\mathbf{g} \colon (Y, \mathscr{B}, \mathscr{M}) \to (X, \mathscr{A}, \mathscr{N})$ be another morphism satisfying the identities $\boldsymbol{\pi}_i \circ \mathbf{g} = \mathbf{f}_i$ for $i \in I$. Let $g \colon Y \to X$ be a representative of $\mathbf{g}$. The coordinate functions $\pi_i \circ g$ must coincide with f_i almost everywhere. As there are only countably many of them, we conclude that f and g are equal almost everywhere, that is, $\mathbf{f} = \mathbf{g}$. ∎

Remark 2.14. In case $(X_i, \mathscr{A}_i, \mathscr{N}_i)$ are associated with measure spaces $(X_i, \mathscr{A}_i, \mu_i)$, $i = 1, 2$, the σ-ideal $\mathscr{N}$ considered in the above proposition may not coincide with $\mathscr{N}_{\mu_1 \otimes \mu_2}$.

This is the case, for instance, when $(X_i, \mathscr{A}_i, \mu_i) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathscr{L}^1)$, $i = 1, 2$, since the diagonal $D = \mathbb{R}^2 \cap \{(x, x) : x \in \mathbb{R}\} \in \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$ is $\mathscr{L}^2$-negligible but does not belong to $\mathscr{N}$.

3. Supremum preserving morphisms

3.1. (Motivation) One of the reasons we were led to introduce MSNs is that the category of measure spaces and (equivalence classes of) measure preserving measurable maps does not have good properties at all. Roughly speaking, this can be attributed to the fact that it has very few arrows. One way to increase their number is to define as morphisms $(X, \mathscr{A}, \mu) \to (Y, \mathscr{B}, \nu)$ the $(\mathscr{A}, \mathscr{B})$-measurable maps $\varphi \colon X \to Y$ such that the pushforward measure $\varphi_{\#}\mu$ is absolutely continuous with respect to ν. If we drop the measures and retain only which sets have measure zero, we get the notion of $[(\mathscr{A}, \mathscr{N}_\mu), (\mathscr{B}, \mathscr{N}_\nu)]$-measurability of 2.6. However, doing so, we may introduce some "irregular" maps. For example, if $\mathscr{A}$ is the σ-algebra of Lebesgue measurable sets of the real line, $\mathscr{L}^1$ the Lebesgue measure and ν the counting measure on $(\mathbb{R}, \mathscr{A})$ then the identity map induces a morphism $(\mathbb{R}, \mathscr{A}, \mathscr{L}^1) \to (\mathbb{R}, \mathscr{A}, \nu)$. But, $\mathscr{L}^1$ does not really compare to ν, although it is absolutely continuous with respect to ν. For instance, $\mathscr{L}^1$ has no Radon-Nikodým density with respect to ν, not even in the generalized sense of Section 9. Forgetting the measures, the morphism of MSNs $(\mathbb{R}, \mathscr{A}, \mathscr{N}_{\mathscr{L}^1}) \to (\mathbb{R}, \mathscr{A}, \{\emptyset\})$ is still somehow inappropriate. To avoid this, we restrict our attention to the supremum preserving morphisms introduced below. This will allow us to define a new category $\mathsf{MSN}_{\mathsf{sp}}$ of saturated MSNs with supremum preserving morphisms. Later, we will be able to define localizable versions of MSNs and similar notions by means of universal properties to be satisfied in $\mathsf{MSN}_{\mathsf{sp}}$.

Definition 3.2. (Boolean algebras) Many of the properties that we will introduce underneath for MSNs are related to their *Boolean algebra*, defined in the following way: given an MSN $(X, \mathscr{A}, \mathscr{N})$, we observe that the σ-algebra $\mathscr{A}$ is a Boolean algebra and $\mathscr{N}$ is an ideal of $\mathscr{A}$ in the ring-theoretic sense; we then associate to $(X, \mathscr{A}, \mathscr{N})$ the quotient Boolean algebra $\mathscr{A}/\mathscr{N}$.

When we restrict our attention to saturated MSNs, this construction becomes functorial. Call $\mathsf{Bool}(X, \mathscr{A}, \mathscr{N}) = \mathscr{A}/\mathscr{N}$ the Boolean algebra of a saturated MSN. Given a morphism $\mathbf{f} \colon (X, \mathscr{A}, \mathscr{N}) \to (Y, \mathscr{B}, \mathscr{M})$ represented by a measurable map $f \colon X \to Y$, we define

$$\mathsf{Bool}(f) \colon \mathsf{Bool}(Y, \mathscr{B}, \mathscr{M}) \to \mathsf{Bool}(X, \mathscr{A}, \mathscr{N})$$

that maps the equivalence class of $B \in \mathscr{B}$ to the equivalence class of $f^{-1}(B)$. This map is well-defined because of the $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurability of f, it is a morphism of Boolean algebras, and it does not depend on the representative of $\mathbf{f}$, as one can easily check.

Definition 3.3. Let $(X, \mathscr{A}, \mathscr{N})$ be an MSN and $\mathscr{E}$ be a subcollection of $\mathscr{A}$. We say that $U \in \mathscr{A}$ is an *$\mathscr{N}$-essential upper bound* of $\mathscr{E}$ whenever $E \setminus U \in \mathscr{N}$ for all $E \in \mathscr{E}$. Furthermore, a measurable set $S \in \mathscr{A}$ is an *$\mathscr{N}$-essential supremum* of $\mathscr{E}$ whenever

(1) S is an $\mathscr{N}$-essential upper bound of $\mathscr{E}$;

(2) If S' is an $\mathscr{N}$-essential upper bound of $\mathscr{E}$, then $S \setminus S' \in \mathscr{N}$.

In particular, if S, S' are both $\mathscr{N}$-essential suprema of $\mathscr{E}$, their symmetric difference $S \ominus S'$ is negligible. In other words, an essential supremum, when it exists, is unique up to negligible sets. In fact, it corresponds to a (unique) supremum in $\mathsf{Bool}(X, \mathscr{A}, \mathscr{N})$. A collection $\mathscr{E} \subseteq \mathscr{A}$ that admits X as an $\mathscr{N}$-essential supremum is called *$\mathscr{N}$-generating*. We will use the following repeatedly. If $\mathscr{E} \subseteq \mathscr{A}$ and $S \in \mathscr{A}$ is an $\mathscr{N}$-essential supremum of $\mathscr{E}$, then $\mathscr{E} \cup \{X \setminus S\}$ is $\mathscr{N}$-generating.

The next ubiquitous lemma expresses that $\cap$ is distributive over the (partially defined) operation of taking essential suprema. It implies the following fact, which we will use frequently: If $\mathscr{E}$ is $\mathscr{N}$-generating and $A \in \mathscr{A} \setminus \mathscr{N}$, then $E \cap A \notin \mathscr{N}$ for some $E \in \mathscr{E}$.

Lemma 3.4. (Distributivity Lemma) *Let $(X, \mathscr{A}, \mathscr{N})$ be an MSN, $\mathscr{E} \subseteq \mathscr{A}$ be a collection that has an $\mathscr{N}$-essential supremum S, and $C \in \mathscr{A}$. Then $C \cap S$ is an $\mathscr{N}$-essential supremum of $\{C \cap E : E \in \mathscr{E}\}$.*

Proof. Condition (1) in Definition 3.3 is met because

$$C \cap E \setminus C \cap S = C \cap (E \setminus S) \in \mathscr{N} \text{ for all } E \in \mathscr{E}.$$

As for (2), we let S' be an $\mathscr{N}$-essential upper bound for $\{C \cap E : E \in \mathscr{E}\}$. We claim that $S'' := S' \cup (X \setminus C)$ is an $\mathscr{N}$-essential upper bound for $\mathscr{E}$. Indeed, for any $E \in \mathscr{E}$, we have $E \setminus S'' = (C \cap E) \setminus S' \in \mathscr{N}$. It follows that $(C \cap S) \setminus S' = S \setminus S'' \in \mathscr{N}$. ∎

3.5. Note that if $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$ are MSNs, $f : X \to Y$ is $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurable, $\mathscr{E} \subseteq \mathscr{B}$, and $S \in \mathscr{B}$ is an $\mathscr{M}$-essential upper bound of $\mathscr{E}$, then $f^{-1}(S)$ is an $\mathscr{N}$-essential upper bound of $f^{-1}(\mathscr{E})$. However, if S is an $\mathscr{M}$-essential supremum of $\mathscr{E}$ then $f^{-1}(S)$ may not be an $\mathscr{N}$-essential supremum of $f^{-1}(\mathscr{E})$. Consider, for instance, $(X, \mathscr{A}, \mathscr{N}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathscr{N}_{\mathscr{L}^1})$, $(Y, \mathscr{B}, \mathscr{M}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \{\emptyset\})$, $f = \mathrm{id}_{\mathbb{R}}$, and $\mathscr{E} = \{\{x\} : x \in \mathbb{R}\}$. Then $\mathbb{R}$ is an $\{\emptyset\}$-essential supremum of $\mathscr{E}$, $\emptyset$ is an $\mathscr{N}_{\mathscr{L}^1}$-essential supremum of $\mathscr{E} = f^{-1}(\mathscr{E})$, and $\mathbb{R} \setminus \emptyset \notin \mathscr{N}_{\mathscr{L}^1}$.

3.6. There are several objects that we can call *supremum preserving.* For saturated MSNs $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$, we define

- A morphism of Boolean algebras $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is called supremum preserving if, for any family $\mathfrak{C} \subseteq \mathfrak{A}$ that admits a supremum, the family $\varphi(\mathfrak{C})$ admits a supremum and $\varphi(\sup \mathfrak{C}) = \sup \varphi(\mathfrak{C})$.
- A morphism $\mathbf{f}\colon (X, \mathscr{A}, \mathscr{N}) \to (Y, \mathscr{B}, \mathscr{M})$ is called supremum preserving whenever $\mathsf{Bool}(\mathbf{f})$ is.
- An $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurable map $f\colon X \to Y$ is called supremum preserving if, for any collection $\mathscr{E} \subseteq \mathscr{B}$ with an $\mathscr{M}$-essential supremum S, $f^{-1}(S)$ is an $\mathscr{N}$-essential supremum of $f^{-1}(\mathscr{E}) := \{f^{-1}(E) : E \in \mathscr{E}\}$.

For a morphism $\mathbf{f}$ represented by $f \in \mathbf{f}$, the supremum preserving characters of f, $\mathbf{f}$ and $\mathsf{Bool}(\mathbf{f})$ are all equivalent. Also, the composition of two supremum preserving morphisms is supremum preserving. We call $\mathsf{MSN}_{\mathsf{sp}}$ the subcategory of MSN that consists of saturated MSNs and supremum preserving morphisms. In the next proposition, we gather some basic facts about supremum preserving morphisms and the category $\mathsf{MSN}_{\mathsf{sp}}$.

Proposition 3.7. *The following hold:*

(A) *Two saturated MSNs are isomorphic in* MSN *if and only if they are isomorphic in* $\mathsf{MSN}_{\mathsf{sp}}$.

(B) *Let* $(X, \mathscr{A}, \mathscr{N})$ *be a saturated MSN and* $Z \in \mathscr{A}$. *The morphism* $\boldsymbol{\iota}_Z : (Z, \mathscr{A}_Z, \mathscr{N}_Z) \to (X, \mathscr{A}, \mathscr{N})$ *induced by the inclusion map* $\iota_Z : Z \to X$ *is supremum preserving.*

(C) *Let* $\mathbf{f}, \mathbf{g} : (X, \mathscr{A}, \mathscr{N}) \to (Y, \mathscr{B}, \mathscr{M})$ *be a pair of morphisms in* $\mathsf{MSN}_{\mathsf{sp}}$, *represented by* $f \in \mathbf{f}$ *and* $g \in \mathbf{g}$. *If* $Z := \{f = g\}$ *is* $\mathscr{A}$*-measurable, then* $\big((Z, \mathscr{A}_Z, \mathscr{N}_Z), \boldsymbol{\iota}_Z\big)$ *is the equalizer of* $\mathbf{f}, \mathbf{g}$ *in* $\mathsf{MSN}_{\mathsf{sp}}$.

(D) *The category* $\mathsf{MSN}_{\mathsf{sp}}$ *has coproducts, which are preserved by the forgetful functor* $\mathsf{MSN}_{\mathsf{sp}} \to \mathsf{MSN}$.

Proof. (A) Let $\mathbf{f}$ be an isomorphism in MSN. Then $\mathsf{Bool}(\mathbf{f})$ is an isomorphism of Boolean algebras. More specifically, it is an isomorphism of posets and for this reason it preserves suprema.

(B) This is the content of Lemma 3.4.

(C) By Proposition 2.11, $((Z, \mathscr{A}_Z, \mathscr{N}_Z), \boldsymbol{\iota}_Z)$ is the equalizer of $\mathbf{f}, \mathbf{g}$ in MSN and by (B) the morphism $\boldsymbol{\iota}_Z$ is a morphism of $\mathsf{MSN}_{\mathsf{sp}}$. Let $\mathbf{h} : (T, \mathscr{C}, \mathscr{P}) \to (X, \mathscr{A}, \mathscr{N})$ be a supremum preserving morphism that satisfies $\mathbf{f} \circ \mathbf{h} = \mathbf{g} \circ \mathbf{h}$. Recalling the proof of Proposition 2.11, there is a representative $h \in \mathbf{h}$ with values in Z, and its restriction $h' : T \to Z$ induces the unique morphism $\mathbf{h}'$ such that $\mathbf{h} = \boldsymbol{\iota}_Z \circ \mathbf{h}'$. The results follows from the fact that h' is easily checked to be supremum preserving.

(D) Let $\langle (X_i, \mathscr{A}_i, \mathscr{N}_i) \rangle_{i \in I}$ be a family of saturated MSNs, $(X, \mathscr{A}, \mathscr{N})$ be their coproduct in the category MSN, and $\langle \mathbf{f}_i \rangle_{i \in I}$ be a family of supremum preserving morphisms from $(X_i, \mathscr{A}_i, \mathscr{N}_i)$ to a saturated $(Y, \mathscr{B}, \mathscr{M})$, each represented by $f_i : X_i \to Y$. We need to show that $f := \coprod_{i \in I} f_i : X \to Y$ is supremum preserving. For this, let $\mathscr{E} \subseteq \mathscr{B}$ be a collection that has an $\mathscr{M}$-essential supremum S. We observe that $f^{-1}(S) = \coprod_{i \in I} f_i^{-1}(S)$ is an $\mathscr{N}$-essential upper bound of $f^{-1}(\mathscr{E})$. Let U be a second $\mathscr{N}$-essential upper bound of $f^{-1}(\mathscr{E})$. Then $X_i \cap U$ is an $\mathscr{N}_i$-essential upper bound of $\{X_i \cap f^{-1}(E) : E \in \mathscr{E}\} = f_i^{-1}(\mathscr{E})$. It follows that $f_i^{-1}(S) \setminus (X_i \cap U) \in \mathscr{N}_i$. As this happens for all $i \in I$, we get that $f^{-1}(S) \setminus U \in \mathscr{N}$. ∎

4. Localizable, 4c and strictly localizable MSNs

Definition 4.1. (Localizable MSN) An MSN $(X, \mathscr{A}, \mathscr{N})$ is *localizable* whenever each collection $\mathscr{E} \subseteq \mathscr{A}$ admits an $\mathscr{N}$-essential supremum. Equivalently, $(X, \mathscr{A}, \mathscr{N})$ is localizable whenever its Boolean algebra $\mathscr{A}/\mathscr{N}$ is *Dedekind complete*, that is, each subset of $\mathscr{A}/\mathscr{N}$ has a supremum.

Originally, localizability was introduced by Segal [15] in the context of measure spaces. Since then, many minor variations over the definition in that context have been proposed (see [13] for an overview). We will follow the definition in [6, Chapter 2]. A measure space $(X, \mathscr{A}, \mu)$ is called *localizable* whenever

(1) it is *semi-finite*, i.e. for all $A \in \mathscr{A}$ with $\mu(A) > 0$, there is a measurable set $A' \subseteq A$ such that $0 < \mu(A') < \infty$;

(2) the underlying MSN $(X, \mathscr{A}, \mathscr{N}_\mu)$ is localizable.

4.2. (Semi-finite measure space) In the definition of localizable measure space, semi-finiteness plays on important rôle. Let us rephrase it. Given $(X, \mathscr{A}, \mu)$ a measure space, we abbreviate $\mathscr{A}^f := \mathscr{A} \cap \{E : \mu(E) < \infty\}$. We say that $N \in \mathscr{A}$ is *locally μ-negligible* whenever $N \cap E \in \mathscr{N}_\mu$ for every $E \in \mathscr{A}^f$. We let $\mathscr{N}_{\mu,\mathrm{loc}}$ be the σ-ideal consisting of locally μ-negligible measurable sets. The following are equivalent:

(1) $\mathscr{N}_\mu = \mathscr{N}_{\mu,\mathrm{loc}}$.

(2) $(X, \mathscr{A}, \mu)$ is semi-finite.

(3) $\mathscr{A}^f$ is $\mathscr{N}_\mu$-generating.

The only non trivial part is (3) $\Rightarrow$ (1). If $N \in \mathscr{A}$ then N is an $\mathscr{N}_\mu$-essential supremum of $\left\{N \cap F : F \in \mathscr{A}^f\right\}$, according to the Distributivity Lemma 3.4. If also $N \in \mathscr{N}_{\mu,\mathrm{loc}}$, then it follows that $N \in \mathscr{N}_\mu$. The notion of locally μ-negligible sets will appear again in 5.2.

Next we introduce some classes of localizable MSNs that will appear throughout the paper.

4.3. (Countable chain condition) Let $(X, \mathscr{A}, \mathscr{N})$ be an MSN. A family $\mathscr{E} \subseteq \mathscr{A} \setminus \mathscr{N}$ is called *almost disjointed* whenever $E \cap E' \in \mathscr{N}$ for any pair of distinct $E, E' \in \mathscr{E}$. The MSN $(X, \mathscr{A}, \mathscr{N})$ is said to have the *countable chain condition* (in short: is *ccc*) whenever an almost disjointed family in $\mathscr{A} \setminus \mathscr{N}$ is at most countable.

The previous notions have counterparts in the realm of Boolean algebras. Given a Boolean algebra $\mathfrak{A}$, a subset $\mathfrak{E} \subseteq \mathfrak{A}$ is called *disjointed* whenever $x \wedge y = 0$ for any pair of distinct elements $x, y \in \mathfrak{E}$. The Boolean algebra $\mathfrak{A}$ has the *countable chain condition* (or: is *ccc*) whenever each of its disjointed families is at most countable. Of course, an MSN $(X, \mathscr{A}, \mathscr{N})$ is ccc if and only if its Boolean algebra $\mathscr{A}/\mathscr{N}$ is. In the following proposition, we show that being ccc is stronger than localizability. It is related to the fact, first established in [18], that a Dedekind σ-complete Boolean algebra (that is, a Boolean algebra where countable collections have suprema) having the countable chain condition is Dedekind complete.

Proposition 4.4. *If an MSN $(X, \mathscr{A}, \mathscr{N})$ is ccc and $\mathscr{E} \subseteq \mathscr{A}$ is a collection, then there is a countable subcollection $\mathscr{E}' \subseteq \mathscr{E}$ such that $\bigcup \mathscr{E}'$ is an $\mathscr{N}$-essential supremum of $\mathscr{E}$. In particular, $(X, \mathscr{A}, \mathscr{N})$ is localizable.*

Proof. Suppose the existence of a collection $\mathscr{E} \subseteq \mathscr{A}$ for which one cannot find a countable subcollection $\mathscr{E}' \subseteq \mathscr{E}$ whose union is an $\mathscr{N}$-essential supremum of $\mathscr{E}$. This assumption allows us to construct transfinitely a sequence $\langle E_\alpha \rangle_{\alpha < \omega_1}$ with values in $\mathscr{E}$ such that for every $\alpha < \omega_1$, one has $F_\alpha := E_\alpha \setminus \bigcup_{\beta < \alpha} E_\beta \notin \mathscr{N}$. But the disjointed family $\{F_\alpha : \alpha < \omega_1\}$ contradicts the fact that $(X, \mathscr{A}, \mathscr{N})$ is ccc. ∎

Proposition 4.5. *Let $(X, \mathscr{A}, \mu)$ be a finite measure space. Then the space $(X, \mathscr{A}, \mathscr{N}_\mu)$ is ccc.*

Proof. Let $\mathscr{E} \subseteq \mathscr{A} \setminus \mathscr{N}$ be an almost disjointed family. For each positive integer n, set $\mathscr{E}_n = \mathscr{E} \cap \{E : \mu(E) > n^{-1}\}$. As $\mu(X) \geqslant \mu\left(\bigcup \mathscr{E}_n\right) \geqslant n^{-1} \operatorname{card} \mathscr{E}_n$, we have that $\mathscr{E}_n$ is finite. Consequently, $\mathscr{E}$ is at most countable. ∎

Proposition 4.6. *A coproduct $\coprod_{i \in I}(X_i, \mathscr{A}_i, \mathscr{N}_i)$ of saturated localizable MSNs is localizable.*

Proof. Let $\mathscr{E}$ be a collection of measurable sets of some coproduct $\coprod_{i \in I}(X_i, \mathscr{A}_i, \mathscr{N}_i)$. For each $i \in I$, the collection $\mathscr{E}_i := \{X_i \cap E : E \in \mathscr{E}\}$

has an $\mathscr{N}_i$-essential supremum $S_i \subseteq X_i$. We then routinely check that $S := \coprod_{i \in I} S_i \in \mathscr{A}$ is an $\mathscr{N}$-essential supremum of $\mathscr{E}$. ∎

Definition 4.7. (Stronger notions of localizability) An MSN is called *strictly localizable* if it is isomorphic to a coproduct of the form $\coprod_{i \in I}(X_i, \mathscr{A}_i, \mathscr{N}_{\mu_i})$, where $(X_i, \mathscr{A}_i, \mu_i)$ are complete finite measure spaces. Examples of strictly localizable MSNs are provided by MSNs associated to complete σ-finite measure spaces $(X, \mathscr{A}, \mu)$. Indeed, denoting $\langle X_i \rangle_{i \in I}$ a countable partition of X into measurable subsets of finite μ measure, one can verify that $(X, \mathscr{A}, \mathscr{N}_\mu)$ is isomorphic to $\coprod_{i \in I}(X_i, \mathscr{A}_{X_i}, \mathscr{N}_{\mu \llcorner X_i})$.

Likewise, we say that an MSN is *cccc* (abbreviated *4c*) whenever it is isomorphic to a coproduct of saturated ccc MSNs. We have the chain of implications

$$\text{strictly localizable} \implies \text{4c} \implies \text{localizable}.$$

The first implication comes from Proposition 4.5, the second one from Propositions 4.4 and 4.6. Examples of non localizable spaces are provided by the next results.

Lemma 4.8. *Let $(X, \mathscr{A}, \mathscr{N})$ be a localizable MSN, $\mathscr{E} \subseteq \mathscr{A} \setminus \mathscr{N}$ an almost disjointed family. Then* $\operatorname{card}(\mathscr{A}/\mathscr{N}) \geqslant 2^{\operatorname{card} \mathscr{E}}$.

Proof. Consider the application $\mathscr{P}(\mathscr{E}) \to \mathscr{A}/\mathscr{N}$ which maps each subcollection $\mathscr{E}' \subseteq \mathscr{E}$ to the equivalence class of its $\mathscr{N}$-essential supremum. We claim that this map is injective. Indeed, suppose $\mathscr{E}', \mathscr{E}'' \subseteq \mathscr{E}$ are distinct. Call S' (resp. S'') an $\mathscr{N}$-essential supremum of $\mathscr{E}'$ (resp. $\mathscr{E}''$). Without loss of generality, there is $F \in \mathscr{E}' \setminus \mathscr{E}''$. By Lemma 3.4, $F \cap S'$ (resp. $F \cap S''$) is an essential supremum of $\{F \cap E : E \in \mathscr{E}'\}$ (resp. $\{F \cap E : E \in \mathscr{E}''\}$). We deduce that $F \setminus S' \in \mathscr{N}$ and, taking the almost disjointed character of $\mathscr{E}$ into account, that $F \cap S'' \in \mathscr{N}$. This implies that S' and S'' do not induce the same equivalence class in $\mathscr{A}/\mathscr{N}$. ∎

We will use the following many times.

Lemma 4.9. *Let $(X, \mathscr{A}, \mathscr{N})$ be an MSN and let $\mathscr{C} \subseteq \mathscr{A}$ be $\mathscr{N}$-generating. There exists $\mathscr{E} \subseteq \mathscr{A} \setminus \mathscr{N}$ with the following properties.*

(A) *$\mathscr{E}$ is almost disjointed.*

(B) *For each $E \in \mathscr{E}$, there exists $C \in \mathscr{C}$ such that $E \subseteq C$.*

(C) *$\mathscr{E}$ is $\mathscr{N}$-generating.*

Proof. There is no restriction to assume that $\mathscr{N} \neq \mathscr{A}$; in particular, $\mathscr{C} \neq \emptyset$. Consider the set $\mathbf{E}$ consisting of those $\mathscr{E} \subseteq \mathscr{A} \setminus \mathscr{N}$ that satisfy conditions (A) and (B) above, ordered by inclusion. Thus, $\mathbf{E}$ is nonempty and one readily checks that every chain in $\mathbf{E}$ possesses a maximal element. Therefore, $\mathbf{E}$ admits a maximal element $\mathscr{E}$, according to Zorn's Lemma. We ought to show that $\mathscr{E}$ is $\mathscr{N}$-generating. If this were not the case, there would exist an $\mathscr{N}$-essential upper bound $U \in \mathscr{A}$ of $\mathscr{E}$ such that $X \setminus U \notin \mathscr{N}$. The latter, together with the fact that $\mathscr{C}$ is $\mathscr{N}$-generating, implies the existence of $C \in \mathscr{C}$ such that $C \cap (X \setminus U) \notin \mathscr{N}$. Then, $\mathscr{E} \cup \{C \cap (X \setminus U)\}$ contradicts the maximality of $\mathscr{E}$. ∎

PROPOSITION 4.10. $(\mathsf{ZFC}+\mathsf{CH})$ *Let X be a Polish space endowed with its Borel σ-algebra $\mathscr{B}(X)$ and $\mu\colon \mathscr{B}(X) \to [0,\infty]$ be a semi-finite Borel measure. Under the Continuum Hypothesis, one has the following dichotomy: either μ is σ-finite, or the MSN $(X, \mathscr{B}(X), \mathscr{N}_\mu)$ is not localizable.*

Proof. Let $\mathscr{E}$ be associated with $\mathscr{C} := \mathscr{B}(X) \cap \{A : \mu(A) < \infty\}$ in Lemma 4.9. Recall 4.2 that $\mathscr{C}$ is $\mathscr{N}_\mu$-generating. If $\mathscr{E}$ is countable, then $\bigcup \mathscr{E}$ is measurable and, accordingly, an $\mathscr{N}_\mu$-essential upper bound of $\mathscr{E}$. Thus $X \setminus \bigcup \mathscr{E} \in \mathscr{N}_\mu$, since $\mathscr{E}$ is $\mathscr{N}_\mu$-generating. We have proven that μ is σ-finite.

On the other hand, if $\mathscr{E}$ is uncountable, the Continuum Hypothesis guarantees that it has cardinal greater or equal to $\mathfrak{c}$. Assume if possible that $(X, \mathscr{B}(X), \mu)$ is localizable. As the map $\mathscr{B}(X) \to \mathscr{B}(X)/\mathscr{N}_\mu$ is onto, we deduce from Lemma 4.8 that $\operatorname{card} \mathscr{B}(X) \geqslant 2^{\mathfrak{c}} > \mathfrak{c}$. However, Borel sets are Suslin, and Suslin sets are continuous images of closed subsets of a particular Polish space, the Baire space, see e.g [17, 3.3.18]. This gives the upper bound $\operatorname{card} \mathscr{B}(X) \leqslant \mathfrak{c}$, contradicting the preceding inequality. ∎

DEFINITION 4.11. ($\mathscr{P}$-VERSION OF AN MSN) Let $\mathscr{P}$ be a property associated to MSNs. We suppose that the property $\mathscr{P}$ is *hereditary*: if $(X, \mathscr{A}, \mathscr{N})$ has $\mathscr{P}$ then the MSNs $\left(Z, \mathscr{A}_Z, \mathscr{N}_Z\right)$ also has $\mathscr{P}$ for all $Z \in \mathscr{A}$. "Being strictly localizable", "being 4c" or "being localizable" are examples of hereditary properties. Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN.

We define a *$\mathscr{P}$-version* of $(X, \mathscr{A}, \mathscr{N})$ to be a couple $\left[\left(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right), \mathbf{p}\right]$ consisting of a saturated MSN $\left(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right)$ with the property $\mathscr{P}$ and a supremum preserving morphism $\mathbf{p}\colon \left(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right) \to (X, \mathscr{A}, \mathscr{N})$ satisfying the following property: For any saturated MSN $(Y, \mathscr{B}, \mathscr{M})$ with the property $\mathscr{P}$ and any supremum preserving morphism $\mathbf{q}\colon (Y, \mathscr{B}, \mathscr{M}) \to (X, \mathscr{A}, \mathscr{N})$, there is a unique supremum preserving morphism $\mathbf{r}\colon (Y,\mathscr{B},\mathscr{M}) \to \big(\hat{X},\hat{\mathscr{A}},\hat{\mathscr{N}}\big)$ such that $\mathbf{q} = \mathbf{p} \circ \mathbf{r}$.

$$\begin{tikzcd}
(Y,\mathscr{B},\mathscr{M}) \arrow[rr, dotted, "\exists!\mathbf{r}"] \arrow[dr, "\mathbf{q}"'] & & \big(\hat{X},\hat{\mathscr{A}},\hat{\mathscr{N}}\big) \arrow[dl, "\mathbf{p}"] \\
& (X,\mathscr{A},\mathscr{N}) &
\end{tikzcd}$$

By this definition, a $\mathscr{P}$-version must satisfy a universal property, and as such it is unique up to a unique isomorphism of the category $\mathsf{MSN}_{\mathsf{sp}}$. More specifically, if $\big[\big(\hat{X},\hat{\mathscr{A}},\hat{\mathscr{N}}\big),\mathbf{p}\big]$ and $\big[\big(\hat{X}',\hat{\mathscr{A}}',\hat{\mathscr{N}}'\big),\mathbf{p}'\big]$ are two $\mathscr{P}$-versions, then we easily check that there is a unique isomorphism $\mathbf{r}\colon \big(\hat{X},\hat{\mathscr{A}},\hat{\mathscr{N}}\big) \to \big(\hat{X}',\hat{\mathscr{A}}',\hat{\mathscr{N}}'\big)$ such that $\mathbf{p}' \circ \mathbf{r} = \mathbf{p}$.

Definition 4.12. (Atomic MSNs) One of our motivations in this article is to find a universal construction that transforms an MSN into something with better localizability properties. As such, it is wise to first have a look at the not so easy case of MSNs $(X,\mathscr{A},\mathscr{N})$ such that all singletons are $\mathscr{A}$-measurable and $\mathscr{N} = \{\emptyset\}$. We call such MSNs *atomic.*

In an atomic MSN $(X,\mathscr{A},\{\emptyset\})$, it is easy to see that a subset $\mathscr{E} \subseteq \mathscr{A}$ has an $\{\emptyset\}$-essential supremum if and only if $\bigcup\mathscr{E} \in \mathscr{A}$, in which case $\bigcup\mathscr{E}$ is the $\{\emptyset\}$-essential supremum. Therefore the MSN $(X,\mathscr{A},\{\emptyset\})$ is localizable if and only if $\mathscr{A} = \mathscr{P}(X)$. In other words, the non localizability of $(X,\mathscr{A},\{\emptyset\})$ can only be due to the lack of measurable sets; therefore it seems sensible to ask for $(X,\mathscr{P}(X),\{\emptyset\})$ to be the "localization" of $(X,\mathscr{A},\{\emptyset\})$.

Unfortunately, Proposition 4.13 gives a negative result. It tells us that the localizable version of an MSN, as defined in 4.11, is not the right notion of "localization". This issue will be addressed in Section 5 by introducing a notion of local determination for MSNs.

Proposition 4.13. *Let X be an uncountable set, $\mathscr{C}(X)$ be its countable-cocountable σ-algebra. Let $\boldsymbol{\iota} : (X,\mathscr{P}(X),\{\emptyset\}) \to (X,\mathscr{C}(X),\{\emptyset\})$ be the morphism induced by the identity map. Then $\big[(X,\mathscr{P}(X),\{\emptyset\}),\boldsymbol{\iota}\big]$ is not a localizable version of $(X,\mathscr{C}(X),\{\emptyset\})$.*

Proof. That $\boldsymbol{\iota}$ is supremum preserving follows from the discussion in Paragraph 4.12. Assume if possible that $((X,\mathscr{P}(X),\{\emptyset\}),\boldsymbol{\iota})$ is a localizable version of $(X,\mathscr{A},\{\emptyset\})$.

We will get a contradiction if we manage to build a localizable saturated MSN $(Y,\mathscr{B},\mathscr{M})$ and a function $q\colon Y\to X$ that is $[(\mathscr{B},\mathscr{M}),(\mathscr{C}(X),\{\emptyset\})]$-measurable, supremum preserving, but not $(\mathscr{B},\mathscr{P}(X))$-measurable.

We choose $Y=X^2\times\{0,1\}$. For any subset $B\subseteq Y$, we call $B[0]$ and $B[1]$ the subsets defined by

$$B[i]:=X^2\cap\{(x_1,x_2)\,:\,(x_1,x_2,i)\in B\}\quad\text{for } i\in\{0,1\}$$

We let $\mathscr{B}=\mathscr{P}(Y)\cap\{B\,:\,B[0]\ominus B[1]\text{ is countable}\}$. We claim that $\mathscr{B}$ is a σ-algebra of Y. The stability of $\mathscr{B}$ under countable unions is a consequence of the formula

$$\left(\bigcup_{n\in\mathbb{N}}B_n\right)[0]\ominus\left(\bigcup_{n\in\mathbb{N}}B_n\right)[1]\subseteq\bigcup_{n\in\mathbb{N}}B_n[0]\ominus B_n[1]$$

that holds for any sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ of subsets in Y, and we leave the other points to the reader. Finally, we define the σ-ideal $\mathscr{M}:=\mathscr{B}\cap\{M:M[0]=\emptyset\}$. Clearly $(Y,\mathscr{B},\mathscr{M})$ is a saturated MSN.

Let us show that $(Y,\mathscr{B},\mathscr{M})$ is localizable. Let $\mathscr{E}\subseteq\mathscr{B}$ be any collection. We set $A:=\bigcup_{E\in\mathscr{E}}E[0]$ and $S:=A\times\{0,1\}$. The set S is $\mathscr{B}$-measurable, because $S[0]=S[1]=A$. For any $E\in\mathscr{E}$, we have $(E\setminus S)[0]=E[0]\setminus S[0]=\emptyset$, meaning that S is an $\mathscr{M}$-essential upper bound of $\mathscr{E}$. Denoting by U another essential upper bound of $\mathscr{E}$, then $(E\setminus U)[0]=E[0]\setminus U[0]=\emptyset$ for all $E\in\mathscr{E}$. It follows that $A\subseteq U[0]$ and $(S\setminus U)[0]=S[0]\setminus U[0]=\emptyset$. Thus, S is an $\mathscr{M}$-essential supremum, as we wanted.

Now, let $\sigma\colon X\to X$ be a bijection of X without fixed points. For example, choose a partition $X=Z\cup Z'$ into subsets Z,Z' that have the same cardinality as X, choose a bijection $f\colon Z\to Z'$ and set σ so that $\sigma(x)=f(x)$ for all $x\in Z$ and $\sigma(x)=f^{-1}(x)$ for all $x\in Z'$. We define the map $q\colon Y\to X$ by

$$\forall(x_1,x_2,i)\in Y,\quad q(x_1,x_2,i)=\begin{cases}x_2 & \text{if } i=1\text{ and } x_2=\sigma(x_1),\\ x_1 & \text{otherwise}.\end{cases}$$

First we show that q is $[(\mathscr{B},\mathscr{M}),(\mathscr{C}(X),\{\emptyset\})]$-measurable. It suffices to show that $q^{-1}(\{x\})\in\mathscr{B}$ for all $x\in X$. But we have

$$\begin{aligned}&q^{-1}(\{x\})[0]=\{x\}\times X,\\&q^{-1}(\{x\})[1]=\big(\{x\}\times(X\setminus\{\sigma(x)\})\big)\cup\{(\sigma^{-1}(x)\,,x)\}.\end{aligned}$$

Consequently, $q^{-1}(\{x\})[0] \ominus q^{-1}(\{x\})[1]$ has only two elements. By the definition of $\mathscr{B}$, this ensures the measurability of $q^{-1}(\{x\})$.

However, we claim that q is not $(\mathscr{B}, \mathscr{P}(X))$-measurable. To this end, we will show that $q^{-1}(Z) \notin \mathscr{B}$. We have

$$q^{-1}(Z)[0] = Z \times X,$$
$$q^{-1}(Z)[1] = \{(x_1, x_2) : x_1 \in Z, x_2 \neq \sigma(x_1)\} \cup \{(\sigma^{-1}(x), x) : x \in Z\}.$$

It follows that $q^{-1}(Z)[0] \ominus q^{-1}(Z)[1] = \{(x, \sigma(x)) : x \in Z\} \cup \{(\sigma^{-1}(x), x) : x \in Z\}$ is uncountable. Thus, $q^{-1}(Z) \notin \mathscr{B}$.

It only remains to prove that q is supremum preserving. Let $\mathscr{E} \subseteq \mathscr{C}(X)$ be a collection that has an $\{\emptyset\}$-essential supremum S. This implies that $S = \bigcup \mathscr{E}$. First suppose that $\mathscr{E}$ consists only of singletons. We wish to prove that $q^{-1}(S)$ is an $\mathscr{M}$-essential supremum of $q^{-1}(\mathscr{E}) = \{q^{-1}\{x\} : x \in S\}$. Of course, $q^{-1}(S)$ is an $\mathscr{M}$-essential upper bound of $q^{-1}(\mathscr{E})$. Let U an arbitrary $\mathscr{M}$-essential upper bound of $q^{-1}(\mathscr{E})$. For all $x \in S$, we have $q^{-1}\{x\} \setminus U \in \mathscr{M}$, meaning that $\{x\} \times X = (q^{-1}\{x\})[0] \subseteq U[0]$. Thus $S \times X \subseteq U[0]$, which implies $(q^{-1}(S) \setminus U)[0] = q^{-1}(S)[0] \setminus U[0] = S \times X \setminus U[0] = \emptyset$. It means that $q^{-1}(S) \setminus U \in \mathscr{M}$. Thus, we have shown that $q^{-1}(S)$ is an $\mathscr{M}$-essential supremum of $\mathscr{E}$.

Now we turn to the general case, where $\mathscr{E}$ need not consist only of singletons. Let $\mathscr{E}' = \{\{x\} : x \in E \in \mathscr{E}\}$. Clearly, $\mathscr{E}$ and $\mathscr{E}'$ have the same $\{\emptyset\}$-essential supremum $S := \bigcup \mathscr{E} = \bigcup \mathscr{E}'$. By what precedes, $q^{-1}(S)$ is an $\mathscr{M}$-essential supremum of $q^{-1}(\mathscr{E}')$ and it is an $\mathscr{M}$-essential upper bound of $q^{-1}(\mathscr{E})$. An $\mathscr{M}$-essential upper bound U of $q^{-1}(\mathscr{E})$ is also an upper bound for $q^{-1}(\mathscr{E}')$, as any member of $q^{-1}(\mathscr{E}')$ is a subset of a member of $q^{-1}(\mathscr{E})$. Therefore, $q^{-1}(S) \setminus U \in \mathscr{M}$, showing that $q^{-1}(S)$ is an $\mathscr{M}$-essential supremum of $q^{-1}(\mathscr{E})$. ∎

4.14. (Example of an MSN with no localizable part) Consider an MSN of the form $(X, \mathscr{P}(X), \mathscr{K}(X))$, where X is a set of cardinality $\aleph_1$ and $\mathscr{K}(X)$ is the σ-ideal of countable subsets. There is a bijection $\varphi\colon X \to X \times X$ and we can use it to construct an uncountable family of "horizontal lines" $H_x := \varphi^{-1}(X \times \{x\})$ indexed by $x \in X$ witnessing that $(X, \mathscr{P}(X), \mathscr{K}(X))$ is not ccc. Actually, we can do better and prove that it is not localizable. Suppose $\{H_x : x \in X\}$ has a $\mathscr{K}(X)$-essential supremum S. For each $x \in X$ choose a point $p_x \in S \cap H_x$. Then it is easy to see that $U := S \setminus \{p_x : x \in X\}$ is an essential upper bound for the family of horizontal lines, however

$S \setminus U = \{p_x : x \in X\}$ is not negligible, contradicting that S is an essential supremum.

Observe that the MSN $(X, \mathscr{P}(X), \mathscr{K}(X))$ is isomorphic to all its non negligible subMSNs. In particular, it has no nontrivial ccc or localizable part, an unpleasant situation that we will rule out in the next paragraph by introducing the notions of "locally localizable" and "locally ccc" MSN.

We will prove nonetheless that $\big(X, \mathscr{P}(X), \mathscr{K}(X)\big)$ has a 4c version, that is disappointingly the trivial MSN $(\emptyset, \{\emptyset\}, \{\emptyset\})$ (with the only morphism from there to $\big(X, \mathscr{P}(X), \mathscr{K}(X)\big)$). To establish this fact, one needs to prove that if $(Y, \mathscr{B}, \mathscr{M})$ is a 4c MSN and $\mathbf{f}\colon (Y, \mathscr{B}, \mathscr{M}) \to \big(X, \mathscr{P}(X), \mathscr{K}(X)\big)$ is a supremum preserving morphism, then $Y \in \mathscr{M}$ (actually, the supremum preserving character of $\mathbf{f}$ will not be used). We can reduce to the case where $(Y, \mathscr{B}, \mathscr{M})$ is ccc.

We reproduce an argument due to Ulam [19], showing that there is a family $\langle A_{n,\alpha}\rangle_{n\in\mathbb{N},\alpha<\omega_1}$ of subsets of X such that

- for all $n \in \mathbb{N}$, the family $\langle A_{n,\alpha}\rangle_{\alpha<\omega_1}$ is disjointed;
- for all $\alpha < \omega_1$, the union $\bigcup_{n\in\mathbb{N}} A_{n,\alpha}$ is conegligible (that is, cocountable).

Any ordinal $\beta < \omega_1$ is countable, so we can select a sequence $\langle k_{\alpha,\beta}\rangle_{\alpha<\beta}$ of distinct integers. Let $\langle x_\beta\rangle_{\beta<\omega_1}$ be an enumeration of all the elements in X. Set $A_{n,\alpha} := \{x_\beta : \beta > \alpha \text{ and } k_{\alpha,\beta} = n\}$ for every $n \in \mathbb{N}$ and $\alpha < \omega_1$. For distinct $\alpha, \alpha' < \omega_1$, there cannot be some $x_\beta \in A_{n,\alpha} \cap A_{n,\alpha'}$, for otherwise we would have $k_{\alpha,\beta} = k_{\alpha',\beta}$. In addition, one has $\bigcup_{n\in\mathbb{N}} A_{n,\alpha} = \{x_\beta : \beta > \alpha\}$ whose complement in X is the countable set $\{x_\beta : \beta \leqslant \alpha\}$.

Now, fix a representative $f \in \mathbf{f}$. The family $\big\langle f^{-1}(A_{n,\alpha})\big\rangle_{\alpha<\omega_1}$ being disjointed, the set $C_n := \omega_1 \cap \big\{\alpha : f^{-1}(A_{n,\alpha}) \notin \mathscr{M}\big\}$ is countable for all $n \in \mathbb{N}$. Hence the existence of some $\alpha \in \omega_1 \setminus \bigcup_{n\in\mathbb{N}} C_n$. Now we see that the set $f^{-1}\big(\bigcup_{n\in\mathbb{N}} A_{n,\alpha}\big)$ is both negligible and conegligible in Y, which can happen only if $Y \in \mathscr{M}$.

Definition 4.15. Let $\mathscr{P}$ be a hereditary property associated to MSNs. We say that an MSN $(X, \mathscr{A}, \mathscr{N})$ is *locally $\mathscr{P}$* whenever one of the following equivalent statements holds:

(A) The collection

$$\mathscr{A} \cap \big\{Z : (Z, \mathscr{A}_Z, \mathscr{N}_Z) \text{ has the property } \mathscr{P}\big\}$$

is $\mathscr{N}$-generating;

(B) for any $Y \in \mathscr{A} \setminus \mathscr{N}$ there is $Z \in \mathscr{A} \setminus \mathscr{N}$ such that $Z \subseteq Y$ and $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ has the property $\mathscr{P}$.

Proof. (Proof of the equivalence) (A) $\implies$ (B) For $Y \in \mathscr{A} \setminus \mathscr{N}$, an application of Lemma 3.4 gives that Y is an essential supremum of

$$\big\{ Y \cap Z \,:\, Z \in \mathscr{A} \text{ and } (Z, \mathscr{A}_Z, \mathscr{N}_Z) \text{ has the property } \mathscr{P} \big\}.$$

Therefore, there must be some $Z \in \mathscr{A}$ such that $Y \cap Z \notin \mathscr{N}$ and $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ has the property $\mathscr{P}$. The subset $Y \cap Z \subseteq Y$ establishes (B).

(B) $\implies$ (A) Clearly, X is an $\mathscr{N}$-essential upper bound of the collection $\mathscr{A} \cap \{Z : (Z, \mathscr{A}_Z, \mathscr{N}_Z) \text{ has the property } \mathscr{P}\}$. Let S be another upper bound. If $X \setminus S$ were not negligible, (B) gives the existence of some measurable $Z \in \mathscr{A} \setminus \mathscr{N}$ such that $Z \subseteq X \setminus S$ and $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ has the property $\mathscr{P}$. But $Z \setminus S = Z \notin \mathscr{N}$, which contradicts that S is an essential upper bound. ∎

For instance, in a semi-finite measure space $(X, \mathscr{A}, \mu)$, any non negligible set $A \in \mathscr{A} \setminus \mathscr{N}_\mu$ contains a measurable subset Z of nonzero finite measure. By (B), this implies that the associated MSN $(X, \mathscr{A}, \mathscr{N}_\mu)$ is locally strictly localizable.

We conclude this section with an important property of "local isomorphism" that holds for $\mathscr{P}$-versions.

Proposition 4.16. *Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN and $\big[\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big), \mathbf{p}\big]$ a $\mathscr{P}$-version of it. Fix a representative map $p \in \mathbf{p}$. For any $F \in \mathscr{A}$, we set $\hat{F} := p^{-1}(F)$ and we call $\mathbf{p}_F \colon \big(\hat{F}, \hat{\mathscr{A}}_{\hat{F}}, \hat{\mathscr{N}}_{\hat{F}}\big) \to (F, \mathscr{A}_F, \mathscr{N}_F)$ the morphism induced by the restriction $p_F \colon \hat{F} \to F$ of p.*

(A) *$\big[\big(\hat{F}, \hat{\mathscr{A}}_{\hat{F}}, \hat{\mathscr{N}}_{\hat{F}}\big), \mathbf{p}_F\big]$ is the $\mathscr{P}$-version of $(F, \mathscr{A}_F, \mathscr{N}_F)$;*

(B) *If $(F, \mathscr{A}_F, \mathscr{N}_F)$ has the property $\mathscr{P}$, then $\mathbf{p}_F$ is an isomorphism.*

Proof. (A) Since the property $\mathscr{P}$ is hereditary, we can assert that $(\hat{F}, \hat{\mathscr{A}}_{\hat{F}}, \hat{\mathscr{N}}_{\hat{F}})$ has it. We also readily check that p_F is supremum preserving. Let $\mathbf{q} \colon (Y, \mathscr{B}, \mathscr{M}) \to (F, \mathscr{A}_F, \mathscr{N}_F)$ be a supremum preserving morphism starting from a saturated MSN with the property $\mathscr{P}$.

Then $\boldsymbol{\iota_F} \circ \mathbf{q}$ is a supremum preserving morphism ending in $(X, \mathscr{A}, \mathscr{N})$. It has a lifting $\mathbf{r} \colon (Y, \mathscr{B}, \mathscr{M}) \to \big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$. Let $r \in \mathbf{r}$ and $q \in \mathbf{q}$ be representatives. As $p(r(y)) = q(y)$ for $\mathscr{M}$-almost all $y \in Y$, we lose no generality in supposing that r has values in $\hat{F}$. Calling its restriction $r' \colon Y \to \hat{F}$, we see that $p_F \circ r'$ and q coincide $\mathscr{M}$-almost everywhere. The induced morphism $\mathbf{r}'$ provides a factorization of $\mathbf{q}$ through $\big(\hat{F}, \hat{\mathscr{A}}_{\hat{F}}, \hat{\mathscr{N}}_{\hat{F}}\big)$.

To establish the uniqueness of this factorization, we proceed as follows. For any morphism $\mathbf{r}''$ such that $\mathbf{p}_F \circ \mathbf{r}'' = \mathbf{q}$ we notice that

$$\boldsymbol{\iota}_F \circ \mathbf{q} = \boldsymbol{\iota}_F \circ \mathbf{p}_F \circ \mathbf{r}'' = \mathbf{p} \circ \boldsymbol{\iota}_{\hat{F}} \circ \mathbf{r}''.$$

Since this holds for $\mathbf{r}'$ we obtain $\mathbf{p} \circ \boldsymbol{\iota}_{\hat{F}} \circ \mathbf{r}' = \mathbf{p} \circ \boldsymbol{\iota}_{\hat{F}} \circ \mathbf{r}''$ and, by uniqueness of the factorization relative to the universal property of $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$, $\boldsymbol{\iota}_{\hat{F}} \circ \mathbf{r}' = \boldsymbol{\iota}_{\hat{F}} \circ \mathbf{r}''$. Thus, r' and r'' coincide $\mathscr{M}$-almost everywhere.

(B) If $(F, \mathscr{A}_F, \mathscr{N}_F)$ has property $\mathscr{P}$, then obviously $\big[(F, \mathscr{A}_F, \mathscr{N}_F), \mathbf{id}\big]$ is a second $\mathscr{P}$-version. From the uniqueness of the $\mathscr{P}$-version, we obtain a isomorphism $\mathbf{r}\colon (F, \mathscr{A}_F, \mathscr{N}_F) \to \big(\hat{F}, \hat{\mathscr{A}}_{\hat{F}}, \hat{\mathscr{N}}_{\hat{F}}\big)$ such that $\mathbf{id} = \mathbf{p}_F \circ \mathbf{r}$, whence $\mathbf{p}_F = \mathbf{r}^{-1}$. ∎

5. Localizable locally determined MSNs

In order to motivate the main definition in this section, we start with the following result, of which we can think as a way of testing whether an MSN has a property $\mathscr{P}$. For instance, if each $F \in \mathscr{F}$ corresponds to a ccc subMSN, then $(X, \mathscr{A}, \mathscr{N})$ is 4c. The difficulty in applying this proposition stems with both hypotheses: conditions (1) and (2) will be turned into a definition in 5.2, whereas condition (3), that $\mathscr{F}$ is *disjointed* rather than merely *almost disjointed*, calls for techniques that transform almost disjointed generating families (whose existence, in applications, follows from Lemma 4.9) into partitions – see the proof of Theorem 7.6 in case $\operatorname{card} \mathscr{E} \leqslant \mathfrak{c}$ and the notion of compatible family of densities introduced in Section 10.

Proposition 5.1. *Let $(X, \mathscr{A}, \mathscr{N})$ be an MSN and $\mathscr{F} \subseteq \mathscr{A}$. Assume that*

(1) *For every $A \subseteq X$ the following holds:*

$$\big[\forall F \in \mathscr{F} : A \cap F \in \mathscr{A}\big] \Rightarrow A \in \mathscr{A};$$

(2) *For every $N \subseteq X$ the following holds:*

$$\big[\forall F \in \mathscr{F} : N \cap F \in \mathscr{N}\big] \Rightarrow N \in \mathscr{N};$$

(3) *$\mathscr{F}$ is a partition of X.*

Then the MSNs $(X, \mathscr{A}, \mathscr{N})$ and $\coprod_{F \in \mathscr{F}} (F, \mathscr{A}_F, \mathscr{N}_F)$ are isomorphic in $\mathsf{MSN_{sp}}$.

Proof. We abbreviate $(Y,\mathscr{B},\mathscr{M})$ for $\coprod_{F\in\mathscr{F}}(F,\mathscr{A}_F,\mathscr{N}_F)$. Since $\mathscr{F}$ is a partition of X, there is a canonical bijection $\varphi : X \to Y$. Its inverse φ^{-1} is $[(\mathscr{B},\mathscr{M}),(\mathscr{A},\mathscr{N})]$-measurable, by definition of coproduct of MSNs. We now show that φ is $[(\mathscr{A},\mathscr{N}),(\mathscr{B},\mathscr{M})]$-measurable. Given $B \in \mathscr{B}$, we note that $\varphi^{-1}(B) = \bigcup_{F\in\mathscr{F}} B\cap F$, whence $\varphi^{-1}(B)\cap F = B\cap F \in \mathscr{A}$ for every $F \in \mathscr{F}$, by definition of $\mathscr{B}$. We infer from hypothesis (1) that $\varphi^{-1}(B) \in \mathscr{A}$. Let $M \in \mathscr{M}$. As above we infer from the definition of $\mathscr{M}$ that $\varphi^{-1}(M)\cap F \in \mathscr{N}$ for every $F \in \mathscr{F}$, whence $\varphi^{-1}(M) \in \mathscr{N}$, in view of hypothesis (2). In other words, $(X,\mathscr{A},\mathscr{N})$ and $(Y,\mathscr{B},\mathscr{M})$ are isomorphic in MSN. The conclusion follows from Proposition 3.7(A). ∎

Definition 5.2. We borrow the following definition from [6, 211H]. A measure space $(X,\mathscr{A},\mu)$ is *locally determined* whenever it is semi-finite and, for every subset $A \subseteq X$,

$$\left[\forall E \in \mathscr{A}^f : A\cap E \in \mathscr{A}\right] \Rightarrow A \in \mathscr{A},$$

where, as usual, $\mathscr{A}^f = \mathscr{A} \cap \{E : \mu(E) < \infty\}$.

The definition relies on the particular collection $\mathscr{A}^f$ (which is $\mathscr{N}_\mu$-generating, recall 4.2). This makes sense because we are dealing with a measure space. It is a rather good surprise that we can define an analogous notion of locally determined MSNs, by substituting for $\mathscr{A}^f$ an arbitrary generating collection. Namely, a saturated MSN $(X,\mathscr{A},\mathscr{N})$ is called *locally determined* whenever the following holds. For every $\mathscr{N}$-generating collection $\mathscr{E} \subseteq \mathscr{A}$ and every $A \subseteq X$,

$$\left[\forall E \in \mathscr{E} : A\cap E \in \mathscr{A}\right] \Rightarrow A \in \mathscr{A}.$$

An MSN that is both localizable and locally determined is called *lld*.

The following is useful as well. We say that a saturated MSN $(X,\mathscr{A},\mathscr{N})$ has *locally determined negligible sets* whenever the following holds. For every $\mathscr{N}$-generating collection $\mathscr{E} \subseteq \mathscr{A}$ and every $N \subseteq X$,

$$\left[\forall E \in \mathscr{E} : N\cap E \in \mathscr{N}\right] \Rightarrow N \in \mathscr{N}.$$

We observe that if $(X,\mathscr{A},\mathscr{N})$ is locally determined, then it has locally determined negligible sets. Indeed, let $\mathscr{E} \subseteq \mathscr{A}$ and $N \subseteq X$ be as above, we first infer from the local determinacy of $(X,\mathscr{A},\mathscr{N})$ that $N \in \mathscr{A}$ and, in turn from the Distributivity Lemma 3.4, that N is an $\mathscr{N}$-essential supremum of $\{N\cap E : E \in \mathscr{E}\}$. Therefore, $N \in \mathscr{N}$.

Next we prove some elementary properties concerning locally determined MSNs. In particular, the consistency between both notions of local determination (for complete semi-finite measure spaces and MSNs) is established in Proposition 5.3(F). Here, the semi-finiteness property of a measure space is critical as the following example shows. We consider $\mathscr{H}^1$, the 1-dimensional Hausdorff measure in $\mathbb{R}^2$ and $\mathscr{A}$ the σ-algebra consisting of $\mathscr{H}^1$-measurable subsets of $\mathbb{R}^2$ in the sense of Carathéodory. The following hold:

(a) $\forall A \subseteq \mathbb{R}^2 : \left[\forall F \in \mathscr{A}^f : A \cap F \in \mathscr{A}\right] \Rightarrow A \in \mathscr{A}$;

(b) the measure space $(\mathbb{R}^2, \mathscr{A}, \mathscr{H}^1)$ is not semi-finite;

(c) the (saturated) MSN $(\mathbb{R}^2, \mathscr{A}, \mathscr{N}_{\mathscr{H}^1})$ does not have locally determined negligible sets and, in particular, is not locally determined.

For (a), see for instance [2, 6.2]. For (b), see [8, 439H]. Now (c) follows for example from [2, 4.4]. It follows from 4.2 that $\mathscr{A}^f$ is not $\mathscr{N}_{\mathscr{H}^1}$-generating.

Proposition 5.3. *The following hold.*

(A) *Being locally determined is a hereditary property.*

(B) *Being locally determined is a property invariant under isomorphisms in* $\mathsf{MSN_{sp}}$.

(C) *A saturated ccc MSN is locally determined.*

(D) *A coproduct of locally determined MSNs is locally determined.*

(E) *A 4c MSN is locally determined.*

(F) *A complete semi-finite measure space $(X, \mathscr{A}, \mu)$ is locally determined (as a measure space) if and only if the MSN $(X, \mathscr{A}, \mathscr{N}_\mu)$ is locally determined.*

Proof. (A) Let $(X, \mathscr{A}, \mathscr{N})$ be a locally determined MSN and $Z \in \mathscr{A}$. Let $\mathscr{E}$ be an $\mathscr{N}_Z$-generating family in the subMSN $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ and $A \subseteq Z$ be such that $E \cap A \in \mathscr{A}_Z$ for any $E \in \mathscr{E}$. The family $\mathscr{E} \cup \{X \setminus Z\}$ is $\mathscr{N}$-generating in $(X, \mathscr{A}, \mathscr{N})$ and $E \cap A \in \mathscr{A}$ for all $E \in \mathscr{E} \cup \{X \setminus Z\}$. It follows that $A \in \mathscr{A}$.

(B) Let $(X, \mathscr{A}, \mathscr{N})$ and $(Y, \mathscr{B}, \mathscr{M})$ be two saturated MSNs, $f \colon X \to Y$ and $g \colon Y \to X$ be two measurable supremum preserving maps that induce reciprocal isomorphisms. Assume that $(X, \mathscr{A}, \mathscr{N})$ is locally determined. Let $\mathscr{E} \subseteq \mathscr{B}$ be an $\mathscr{M}$-generating collection and $B \subseteq Y$ be such that $E \cap B \in \mathscr{B}$ for all $E \in \mathscr{E}$. Then $f^{-1}(E) \cap f^{-1}(B) = f^{-1}(E \cap B) \in \mathscr{A}$. As f is supremum preserving, $f^{-1}(\mathscr{E})$ is $\mathscr{N}$-generating. And as $(X, \mathscr{A}, \mathscr{N})$ is locally determined, we infer that $f^{-1}(B) \in \mathscr{A}$. Therefore $g^{-1}(f^{-1}(B)) \in \mathscr{B}$. But $B \ominus g^{-1}(f^{-1}(B)) \in \mathscr{M}$ and as $(Y, \mathscr{B}, \mathscr{M})$ is saturated we conclude that $B \in \mathscr{B}$.

(C) Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated ccc MSN, $\mathscr{E} \subseteq \mathscr{A}$ an $\mathscr{N}$-generating family and $A \in \mathscr{P}(X)$ be such that $E \cap A \in \mathscr{A}$ for all $E \in \mathscr{E}$. By Proposition 4.4, there is a countable subset $\mathscr{E}' \subseteq \mathscr{E}$ that is $\mathscr{N}$-generating. Then

$$A = \left(\bigcup_{E \in \mathscr{E}'} E \cap A \right) \cup \left(A \setminus \bigcup \mathscr{E}' \right)$$

Since $X \setminus \bigcup \mathscr{E}'$ is $\mathscr{N}$-negligible and $(X, \mathscr{A}, \mathscr{N})$ is saturated, we infer that $A \setminus \bigcup \mathscr{E}'$ is $\mathscr{A}$-measurable. Therefore, $A \in \mathscr{A}$.

(D) Let $(X, \mathscr{A}, \mathscr{N})$ be the coproduct of a family $\langle (X_i, \mathscr{A}_i, \mathscr{N}_i) \rangle_{i \in I}$ of locally determined MSNs. It is readily saturated. Let $\mathscr{E} \subseteq \mathscr{A}$ be an $\mathscr{N}$-generating family and $A \subseteq X$ such that $E \cap A \in \mathscr{A}$ for all $E \in \mathscr{E}$. For all $i \in I$, the family $\mathscr{E}_i := \{E \cap X_i : E \in \mathscr{E}\}$ is $\mathscr{N}_i$-generating in $(X_i, \mathscr{A}_i, \mathscr{N}_i)$ by Lemma 3.4. This observation leads to the fact that $A \cap X_i \in \mathscr{A}_i$ for all $i \in I$, in other words, $A \in \mathscr{A}$.

(E) This obviously follows from (C) and (D).

(F) Suppose that the measure space $(X, \mathscr{A}, \mu)$ is locally determined. Let $\mathscr{E} \subseteq \mathscr{A}$ be an $\mathscr{N}_\mu$-generating family and $A \subseteq X$ be such that $E \cap A \in \mathscr{A}$ for all $E \in \mathscr{E}$. Let $F \in \mathscr{A}^f$. By Lemma 3.4, the collection $\{F \cap E : E \in \mathscr{E}\}$ is $\mathscr{N}_F$-generating in $\big(F, \mathscr{A}_F, (\mathscr{N}_\mu)_F\big)$ and of course $F \cap E \cap A \in \mathscr{A}_F$ for all $E \in \mathscr{E}$. On top of that, $\big(F, \mathscr{A}_F, (\mathscr{N}_\mu)_F\big)$ is a ccc MSN by Proposition 4.5 and it is saturated. We get from (C) above that $A \cap F$ is measurable. As this happens for all $F \in \mathscr{A}^f$, we conclude that $A \in \mathscr{A}$.

Conversely, suppose the MSN $(X, \mathscr{A}, \mathscr{N}_\mu)$ is locally determined. Owing to the semi-finiteness of $(X, \mathscr{A}, \mu)$, the collection $\mathscr{A}^f$ is $\mathscr{N}_\mu$-generating. Then $(X, \mathscr{A}, \mu)$ is easily seen to be locally determined: if $A \in \mathscr{P}(X)$ satisfies $A \cap F \in \mathscr{A}$ for all $F \in \mathscr{A}^f$, then $A \in \mathscr{A}$. ∎

5.4. (LLD VERSION OF AN ATOMIC MSN) As a first result, we mention that the lld version of an atomic MSN $(X, \mathscr{A}, \{\emptyset\})$ is the space $[(X, \mathscr{P}(X), \{\emptyset\}), \boldsymbol{\iota}]$, where $\boldsymbol{\iota}$ is the morphism induced by the identity map (that $\boldsymbol{\iota}$ is supremum preserving follows from 4.12). This amounts to prove that, for any lld MSN $(Y, \mathscr{B}, \mathscr{M})$, a $[(\mathscr{B}, \mathscr{M}), (\mathscr{A}, \{\emptyset\})]$-measurable supremum preserving map $q \colon Y \to X$ is automatically $(\mathscr{B}, \mathscr{P}(X))$-measurable.

Indeed, let $S \in \mathscr{P}(X)$. Then $q^{-1}(S) \cap q^{-1}\{x\}$ is either $q^{-1}\{x\}$ or $\emptyset$, hence $q^{-1}(S) \cap q^{-1}\{x\}$ is $\mathscr{B}$-measurable for every $x \in X$. Besides, q is supremum preserving, thus the collection $\{q^{-1}\{x\} : x \in X\}$ is $\mathscr{M}$-generating. By local determination in $(Y, \mathscr{B}, \mathscr{M})$ we conclude that $q^{-1}(S) \in \mathscr{B}$.

5.5. Call $\mathsf{LLD_{sp}}$ the full subcategory of $\mathsf{MSN_{sp}}$ that consists of lld MSNs, and consider the forgetful functor $\mathsf{Forget}\colon \mathsf{LLD_{sp}} \to \mathsf{MSN_{sp}}$. In categorical terms, an lld version is the coreflection of a saturated MSN $(X,\mathscr{A},\mathscr{N})$ along the functor Forget, see [1, Chapter 3]. In this paper, we do not answer the question whether there exists an lld version for each saturated MSN. This is equivalent to the existence of a right adjoint R of Forget. As a matter of fact, if such an adjoint exists, there would be a natural transformation $\varepsilon\colon \mathsf{Forget}\circ\mathsf{R} \implies \mathrm{id}_{\mathsf{MSN_{sp}}}$ such that the pair $\left[\mathsf{R}(X,\mathscr{A},\mathscr{N}),\varepsilon_{(X,\mathscr{A},\mathscr{N})}\right]$ gives the lld version of any saturated MSN $(X,\mathscr{A},\mathscr{N})$.

In search for an abstract proof of the existence of R, one might think of using Freyd's Adjunction Theorem, [1, Theorem 3.3.3]. Following this path, one needs to establish (setting aside the solution set condition) that:

(A) the category $\mathsf{MSN_{sp}}$ is cocomplete;

(B) the forgetful functor Forget preserves small colimits.

Assertion (A) boils down to showing that $\mathsf{MSN_{sp}}$ has two types of small colimits: coproducts and coequalizers. The existence of the former is shown in Proposition 3.7(D). We do not know whether coequalizers exist in $\mathsf{MSN_{sp}}$ and it is the main difficulty here.

As for (B), which, regarding the existence of lld versions, is a necessary condition even if (A) were to be false, we have already proven in Propositions 4.6 and 5.3(D) that coproducts of lld MSNs are lld. Our next goal is Proposition 5.7 which states that coequalizers of lld MSNs are lld. Before that, we need to introduce some notation and a lemma.

For a saturated MSN $(X,\mathscr{A},\mathscr{N})$ and an arbitrary $\mathscr{E}\subseteq\mathscr{A}$, we define

$$\begin{aligned}\mathscr{A}_{\mathscr{E}} &:= \mathscr{P}(X)\cap\{A : E\cap A\in\mathscr{A} \text{ for all } E\in\mathscr{E}\},\\ \mathscr{N}_{\mathscr{E}} &:= \mathscr{P}(X)\cap\{N : E\cap N\in\mathscr{N} \text{ for all } E\in\mathscr{E}\}.\end{aligned}$$

It is clear that $(X,\mathscr{A}_{\mathscr{E}},\mathscr{N}_{\mathscr{E}})$ is a saturated MSN.

LEMMA 5.6. *Let $(X,\mathscr{A},\mathscr{N})$ be a localizable saturated MSN and $\mathscr{E}\subseteq\mathscr{A}$ an $\mathscr{N}$-generating family. Let $\boldsymbol{\iota}$ be the morphism $(X,\mathscr{A}_{\mathscr{E}},\mathscr{N}_{\mathscr{E}})\to(X,\mathscr{A},\mathscr{N})$ induced by the identity map on X. Then $\mathsf{Bool}(\boldsymbol{\iota})$ is an isomorphism. In particular, $\boldsymbol{\iota}$ is supremum preserving and $(X,\mathscr{A}_{\mathscr{E}},\mathscr{N}_{\mathscr{E}})$ is localizable.*

Proof. First we make the following observation, to be used later in the proof: $\mathscr{A}\cap\mathscr{N}_{\mathscr{E}}=\mathscr{N}$. Indeed, if $N\in\mathscr{A}$ is such that $E\cap N\in\mathscr{N}$ for all

$E \in \mathscr{E}$, then by the Distributivity Lemma 3.4, we conclude that $N \in \mathscr{N}$. This proves the inclusion $\mathscr{A} \cap \mathscr{N}_{\mathscr{E}} \subseteq \mathscr{N}$, the reciprocal being trivial.

The identity map $\iota\colon X \to X$ is $[(\mathscr{A}_{\mathscr{E}}, \mathscr{N}_{\mathscr{E}}), (\mathscr{A}, \mathscr{N})]$-measurable because $\mathscr{A} \subseteq \mathscr{A}_{\mathscr{E}}$ and $\mathscr{N} \subseteq \mathscr{N}_{\mathscr{E}}$. Let us show that $\mathsf{Bool}(\boldsymbol{\iota})\colon \mathscr{A}/\mathscr{N} \to \mathscr{A}_{\mathscr{E}}/\mathscr{N}_{\mathscr{E}}$ is injective by inspecting its kernel. Let $\boldsymbol{A} \in \mathscr{A}/\mathscr{N}$ be a class represented by $A \in \mathscr{A}$ such that $\mathsf{Bool}(\boldsymbol{\iota})(\boldsymbol{A}) = 0$, in other words $A = \iota^{-1}(A) \in \mathscr{N}_{\mathscr{E}}$. Then $A \in \mathscr{A} \cap \mathscr{N}_{\mathscr{E}} = \mathscr{N}$. This means that $\boldsymbol{A} = 0$ in $\mathscr{A}/\mathscr{N}$. Therefore, $\mathsf{Bool}(\boldsymbol{\iota})$ is injective.

Now let us show that $\mathsf{Bool}(\boldsymbol{\iota})$ is surjective. To this end, let $\boldsymbol{H} \in \mathscr{A}_{\mathscr{E}}/\mathscr{N}_{\mathscr{E}}$ be a class represented by $H \in \mathscr{A}_{\mathscr{E}}$. We ought to prove that $\boldsymbol{H}$ is in the range of $\mathsf{Bool}(\boldsymbol{\iota})$. Set $\mathscr{F} := \{E \cap H : E \in \mathscr{E}\}$. Note that $\mathscr{F} \subseteq \mathscr{A}$. The localizability of $(X, \mathscr{A}, \mathscr{N})$ guarantees that $\mathscr{F}$ has an $\mathscr{N}$-essential supremum $S \in \mathscr{A}$. In particular, $E \cap H \setminus S \in \mathscr{N}$ for all $E \in \mathscr{E}$, meaning that $H \setminus S \in \mathscr{N}_{\mathscr{E}}$.

We also claim that $S \setminus H \in \mathscr{N}_{\mathscr{E}}$. Indeed, let $E_0 \in \mathscr{E}$. Set $S' := S \setminus (E_0 \cap S \setminus H)$. We note that $S' \in \mathscr{A}$. For all $E \in \mathscr{E}$, we have $E \cap H \setminus S' = E \cap H \setminus S \in \mathscr{N}$, as $H \setminus S \in \mathscr{N}_{\mathscr{E}}$. This means that S' is an $\mathscr{N}$-essential upper bound of $\mathscr{F}$. It follows that $S \setminus S' = E_0 \cap S \setminus H \in \mathscr{N}$. As $E_0 \in \mathscr{E}$ is arbitrary, we obtain $S \setminus H \in \mathscr{N}_{\mathscr{E}}$, as required.

We proved that $H \ominus S \in \mathscr{N}_{\mathscr{E}}$. Calling $\boldsymbol{S}$ the equivalence class of S in $\mathscr{A}/\mathscr{N}$, we have that $\mathsf{Bool}(\boldsymbol{\iota})(\boldsymbol{S}) = \boldsymbol{H}$. ∎

PROPOSITION 5.7. *Consider the following diagram in* $\mathsf{MSN}_{\mathsf{sp}}$, *where* $((Z, \mathscr{C}, \mathscr{P}), \mathbf{h})$ *is the coequalizer of* $\mathbf{f}, \mathbf{g}$.

$$(X, \mathscr{A}, \mathscr{N}) \underset{\mathbf{g}}{\overset{\mathbf{f}}{\rightrightarrows}} (Y, \mathscr{B}, \mathscr{M}) \xrightarrow{\mathbf{h}} (Z, \mathscr{C}, \mathscr{P})$$

(A) If $(Y, \mathscr{B}, \mathscr{M})$ *is localizable, so is* $(Z, \mathscr{C}, \mathscr{P})$.

(B) If $(Y, \mathscr{B}, \mathscr{M})$ *is lld, so is* $(Z, \mathscr{C}, \mathscr{P})$.

Proof. (A) Let us call $\mathbf{2}$ the special MSN $\big(\{0,1\}, \mathscr{P}(\{0,1\}), \{\emptyset\}\big)$. First we show the following intermediate result: for any MSN $(X, \mathscr{A}, \mathscr{N})$, there is a one-to-one correspondence Υ_X between the Boolean algebra $\mathsf{Bool}(X, \mathscr{A}, \mathscr{N})$ and the set of morphisms $\mathrm{Hom}\big((X, \mathscr{A}, \mathscr{N}), \mathbf{2}\big)$ (those are automatically supremum preserving since the Boolean algebra of $\mathbf{2}$ is finite). Given a class $\boldsymbol{A} \in \mathsf{Bool}(X, \mathscr{A}, \mathscr{N})$, represented by a set A, the characteristic function $\mathbb{1}_A\colon X \to \{0,1\}$ induces a morphism $\mathbf{1}_{\boldsymbol{A}}$ which only depends on the equivalence class $\boldsymbol{A}$. Indeed, if A' is another representative of $\boldsymbol{A}$, then $\mathbb{1}_A$ and $\mathbb{1}_{A'}$ coincide $\mathscr{N}$-almost everywhere. We set $\Upsilon_X(\boldsymbol{A}) := \mathbf{1}_{\boldsymbol{A}}$.

This map is surjective because each morphism $\boldsymbol{\varphi}\colon (X,\mathscr{A},\mathscr{N}) \to \mathbf{2}$ is represented by a map $\varphi \in \boldsymbol{\varphi}$ which has the form $\varphi = \mathbb{1}_{\varphi^{-1}(\{1\})}$. It is injective because if $\mathbb{1}_A$ coincides with $\mathbb{1}_B$ almost everywhere, for measurable sets A and B, then A and B yield the same equivalence class in $\mathsf{Bool}(X,\mathscr{A},\mathscr{N})$.

Now we turn to the proof of conclusion (A). By naturality of Υ, the following diagram is commutative, where $\operatorname{Hom}(\mathbf{h},\mathbf{2})$, $\operatorname{Hom}(\mathbf{f},\mathbf{2})$ and $\operatorname{Hom}(\mathbf{g},\mathbf{2})$ denote the right composition with $\mathbf{h}$, $\mathbf{f}$, and $\mathbf{g}$, respectively:

$$\begin{tikzcd}
\operatorname{Hom}((Z,\mathscr{C},\mathscr{P}),\mathbf{2}) \arrow[r, "\operatorname{Hom}(\mathbf{h},\mathbf{2})"] & \operatorname{Hom}((Y,\mathscr{B},\mathscr{M}),\mathbf{2}) \arrow[r, shift left, "\operatorname{Hom}(\mathbf{f},\mathbf{2})"] \arrow[r, shift right, "\operatorname{Hom}(\mathbf{g},\mathbf{2})"'] & \operatorname{Hom}((X,\mathscr{A},\mathscr{N}),\mathbf{2}) \\
\mathsf{Bool}(Z,\mathscr{C},\mathscr{P}) \arrow[u, "\Upsilon_Z"] \arrow[r, "\mathsf{Bool}(\mathbf{h})"] & \mathsf{Bool}(Y,\mathscr{B},\mathscr{M}) \arrow[u, "\Upsilon_Y"] \arrow[r, shift left, "\mathsf{Bool}(\mathbf{f})"] \arrow[r, shift right, "\mathsf{Bool}(\mathbf{g})"'] & \mathsf{Bool}(X,\mathscr{A},\mathscr{N}) \arrow[u, "\Upsilon_X"]
\end{tikzcd}$$

We show that $\operatorname{Hom}(\mathbf{h},\mathbf{2})$ is injective. Indeed, if $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are such that $\operatorname{Hom}(\mathbf{h},\mathbf{2})(\boldsymbol{\varphi}) = \operatorname{Hom}(\mathbf{h},\mathbf{2})(\boldsymbol{\psi})$ then, upon letting $\mathbf{k} = \boldsymbol{\varphi}\circ\mathbf{h} = \boldsymbol{\psi}\circ\mathbf{h}$, we infer that $\mathbf{k}\circ\mathbf{f} = \mathbf{k}\circ\mathbf{g}$. By the universal property of $(Z,\mathscr{C},\mathscr{P})$, there exists a unique $\boldsymbol{\ell} \in \operatorname{Hom}\big((Z,\mathscr{C},\mathscr{P}),\mathbf{2}\big)$ such that $\boldsymbol{\ell}\circ\mathbf{h} = \mathbf{k}$. Since $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ have the property of $\boldsymbol{\ell}$, we conclude that they coincide. Similarly, the universal property of coequalizers tells us that the range of $\operatorname{Hom}(\mathbf{h},\mathbf{2})$ consists of those morphisms $\mathbf{k}$ such that $\operatorname{Hom}(\mathbf{f},\mathbf{2})(\mathbf{k}) = \operatorname{Hom}(\mathbf{g},\mathbf{2})(\mathbf{k})$. On the second line of the diagram, these two observations translate to the fact that $\mathsf{Bool}(\mathbf{h})$ induces an isomorphism of Boolean algebras from $\mathsf{Bool}(Z,\mathscr{C},\mathscr{P})$ onto the Boolean subalgebra

$$\mathfrak{A} := \mathsf{Bool}(Y,\mathscr{B},\mathscr{M}) \cap \big\{\xi : \mathsf{Bool}(\mathbf{f})(\xi) = \mathsf{Bool}(\mathbf{g})(\xi)\big\}.$$

It remains to prove that $\mathfrak{A}$ is Dedekind complete. Let $\mathfrak{E} \subseteq \mathfrak{A}$ be a collection. It has a supremum s in $\mathsf{Bool}(Y,\mathscr{B},\mathscr{M})$, as $(Y,\mathscr{B},\mathscr{M})$ is localizable. Since $\mathbf{f}$ and $\mathbf{g}$ are supremum preserving, we have

$$\mathsf{Bool}(\mathbf{f})(s) = \sup \mathsf{Bool}(\mathbf{f})(\mathfrak{E}) = \sup \mathsf{Bool}(\mathbf{g})(\mathfrak{E}) = \mathsf{Bool}(\mathbf{g})(s).$$

Hence $s \in \mathfrak{A}$ and $\mathfrak{A}$ is Dedekind complete.

(B) That $(Z,\mathscr{C},\mathscr{P})$ is localizable follows from (A). Let $\mathscr{G} \subseteq \mathscr{C}$ be any $\mathscr{P}$-generating family. We wish to prove that $\mathscr{C}_\mathscr{G} \subseteq \mathscr{C}$. If we manage to do so, then $(Z,\mathscr{C},\mathscr{P})$ is locally determined, as $\mathscr{G}$ is arbitrary.

Let h be a representative of $\mathbf{h}$. By definition, it is $[(\mathscr{B},\mathscr{M}),(\mathscr{C},\mathscr{P})]$-measurable and supremum preserving. We claim that it is, in fact, $[(\mathscr{B},\mathscr{M}),(\mathscr{C}_\mathscr{G},\mathscr{P}_\mathscr{G})]$-measurable. Indeed, let $C \in \mathscr{C}_\mathscr{G}$. For all $G \in \mathscr{G}$, we have $G\cap C \in \mathscr{C}$ which implies that $h^{-1}(G)\cap h^{-1}(C) = h^{-1}(G\cap C) \in \mathscr{B}$.

Moreover, $h^{-1}(\mathscr{G})$ is $\mathscr{M}$-generating, as $\mathscr{G}$ is $\mathscr{P}$-generating and h is supremum preserving. Thus, since $(Y,\mathscr{B},\mathscr{M})$ is locally determined, we have that $h^{-1}(C) \in \mathscr{B}$. Next, if $P \in \mathscr{P}_\mathscr{G}$, then $h^{-1}(P) \in \mathscr{B}$ by what precedes and $h^{-1}(P) \cap h^{-1}(G) = h^{-1}(P \cap G) \in \mathscr{M}$ for all $G \in \mathscr{G}$. By the Distributivity Lemma 3.4, we obtain $h^{-1}(P) \in \mathscr{M}$.

Denote as $\mathbf{h}' \colon (Y,\mathscr{B},\mathscr{M}) \to (Z,\mathscr{C}_\mathscr{G},\mathscr{P}_\mathscr{G})$ the morphism induced by h, and denote as $\boldsymbol{\iota} \colon (Z,\mathscr{C}_\mathscr{G},\mathscr{P}_\mathscr{G}) \to (Z,\mathscr{C},\mathscr{G})$ the morphism induced by the identity map id_Z. By Lemma 5.6, we have $\mathsf{Bool}(\mathbf{h}') = \mathsf{Bool}(\mathbf{h}) \circ \mathsf{Bool}(\boldsymbol{\iota})^{-1}$, which is the composition of two supremum preserving morphisms of Boolean algebras. Thus, $\mathbf{h}'$ is a supremum preserving as well. Also, we recall $\mathbf{h} \circ \mathbf{f} = \mathbf{h} \circ \mathbf{g}$. As $\mathbf{h}$ and $\mathbf{h}'$ are induced by the same map, we deduce that $\mathbf{h}' \circ \mathbf{f} = \mathbf{h}' \circ \mathbf{g}$. By the universal property of coequalizers, there is a morphism $\mathbf{k} \colon (Z,\mathscr{C},\mathscr{P}) \to (Z,\mathscr{C}_\mathscr{G},\mathscr{P}_\mathscr{G})$ such that $\mathbf{h}' = \mathbf{k} \circ \mathbf{h}$.

$$\begin{tikzcd} & & (Z,\mathscr{C}_\mathscr{G},\mathscr{P}_\mathscr{G}) \arrow[d, shift left, "\boldsymbol{\iota}"] \\ (X,\mathscr{A},\mathscr{N}) \arrow[r, shift left, "\mathbf{f}"] \arrow[r, shift right, "\mathbf{g}"'] & (Y,\mathscr{B},\mathscr{M}) \arrow[ur, "\mathbf{h}'"] \arrow[r, "\mathbf{h}"] & (Z,\mathscr{C},\mathscr{P}) \arrow[u, shift left, "\mathbf{k}"] \end{tikzcd}$$

Hence $\boldsymbol{\iota} \circ \mathbf{k} \circ \mathbf{h} = \boldsymbol{\iota} \circ \mathbf{h}' = \mathbf{h} = \mathrm{id}_{(Z,\mathscr{C},\mathscr{P})} \circ \mathbf{h}$. The uniqueness in the universal property of equalizers implies that $\mathbf{h}$ is an epimorphism. Thus $\boldsymbol{\iota} \circ \mathbf{k} = \mathrm{id}_{(Z,\mathscr{C},\mathscr{P})}$. A representative $k \in \mathbf{k}$ must satisfy $z = \mathrm{id}_Z(k(z)) = k(z)$ for $\mathscr{P}$-almost all $z \in Z$, i.e. $P = Z \cap \{z : z \neq k(z)\} \in \mathscr{P}$. Let $C \in \mathscr{C}_\mathscr{G}$. Since k is $(\mathscr{C},\mathscr{C}_\mathscr{G})$-measurable, it follows that $k^{-1}(C) \in \mathscr{C}$. Since $k^{-1}(C) \ominus C \subseteq P$, we deduce that $k^{-1}(C) \ominus C \in \mathscr{C}$ and, in turn, $C \in \mathscr{C}$. ∎

5.8. The last two results of this section show that the category $\mathsf{LLD}_{\mathsf{sp}}$ has better categorical properties than $\mathsf{MSN}_{\mathsf{sp}}$: it has equalizers in full generality. This result is reminiscent of [6, 214Ie].

Proposition 5.9. *Let $(X,\mathscr{A},\mathscr{N})$ be an lld MSN and $Y \subseteq X$ any subset. Then the subMSN $(Y,\mathscr{A}_Y,\mathscr{N}_Y)$ is lld and the canonical morphism $\boldsymbol{\iota}_Y \colon (Y,\mathscr{A}_Y,\mathscr{N}_Y) \to (X,\mathscr{A},\mathscr{N})$ is supremum preserving.*

Proof. First we show that the map $\iota_Y \colon Y \to X$ is supremum preserving. Let $\mathscr{E} \subseteq \mathscr{A}$ and assume $S \in \mathscr{A}$ is an $\mathscr{N}$-essential supremum of $\mathscr{E}$. The set $S \cap Y = \iota_Y^{-1}(S)$ is an $\mathscr{N}_Y$-essential upper bound of $\iota_Y^{-1}(\mathscr{E})$. Let $U \in \mathscr{A}_Y$ be an arbitrary $\mathscr{N}_Y$-essential upper bound of $\iota_Y^{-1}(\mathscr{E})$. We ought to show that $S \cap Y \setminus U \in \mathscr{N}_Y$. For all $E \in \mathscr{E}$, one has $E \cap S \cap Y \setminus U \subseteq E \cap Y \setminus U \in \mathscr{N}_Y$. As

$(X,\mathscr{A},\mathscr{N})$ is saturated, $\mathscr{N}_Y \subseteq \mathscr{N}$ and, also, $E \cap S \cap Y \setminus U \in \mathscr{N}$. Of course $(X \setminus S) \cap S \cap Y \setminus U = \emptyset$ is also $\mathscr{N}$-negligible. Since the family $\mathscr{E} \cup \{X \setminus S\}$ is $\mathscr{N}$-generating and $(X,\mathscr{A},\mathscr{N})$ is locally determined, we deduce that $S \cap Y \setminus U$ is $\mathscr{A}$-measurable and, in turn, that it is $\mathscr{N}$-negligible by the Distributivity Lemma 3.4. The proof that $\boldsymbol{\iota}_Y$ is supremum preserving is complete.

Since the morphism $\mathsf{Bool}(\boldsymbol{\iota}_Y)\colon \mathscr{A}/\mathscr{N} \to \mathscr{A}_Y/\mathscr{N}_Y$ is onto and supremum preserving, and $\mathscr{A}/\mathscr{N}$ is Dedekind complete, so is $\mathscr{A}_Y/\mathscr{N}_Y$, meaning that $(Y,\mathscr{A}_Y,\mathscr{N}_Y)$ is localizable. It remains to show that $(X,\mathscr{A},\mathscr{N})$ is locally determined.

We claim the following: *If $\mathscr{E} \subseteq \mathscr{A}_Y$ is $\mathscr{N}_Y$-generating and $N \in \mathscr{P}(Y)$ satisfies $E \cap N \in \mathscr{N}_Y$ for all $E \in \mathscr{E}$, then $N \in \mathscr{N}_Y$.* By definition of $\mathscr{A}_Y$, any set $E \in \mathscr{E}$ can be written as $E = E' \cap Y$, for some $E' \in \mathscr{A}$, so there is a subset $\mathscr{E}' \subseteq \mathscr{A}$ such that $\mathscr{E} = \iota_Y^{-1}(\mathscr{E}')$. The localizability of $(X,\mathscr{A},\mathscr{N})$ guarantees the existence of an $\mathscr{N}$-essential supremum S of $\mathscr{E}'$. For all $E' \in \mathscr{E}'$ one has $E' \cap N = E' \cap Y \cap N \in \mathscr{N}_Y \subseteq \mathscr{N}$, because $E' \cap Y \in \mathscr{E}$. Also $S \cap Y = \iota_Z^{-1}(S)$ is an $\mathscr{N}_Y$-essential supremum of $\mathscr{E} = \iota_Z^{-1}(\mathscr{E}')$, by the first paragraph. Recalling that $\mathscr{E}$ is $\mathscr{N}_Y$-generating, we find that $Y \setminus S = Y \setminus (S \cap Y) \in \mathscr{N}_Y$. Consequently, $(X \setminus S) \cap N \subseteq Y \setminus S \in \mathscr{N}_Y \subseteq \mathscr{N}$. As $(X,\mathscr{A},\mathscr{N})$ is saturated, we find that $(X \setminus S) \cap N \in \mathscr{N}$. In conclusion, $E' \cap N \in \mathscr{N}$ for any E' that belongs to the $\mathscr{N}$-generating family $\mathscr{E}' \cup \{X \setminus S\}$. Since $(X,\mathscr{A},\mathscr{N})$ is locally determined, we infer that $N \in \mathscr{A}$ and then that $N \in \mathscr{N}$ by the Distributivity Lemma 3.4. As $N \subseteq Y$, we conclude that $N \in \mathscr{N}_Y$.

Now let $\mathscr{E} \subseteq \mathscr{A}_Y$ be an $\mathscr{N}_Y$-generating collection and $A \in \mathscr{P}(Y)$ be such that $E \cap A \in \mathscr{A}_Y$ for all $E \in \mathscr{E}$. We want to prove that $A \in \mathscr{A}_Y$. As $(Y,\mathscr{A}_Y,\mathscr{N}_Y)$ is localizable, $\{E \cap A : E \in \mathscr{E}\}$ has an $\mathscr{N}_Y$-essential supremum S. This implies that $E \cap A \setminus S \in \mathscr{N}_Y$ for all $E \in \mathscr{E}$. By the claim above, $A \setminus S \in \mathscr{N}_Y$.

Fix $E_0 \in \mathscr{E}$. Note that $E_0 \cap (S \setminus A) = (E_0 \cap S) \setminus (E_0 \cap A) \in \mathscr{A}_Y$. Also,

$$\begin{aligned} E \cap A \setminus ((S \setminus (E_0 \cap S \setminus A)) &= E \cap A \cap ((Y \setminus S) \cup (E_0 \cap S \setminus A)) \\ &= E \cap A \setminus S \in \mathscr{N}_Y, \end{aligned}$$

for all $E \in \mathscr{E}$. In other words, $S \setminus (E_0 \cap S \setminus A)$ is an $\mathscr{N}_Y$-essential upper bound of $\{E \cap A : E \in \mathscr{E}\}$. As S is an $\mathscr{N}_Y$-essential supremum of this family, $S \setminus (S \setminus (E_0 \cap S \setminus A)) = E_0 \cap S \setminus A \in \mathscr{N}_Y$. Applying again the claim above, we deduce that $S \setminus A \in \mathscr{N}_Y$ from the arbitrariness of E_0. Summing up, $A \ominus S \in \mathscr{N}_Y$. As $S \in \mathscr{A}_Y$ we infer that $A \in \mathscr{A}_Y$. The proof that $(Y,\mathscr{A}_Y,\mathscr{N}_Y)$ is locally determined is now complete. ∎

Corollary 5.10. $\mathsf{LLD}_{\mathsf{sp}}$ *has equalizers preserved by the forgetful functor* $\mathsf{LLD}_{\mathsf{sp}} \to \mathsf{MSN}$.

Proof. Consider a pair of supremum preserving morphisms $\mathbf{f}, \mathbf{g} \colon (X, \mathscr{A}, \mathscr{N}) \to (Y, \mathscr{B}, \mathscr{M})$ in the category $\mathsf{LLD}_{\mathsf{sp}}$, represented by maps $f \in \mathbf{f}$ and $g \in \mathbf{g}$. Let $\mathbf{h} \colon (T, \mathscr{C}, \mathscr{P}) \to (X, \mathscr{A}, \mathscr{N})$ be another supremum preserving morphism in $\mathsf{LLD}_{\mathsf{sp}}$, such that $\mathbf{f} \circ \mathbf{h} = \mathbf{g} \circ \mathbf{h}$.

Set $Z = \{f = g\}$. We know since Proposition 2.11 that $\big((Z, \mathscr{A}_Z, \mathscr{N}_Z), \boldsymbol{\iota}_Z\big)$ is the equalizer of the pair $\mathbf{f}, \mathbf{g}$ in the category MSN, so there is a unique morphism $\mathbf{h}' \colon (T, \mathscr{C}, \mathscr{P}) \to (X, \mathscr{A}, \mathscr{N})$ such that $\mathbf{h} = \boldsymbol{\iota}_Z \circ \mathbf{h}'$. By the proposition 5.9, $\boldsymbol{\iota}_Z$ is supremum-preserving and $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is lld. It remains to prove that $\mathbf{h}'$ is supremum preserving. This follows from the fact that $\mathsf{Bool}(\mathbf{h}) = \mathsf{Bool}(\mathbf{h}') \circ \mathsf{Bool}(\boldsymbol{\iota}_Z)$, where $\mathsf{Bool}(\mathbf{h})$ is supremum preserving and $\mathsf{Bool}(\boldsymbol{\iota}_Z)$ is supremum preserving and surjective. ∎

6. Gluing measurable functions

Definition 6.1. Let $(X, \mathscr{A}, \mathscr{N})$ be an MSN and $(Y, \mathscr{B})$ a measurable space. Let $\mathscr{E} \subseteq \mathscr{A}$ be a collection. A *family subordinated to* $\mathscr{E}$ is a family of functions $\langle f_E \rangle_{E \in \mathscr{E}}$ such that:

(1) $f_E \colon E \to Y$ is $(\mathscr{A}_E, \mathscr{B})$-measurable for every $E \in \mathscr{E}$.

We further say that $\langle f_E \rangle_{E \in \mathscr{E}}$ is *compatible* whenever

(2) for all pairs $E, E' \in \mathscr{E}$ one has $E \cap E' \cap \{f_E \neq f_{E'}\} \in \mathscr{N}$.

A *gluing* of a compatible family $\langle f_E \rangle_{E \in \mathscr{E}}$ subordinated to $\mathscr{E}$ is a function $f \colon X \to Y$ such that:

(3) f is $(\mathscr{A}, \mathscr{B})$-measurable;

(4) $E \cap \{f \neq f_E\} \in \mathscr{N}$ for every $E \in \mathscr{E}$.

In this section, we will be mainly concerned about the existence of gluings, as they will be of use in the construction of the 4c version of a locally ccc MSN in Section 7. This turns out to depend both on the domain and the target space. In case where $(Y, \mathscr{B})$ is the real line equipped with its Borel σ-algebra $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, we can glue measurable functions together if $(X, \mathscr{A}, \mathscr{N})$ is localizable. In fact, this important property is a characterization of localizability. The interested reader may find a proof of this classical result expressed in the language of MSNs in [2, Proposition 3.13]. Only the measurable structure of $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is involved, thus, the result holds in the more general case where

$(Y, \mathscr{B})$ is a standard Borel space, see [17, Chapter 3].

Many questions arise when we remove the condition that $(Y, \mathscr{B})$ is a standard Borel space. In this case, we need some additional assumptions on $(X, \mathscr{A}, \mathscr{N})$. We will focus on two cases: $(X, \mathscr{A}, \mathscr{N})$ is 4c or lld. But first, we prove that a gluing inherits some of the properties of the functions f_E.

LEMMA 6.2. *Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN, $(Y, \mathscr{B}, \mathscr{M})$ an MSN, and $\mathscr{E} \subseteq \mathscr{A}$ an $\mathscr{N}$-generating collection. We let $\langle f_E \rangle_{E \in \mathscr{E}}$ be a compatible family of functions subordinated to $\mathscr{E}$ and we assume that:*

(1) *for every $E \in \mathscr{E}$, the map f_E is $[(\mathscr{A}_E, \mathscr{N}_E), (\mathscr{B}, \mathscr{M})]$-measurable;*

(2) *the family $\langle f_E \rangle_{E \in \mathscr{E}}$ has a gluing f.*

Then:

(A) *the gluing f is $[(\mathscr{A}, \mathscr{N}), (\mathscr{B}, \mathscr{M})]$-measurable;*

(B) *if f_E is supremum preserving, for every $E \in \mathscr{E}$, then so is f.*

Proof. We start with the following easy observation. For each $E \in \mathscr{E}$ and $B \in \mathscr{B}$ one has $f_E^{-1}(B) \ominus (E \cap f^{-1}(B)) \subseteq E \cap \{f_E \neq f\} \in \mathscr{N}$.

(A) As the gluing f is $(\mathscr{A}, \mathscr{B})$-measurable by definition, we need only show that $f^{-1}(M) \in \mathscr{N}$ for $M \in \mathscr{M}$. Since $f_E^{-1}(M)$ is $\mathscr{N}$-negligible, the above observation applied with $B = M$ ensures that $E \cap f^{-1}(M) \in \mathscr{N}$ for any $E \in \mathscr{E}$. We next use Lemma 3.4 to assert that $f^{-1}(M)$ is an $\mathscr{N}$-essential supremum of $\{E \cap f^{-1}(M) : E \in \mathscr{E}\}$. This forces $f^{-1}(M)$ to be $\mathscr{N}$-negligible.

(B) Let $\mathscr{F} \subseteq \mathscr{B}$ be a collection that admits an $\mathscr{M}$-essential supremum S. Since f_E is supremum preserving for every $E \in \mathscr{E}$, $f_E^{-1}(S)$ is an $\mathscr{N}$-essential supremum of $\{f_E^{-1}(F) : F \in \mathscr{F}\}$ and it ensues from the observation above, applied with $B \in \{S\} \cup \mathscr{F}$, that $E \cap f^{-1}(S)$ is an $\mathscr{N}$-essential supremum of $\{E \cap f^{-1}(F) : F \in \mathscr{F}\}$. Therefore,

$$f^{-1}(S) = \mathscr{N}\text{-}\operatorname{ess\,sup}_{E \in \mathscr{E}} E \cap f^{-1}(S)$$

(by Lemma 3.4)

$$= \mathscr{N}\text{-}\operatorname{ess\,sup}_{E \in \mathscr{E}} \left(\mathscr{N}\text{-}\operatorname{ess\,sup}_{F \in \mathscr{F}} E \cap f^{-1}(F) \right)$$

(from what precedes)

$$= \mathscr{N}\text{-}\operatorname{ess\,sup}_{F \in \mathscr{F}} \left(\mathscr{N}\text{-}\operatorname{ess\,sup}_{E \in \mathscr{E}} E \cap f^{-1}(F) \right)$$
$$= \mathscr{N}\text{-}\operatorname{ess\,sup}_{F \in \mathscr{F}} f^{-1}(F)$$

(by Lemma 3.4). ∎

PROPOSITION 6.3. *Let $(X, \mathscr{A}, \mathscr{N})$ be a locally determined MSN and $(Y,\mathscr{B})$ be any nonempty measurable space. Let $\mathscr{E} \subseteq \mathscr{A}$ be an $\mathscr{N}$-generating collection. If a compatible family $\langle f_E \rangle_{E\in\mathscr{E}}$ has a gluing, then it is unique up to equality almost everywhere.*

Proof. Let $f, g\colon X \to Y$ be two gluings of $\langle f_E \rangle_{E\in\mathscr{E}}$. We warn the reader that the measurability of $\{f \neq g\}$ is not immediate, since the diagonal $\{(y, y) : y \in Y\}$ may not be measurable in $(Y^2, \mathscr{B} \otimes \mathscr{B})$. Notwithstanding, for all $E \in \mathscr{E}$, we have $E \cap \{f \neq g\} \subseteq (E \cap \{f \neq f_E\}) \cup (E \cap \{g \neq f_E\})$. Since f, g are gluings and $(X, \mathscr{A}, \mathscr{N})$ is saturated, it follows that $E \cap \{f \neq g\} \in \mathscr{N}$. This happens for any E in the $\mathscr{N}$-generating set $\mathscr{E}$. By local determination and the Distributivity Lemma 3.4, $\{f \neq g\} \in \mathscr{N}$. ∎

PROPOSITION 6.4. *Let $(X, \mathscr{A}, \mathscr{N})$ be a 4c MSN and $(Y, \mathscr{B})$ be any nonempty measurable space. Let $\mathscr{E} \subseteq \mathscr{A}$ be an $\mathscr{N}$-generating collection. Any compatible family $\langle f_E \rangle_{E\in\mathscr{E}}$ subordinated to $\mathscr{E}$ admits a unique gluing f up to equality $\mathscr{N}$-almost everywhere.*

Proof. First observe that the uniqueness of the gluing up to almost everywhere equality follows from Proposition 6.3, as a 4c MSN is locally determined by Proposition 5.3(E).

Let us treat the special case where $(X, \mathscr{A}, \mathscr{N})$ is a saturated ccc MSN. According to Proposition 4.4, we can find a sequence of sets $\langle E_i \rangle_{i\in\mathbb{N}}$ in $\mathscr{E}$ such that $\bigcup_{i\in\mathbb{N}} E_i$ provides an $\mathscr{N}$-essential supremum of $\mathscr{E}$. We then define the $(\mathscr{A}, \mathscr{B})$-measurable map $f\colon X \to Y$ which, for all $i \in \mathbb{N}$, coincides with f_i on the set $E_i \setminus \bigcup_{j<i} E_j$, and maps the negligible set $N := X \setminus \bigcup_{i\in\mathbb{N}} E_i$ to some arbitrary point. Let $E \in \mathscr{E}$. For every $i \in \mathbb{N}$, we have

$$N_i := E \cap E_i \cap \{f_E \neq f_{E_i}\} \in \mathscr{N},$$

by hypothesis. Thus, $E \cap \{f \neq f_E\} \subseteq N \cup \bigcup_{i\in\mathbb{N}} N_i$ is negligible.

Suppose now that $(X, \mathscr{A}, \mathscr{N})$ is a coproduct $\coprod_{i\in I}(X_i, \mathscr{A}_i, \mathscr{N}_i)$ of saturated ccc MSNs. For each $E \in \mathscr{E}$ and $i \in I$, call $f_{E,i}$ the restriction of f_E to $E \cap X_i$. By Lemma 3.4, X_i is an $\mathscr{N}_i$-essential supremum of the collection $\{E \cap X_i : E \in \mathscr{E}\}$. Also, $\langle f_{E,i} \rangle_{E\in\mathscr{E}}$ is a compatible family of measurable maps subordinated to $\langle E \cap X_i \rangle_{E\in\mathscr{E}}$. From what precedes, it admits a gluing $f_i\colon X_i \to Y$. Define $f := \coprod_{i\in I} f_i$. The verification that f is a gluing of $\langle f_E \rangle_{E\in\mathscr{E}}$ is routine. ∎

6.5. It would be interesting to know whether 4c MSNs are the only lld MSNs such that gluings can always be performed, with no restriction on the target space $(Y, \mathscr{B})$. This property will be used in the next section and justifies the special role played by 4c MSN.

DEFINITION 6.6. (COUNTABLY SEPARATED MEASURABLE SPACES) One measurable space $(Y, \mathscr{B})$ is called *countably separated* whenever there is a countable set $\mathscr{C} \subseteq \mathscr{B}$ such that for any distinct $y_1, y_2 \in Y$ there exists $C \in \mathscr{C}$ such that $y_1 \in C \not\ni y_2$ or $y_1 \notin C \ni y_2$. Here is a well-known characterization of countably separated spaces.

PROPOSITION 6.7. *Let $(Y, \mathscr{B})$ be a measurable space. The following statements are equivalent:*

(A) *$(Y, \mathscr{B})$ is countably separated;*

(B) *there is an injective measurable map $(Y, \mathscr{B}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$;*

(C) *there is an injective measurable map $(Y, \mathscr{B}) \to (X, \mathscr{B}(X))$ to a Polish space X.*

Proof. (A) $\Rightarrow$ (B) Let $\mathscr{C} \subseteq \mathscr{B}$ be a countable set that separates the points of Y. Let $\langle C_n \rangle_{n \in \mathbb{N}}$ be a enumeration of $\mathscr{C}$ and $h \colon Y \to \mathbb{R}$ be the map $h = \sum_{n=0}^{\infty} 3^{-n} \mathbb{1}_{C_n}$. The sets C_n are measurable, therefore h is measurable. Let y_1, y_2 be distinct points in Y and $n_0 := \min\{n \in \mathbb{N} : \mathbb{1}_{C_n}(y_1) \neq \mathbb{1}_{C_n}(y_2)\}$. Then

$$|h(y_1) - h(y_2)| \geqslant 3^{-n_0} - \sum_{n=n_0+1}^{\infty} 3^{-n} |\mathbb{1}_{C_n}(y_1) - \mathbb{1}_{C_n}(y_2)| \geqslant 3^{-n_0} - \frac{3^{-n_0}}{2} > 0$$

thus $h(y_1) \neq h(y_2)$, which shows that f is injective.

(B) $\Rightarrow$ (C) is obvious.

(C) $\Rightarrow$ (A) Let $\mathscr{U}$ be a countable basis for the topology of X. If there is an injective measurable map $h \colon (Y, \mathscr{B}) \to (X, \mathscr{B}(X))$, then $h^{-1}(\mathscr{U}) \subseteq \mathscr{B}$ is a countable set that separates points. ∎

PROPOSITION 6.8. *Let $\langle (Y_i, \mathscr{B}_i) \rangle_{i \in I}$ be a family of countably separated measurable spaces. If $\operatorname{card} I \leqslant \mathfrak{c}$, then $\coprod_{i \in I} (Y_i, \mathscr{B}_i)$ is countably separated.*

Proof. For each $i \in I$, there is an injective $(\mathscr{B}_i, \mathscr{B}(\mathbb{R}))$-measurable map $h_i \colon Y_i \to \mathbb{R}$ by Proposition 6.7. Choose an arbitrary injective map $g \colon I \to \mathbb{R}$.

Let $h\colon \coprod_{i\in I} Y_i \to \mathbb{R}^2$ be the map defined by $h(y_i) := (h_i(y_i), g(i))$ for all $i \in I$ and $y_i \in Y_i$. Let $B \subseteq \mathbb{R}^2$ be a Borel set. Then, for any $i \in I$, we have

$$h^{-1}(B) \cap Y_i = h_i^{-1}(\mathbb{R} \cap \{x : (x, g(i)) \in B\}) = h_i^{-1}(B^{g(i)}).$$

This last set is $\mathscr{B}_i$-measurable as h_i is measurable and the horizontal section $B^{g(i)}$ is Borel. As i is arbitrary, we conclude that $h^{-1}(B)$ is measurable. This means that h is measurable. By Proposition 6.7, it follows that $\coprod_{i\in I}(Y_i, \mathscr{B}_i)$ is countably separated. ∎

Remark 6.9. The restriction on the cardinal of I is necessary, since a countably measurable space must have cardinal less or equal than $\mathfrak{c}$ by Proposition 6.7(B).

PROPOSITION 6.10. *Let $(X, \mathscr{A}, \mathscr{N})$ be an lld MSN and $(Y, \mathscr{B})$ be a non-empty countably separated measurable space. Let $\mathscr{E} \subseteq \mathscr{A}$ be $\mathscr{N}$-generating. Any compatible family $\langle f_E \rangle_{E\in\mathscr{E}}$ subordinated to $\mathscr{E}$ admits a gluing, unique up to equality almost everywhere.*

Proof. Let h be a measurable injective map $(Y, \mathscr{B}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$, whose existence follows from Proposition 6.7. Now, $\langle h \circ f_E \rangle_{E\in\mathscr{E}}$ is still a compatible family of measurable functions, this time with values in $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. As $(X, \mathscr{A}, \mathscr{N})$ is localizable, it admits a gluing $g\colon X \to \mathbb{R}$. For all $E \in \mathscr{E}$, one has $E \cap g^{-1}(\mathbb{R} \setminus h(Y)) \subseteq E \cap \{g \neq h \circ f_E\}$. Therefore $E \cap g^{-1}(\mathbb{R} \setminus h(Y))$ is negligible. This holds for any E in the $\mathscr{N}$-generating set $\mathscr{E}$. By local determination and Lemma 3.4, we deduce that $g^{-1}(\mathbb{R} \setminus h(Y)) \in \mathscr{N}$. Thus, we lose no generality in supposing, from now on, that g takes values in $h(Y)$. Define $f := h^{-1} \circ g$. We claim that f is a gluing. For $E \in \mathscr{E}$, we observe that $E \cap \{f \neq f_E\} \subseteq E \cap \{g \neq h \circ f_E\} \in \mathscr{N}$, since h is injective. Therefore, condition (4) of 6.1 is satisfied.

Also, let $B \in \mathscr{B}$, then $(E \cap f^{-1}(B)) \ominus f_E^{-1}(B) \subseteq E \cap \{f \neq f_E\} \in \mathscr{N}$. Since f_E is measurable, we have $f_E^{-1}(B) \in \mathscr{A}$ and, in turn, $E \cap f^{-1}(B) \in \mathscr{A}$. Since E is arbitrary, we deduce that $f^{-1}(B) \in \mathscr{A}$, by local determination, showing that f is measurable. Of course, the uniqueness of the gluing is given by Proposition 6.3. ∎

6.11. In this paragraph, we exhibit an lld MSN $(X, \mathscr{A}, \mathscr{N})$, a measurable space $(Y, \mathscr{B})$ and, within this setting, a compatible family of measurable maps that cannot be glued. With regards to Proposition 6.4, it is natural to turn towards Fremlin's example in [6, §216E] of a localizable, locally determined but not strictly localizable[1] measure space $(X, \mathscr{A}, \mu)$. Let us recall its construction. Fix a set Y with cardinal greater than $\mathfrak{c}$ and we set $X := \{0,1\}^{\mathscr{P}(Y)}$. For any $y \in Y$, we define $x_y \in X$ by

$$\forall Z \in \mathscr{P}(Y), \qquad x_y(Z) = \begin{cases} 1 & \text{if } y \in Z, \\ 0 & \text{if } y \notin Z. \end{cases}$$

Let $\mathscr{K} \subseteq \mathscr{P}(\mathscr{P}(Y))$ be the family of countable subsets of $\mathscr{P}(Y)$. For any $K \in \mathscr{K}$ and $y \in Y$, we define $F_{y,K} := X \cap \{x : x(Z) = x_y(Z) \text{ for all } Z \in K\}$. Then we define, for all $y \in Y$,

$$\mathscr{A}_y = \mathscr{P}(X) \cap \big\{A : \text{there is } K \in \mathscr{K} \text{ such that } F_{y,K} \subseteq A \text{ or } F_{y,K} \subseteq X \setminus A\big\}.$$

Let us prove that $\mathscr{A}_y$ is a σ-algebra. Clearly $\emptyset \in \mathscr{A}_y$ and $\mathscr{A}_y$ is closed under complementations. Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathscr{A}_y$. Suppose there is some $n_0 \in \mathbb{N}$ and $K \in \mathscr{K}$ such that $F_{y,K} \subseteq A_{n_0}$. Then $F_{y,K} \subseteq \bigcup_{n \in \mathbb{N}} A_n$, which implies $\bigcup_{n \in \mathbb{N}} A_n \in \mathscr{A}_y$.

Suppose on the contrary that for all $n \in \mathbb{N}$, there is $K_n \in \mathscr{K}$ such that $F_{y,K_n} \subseteq X \setminus A_n$. Then $\bigcap_{n \in \mathbb{N}} F_{y,K_n} = F_{y, \bigcup_{n \in \mathbb{N}} K_n} \subseteq X \setminus \bigcup_{n \in \mathbb{N}} A_n$ which also gives that $\bigcup_{n \in \mathbb{N}} A_n \in \mathscr{A}_y$.

Finally set $\mathscr{A} := \bigcap_{y \in Y} \mathscr{A}_y$ and define the measure $\mu \colon \mathscr{A} \to [0, \infty]$ by

$$\forall A \in \mathscr{A}, \qquad \mu(A) = \operatorname{card}\big(Y \cap \{y : x_y \in A\}\big).$$

For the rest of the discussion, we admit that $(X, \mathscr{A}, \mu)$ is complete, localizable, locally determined and not strictly localizable. The proof of the latter relies on a non trivial fact in infinitary combinatorics; we refer to [6, 216E(f)(g)] for more details. The associated MSN $(X, \mathscr{A}, \mathscr{N}_\mu)$ is saturated, localizable, and it is locally determined, by Proposition 5.3(E).

Define $E_y = X \cap \{x : x(\{y\}) = 1\}$ for all $y \in Y$. This set is $\mathscr{A}$ measurable, because $F_{y,\{\{y\}\}} = E_y$ (hence $E_y \in \mathscr{A}_y$), and for any $z \in Y \setminus \{y\}$, we have $F_{z,\{\{y\}\}} = X \setminus E_y$ (hence $E_y \in \mathscr{A}_z$). Note that $Y \cap \{z : x_z \in E_y\} = \{y\}$.

We now choose $\mathscr{B}$ to be the countable cocountable σ-algebra of Y. For any $y \in Y$, we define the measurable map $f_y \colon (E_y, \mathscr{A}_{E_y}) \to (Y, \mathscr{B})$ that is constant equal to y. We claim that $\langle f_y \rangle_{y \in Y}$ is a compatible family of measurable maps subordinated to $\langle E_y \rangle_{y \in Y}$. This ensues from the fact that $E_y \cap E_z \in \mathscr{N}_\mu$ for

[1] In the context of measure spaces, we follow the terminology of [6]: $(X, \mathscr{A}, \mu)$ is strictly localizable whenever there is a measurable partition $\langle X_i \rangle_{i \in I}$ such that a set $A \subseteq X$ is measurable whenever the sets $A \cap X_i$ are, and in that case $\mu(A) = \sum_{i \in I} \mu(A \cap X_i)$.

any distinct $y, z \in Y$. Assume by contradiction that we can find a gluing $f : X \to Y$. We will use the decomposition $\langle f^{-1}(\{y\})\rangle_{y \in Y}$ to show $(X, \mathscr{A}, \mu)$ is strictly localizable.

Let $A \subseteq X$ such that $A \cap f^{-1}(\{y\}) \in \mathscr{A}$ for all y. We want to show that $A \in \mathscr{A}$. For $y \in Y$, we have

- Case $x_y \in A$: as $A \cap f^{-1}(\{y\}) \in \mathscr{A}_y$ and $x_y \in A \cap f^{-1}(\{y\})$, there is $K \in \mathscr{K}$ such that $F_{y,K} \subseteq A \cap f^{-1}(\{y\}) \subseteq A$ (because $x_y \in F_{y,K}$, this is the branch of the dichotomy, in the definition of $\mathscr{A}_y$, that occurs). Therefore $A \in \mathscr{A}_y$.
- Case $x_y \notin A$: since $x_y \notin A \cap f^{-1}(\{y\}) \in \mathscr{A}_y$, we can find $K \in \mathscr{K}$ such that $F_{y,K} \subseteq X \setminus \big(A \cap f^{-1}(\{y\})\big)$. We deduce that $F_{y,K} \cap f^{-1}(\{y\}) \subseteq X \setminus A$. But $x_y \in f^{-1}(\{y\}) \in \mathscr{A}_y$, so there is $K' \in \mathscr{K}$ such that $F_{y,K'} \subseteq f^{-1}(\{y\})$. Whence $F_{y,K \cup K'} = F_{y,K} \cap F_{y,K'} \subseteq F_{y,K} \cap f^{-1}(\{y\}) \subseteq X \setminus A$. It follows that $A \in \mathscr{A}_y$.

In any case, we have shown that $A \in \mathscr{A}_y$. As $y \in Y$ is arbitrary, $A \in \mathscr{A}$.

Now, one observes that the only $z \in Y$ such that $x_z \in f^{-1}(\{y\})$ is y. Therefore $\mu(A \cap f^{-1}(\{y\}))$ equals 1 if $x_y \in A$ and 0 otherwise. In consequence, we have $\mu(A) = \sum_{y \in Y} \mu(A \cap f^{-1}(\{y\}))$ as desired.

7. Existence of 4c and lld versions

THEOREM 7.1. *Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN and $\mathscr{E} \subseteq \mathscr{A} \setminus \mathscr{N}$. We suppose that*

(1) *$(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is 4c for every $Z \in \mathscr{E}$;*

(2) *$\mathscr{E}$ is almost disjointed;*

(3) *$\mathscr{E}$ is $\mathscr{N}$-generating.*

Then the pair consisting of the MSN

$$\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big) = \coprod_{Z \in \mathscr{E}} (Z, \mathscr{A}_Z, \mathscr{N}_Z)$$

and the morphism $\mathbf{p} = \coprod_{Z \in \mathscr{E}} \boldsymbol{\iota}_Z$ is the 4c version of $(X, \mathscr{A}, \mathscr{N})$ (as usual $\boldsymbol{\iota}_Z$ is the morphism induced by the inclusion map $\iota_Z : Z \to X$).

Proof. The MSN $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$ is 4c as a coproduct of 4c MSNs (this is a general fact, in any category, a coproduct of coproducts is a coproduct, see [1, Proposition 2.2.3]), and $\mathbf{p}$ is supremum preserving, according to 3.7(B) and (D). Observe that each $Z \in \mathscr{E}$ is also a subset of $\hat{X}$ and we denote by $\hat{\iota}_Z \colon Z \to \hat{X}$ the corresponding inclusion map.

Let $(Y, \mathscr{B}, \mathscr{M})$ be a 4c MSN and $\mathbf{q} \colon (Y, \mathscr{B}, \mathscr{M}) \to (X, \mathscr{A}, \mathscr{N})$ be a supremum preserving morphism, represented by $q \in \mathbf{q}$. For all $Z \in \mathscr{E}$, call $q_Z := \hat{\iota}_Z \circ (q \llcorner q^{-1}(Z)) \colon q^{-1}(Z) \to \hat{X}$. Because $\mathbf{q}$ is supremum preserving, $Y = q^{-1}(X)$ is an $\mathscr{M}$-essential supremum of the family $\langle q^{-1}(Z)\rangle_{Z \in \mathscr{E}}$. The family $\mathscr{E}$ being almost disjointed and q being measurable, $q^{-1}(Z) \cap q^{-1}(Z') = q^{-1}(Z \cap Z') \in \mathscr{M}$ for any distinct $Z, Z' \in \mathscr{E}$. As a result, the family $\langle q_Z \rangle_{Z \in \mathscr{E}}$ subordinated to $\langle q^{-1}(Z)\rangle_{Z \in \mathscr{E}}$ is compatible. By Proposition 6.4, this family has a gluing $r \colon Y \to \hat{X}$ and, by Lemma 6.2, r is $[(\mathscr{B}, \mathscr{M}), (\hat{\mathscr{A}}, \hat{\mathscr{N}})]$-measurable and supremum preserving. Call $p := \coprod_{Z \in \mathscr{E}} \iota_Z$. For each $Z \in \mathscr{E}$, we have

$$q^{-1}(Z) \cap \{p \circ r \neq q\} \subseteq q^{-1}(Z) \cap \{r \neq \hat{\iota}_Z \circ (q \llcorner q^{-1}(Z))\} \in \mathscr{M}.$$

The family $\{q^{-1}(Z) : Z \in \mathscr{E}\}$ is $\mathscr{M}$-generating and $(Y, \mathscr{B}, \mathscr{M})$ is locally determined, so we conclude that $\{p \circ r \neq q\} \in \mathscr{M}$, that is, $\mathbf{p} \circ \mathbf{r} = \mathbf{q}$.

As for uniqueness, let $\mathbf{r}$ be any morphism such that $\mathbf{p} \circ \mathbf{r} = \mathbf{q}$, and call $r \in \mathbf{r}$ one of its representatives. Fix $Z \in \mathscr{E}$. Observe that $\hat{\iota}_Z(p(z)) = z$ for all $z \in Z$. For $\mathscr{M}$-almost every $x \in q^{-1}(Z)$, we have $p(r(x)) = q(x)$ which implies that $r(x) \in Z \subseteq \hat{X}$. For such an x, we find that

$$r(x) = \hat{\iota}_Z(p(r(x))) = \hat{\iota}_Z(q(x)) = q_Z(x).$$

Hence, r is a gluing of the compatible family $\langle q_Z \rangle_{Z \in \mathscr{E}}$ and we invoke the uniqueness part of Proposition 6.4 to conclude. ∎

7.2. Consider the following example, taken from [6, 216D]. Let X be a set of cardinality greater or equal than $\aleph_2$. For each $x, y \in X$, we define $H_y = X \times \{y\}$ and $V_x = \{x\} \times X$. Sets of this form are respectively called horizontal and vertical lines. We define a σ-algebra $\mathscr{A}$ of X^2 by declaring that $A \in \mathscr{A}$ iff for all $x, y \in X$, the trace $A \cap H_y$ (resp. $A \cap V_x$) is either countable or cocountable in H_y (resp. V_x). Also, we define the σ-ideal $\mathscr{N}$ of $\mathscr{A}$ as follows: $N \in \mathscr{N}$ if and only if the intersection of N with any line is countable. Clearly, $(X^2, \mathscr{A}, \mathscr{N})$ is saturated.

We assert that it is not localizable. Suppose if possible that the family of horizontal lines $\{H_y : y \in X\}$ has an $\mathscr{N}$-essential supremum S. Then for all $y \in X$, the intersection $S \cap H_y$ is cocountable in H_y, that is, $N_y := X \cap \{x : (x, y) \notin S\}$ is countable. Let Z be a subset of X of cardinality $\aleph_1$. Then $\operatorname{card} \bigcup_{y \in Z} N_y \leqslant \aleph_1$, hence the existence of $x \in X \setminus \bigcup_{y \in Z} N_y$.

This implies that $V_x \cap S$ is not countable, so it is cocountable in V_x. However, $S \setminus V_x$ is easily checked to be an $\mathscr{N}$-essential upper bound of $\{H_y : y \in X\}$, as $H_y \cap V_x$ is negligible for all y. Since $V_x \cap S = S \setminus (S \setminus V_x) \notin \mathscr{N}$, we get a contradiction.

The family of all lines $\mathscr{E} := \{H_y : y \in X\} \cup \{V_x : x \in X\}$ satisfies the three hypotheses of Theorem 7.1. Applying the theorem, we see that the 4c version of $(X^2, \mathscr{A}, \mathscr{N})$ can be described as the coproduct of all lines. Doing so, we see that each point (x, y) in the base MSN $(X^2, \mathscr{A}, \mathscr{N})$ is duplicated in the 4c version: the "fibers" $p^{-1}(\{(x, y)\})$ contains two elements, which represent the horizontal and vertical directions emanating from the point (x, y).

If a given MSN has no obvious choice of a family satisfying the conditions of Theorem 7.1, we can justify the existence of a 4c version in a non constructive way.

Lemma 7.3. *Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN and $\mathscr{C} \subseteq A$ an $\mathscr{N}$-generating collection such that $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is ccc for all $Z \in \mathscr{C}$. Then we can find a collection $\mathscr{E} \subseteq \mathscr{A} \setminus \mathscr{N}$ that satisfies conditions* (1), (2) *and* (3) *of Theorem* 7.1 *and such that each of its members is a subset of some member of $\mathscr{C}$. Moreover, we can suppose* $\operatorname{card} \mathscr{E} \leqslant \max\{\aleph_0, \operatorname{card} \mathscr{C}\}$.

Proof. Let $\mathscr{E}$ be associated with $\mathscr{C}$ in Lemma 4.9. It clearly satisfies conditions (1), (2), and (3) of 7.1, since a subMSN of a ccc MSN is ccc as well. If $\mathscr{C}$ is infinite, then for all $Z \in \mathscr{C}$, call $\mathscr{E}_Z := \mathscr{E} \cap \{Z' : Z' \subseteq Z\}$. Then, each $\mathscr{E}_Z$ is at most countable, since it is an almost disjointed family in the ccc MSN $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$. As $\mathscr{E} = \bigcup_{Z \in \mathscr{C}} \mathscr{E}_Z$, we conclude that $\operatorname{card} \mathscr{E} \leqslant \operatorname{card} \mathscr{C}$. ∎

Corollary 7.4. *Every saturated locally ccc MSN admits a 4c version.*

Proof. Apply Lemma 7.3 to the family $\mathscr{C} := \mathscr{A} \cap \{Z : (Z, \mathscr{A}_Z, \mathscr{N}_Z) \text{ is ccc}\}$ and then Theorem 7.1. ∎

7.5. It is worth noticing that all the arguments contained in Theorem 7.1 and Corollary 7.4 remain valid provided we replace "ccc" by "strictly localizable", "locally ccc" by "locally strictly localizable", and "4c" by "strictly localizable". Summing up, a saturated locally strictly localizable MSN $(X, \mathscr{A}, \mathscr{N})$ has a strictly localizable version, which is constructed as the coproduct of subMSNs whose underlying sets belongs to a family $\mathscr{E}$ that satisfies hypotheses (2), (3) of Theorem 7.1 and

(1') $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is strictly localizable for every $Z \in \mathscr{C}$.

Since (1') implies (1) we can apply Theorem 7.1 again to conclude that the 4c and strictly localizable versions of $(X, \mathscr{A}, \mathscr{N})$ are the same.

As for the existence of lld versions, we have a partial result, which applies for most locally ccc MSNs that one is likely to encounter in analysis.

THEOREM 7.6. *Let* $(X, \mathscr{A}, \mathscr{N})$ *be a saturated MSN with a collection* $\mathscr{C} \subseteq \mathscr{A}$ *such that*

(1) $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ *is ccc for all* $Z \in \mathscr{C}$*;*

(2) $\mathscr{C}$ *is* $\mathscr{N}$*-generating;*

(3) $\operatorname{card} \mathscr{C} \leqslant \mathfrak{c}$.

The following hold:

(A) *If* $(X, \mathscr{A}, \mathscr{N})$ *has an lld version, then it coincides with the 4c version.*

(B) *If moreover* $(Z, \mathscr{A}_Z)$ *is countably separated for all* $Z \in \mathscr{C}$*, then the lld version exists.*

Proof. (A) Recall $(X, \mathscr{A}, \mathscr{N})$ has a 4c version, according to Corollary 7.4. Suppose $(X, \mathscr{A}, \mathscr{N})$ has an lld version $\big[(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}), \mathbf{p}\big]$. In view of Proposition 5.3(E), conclusion (A) will be established if we prove that $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$ is 4c. To this end, we need to find a suitable decomposition in $\hat{X}$. Apply Lemma 7.3 to get an almost disjointed $\mathscr{N}$-generating family $\mathscr{E}$ such that $\operatorname{card} \mathscr{E} \leqslant \mathfrak{c}$ and $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is ccc for all $Z \in \mathscr{E}$. Choose an injection $c : \mathscr{E} \to \mathbb{R} : Z \mapsto c_Z$ and $p \in \mathbf{p}$. Let $f_Z : p^{-1}(Z) \to \mathbb{R}$ be the constant map equal to c_Z; it is readily $\big(\hat{\mathscr{A}}_{p^{-1}(Z)}, \mathscr{B}(\mathbb{R})\big)$-measurable. The family $\langle f_Z \rangle_{Z \in \mathscr{E}}$ is obviously compatible, since $\mathscr{E}$ is almost disjointed. As $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$ is localizable, this family has an $(\hat{\mathscr{A}}, \mathscr{B}(\mathbb{R}))$-measurable gluing $f : \hat{X} \to \mathbb{R}$.

We now show that $\langle f^{-1}\{c_Z\} \rangle_{Z \in \mathscr{E}}$ is a partition of $\hat{X}$ into ccc measurable pieces. Since c is injective, the family $\langle f^{-1}\{c_Z\} \rangle_{Z \in \mathscr{E}}$ is, indeed, a partition of $\hat{X}$ and, since f is $(\hat{\mathscr{A}}, \mathscr{B}(\mathbb{R}))$-measurable, $f^{-1}\{c_Z\} \in \hat{\mathscr{A}}$ for all $Z \in \mathscr{E}$. As f is a gluing of $\langle f_Z \rangle_{Z \in \mathscr{E}}$, we have

$$p^{-1}(Z) \setminus f^{-1}\{c_Z\} = p^{-1}(Z) \cap \{f \neq f_Z\} \in \hat{\mathscr{N}} \qquad \text{for all } Z \in \mathscr{E}.$$

Moreover, since c is injective, for all $Z' \in \mathscr{E}$ distinct from Z, one has

$$p^{-1}(Z') \cap (f^{-1}\{c_Z\} \setminus p^{-1}(Z)) \subseteq p^{-1}(Z') \cap \{f \neq f_{Z'}\} \in \hat{\mathscr{N}}.$$

Also, $p^{-1}(Z) \cap \big(f^{-1}\{c_Z\} \setminus p^{-1}(Z)\big) = \emptyset$ is clearly negligible. Recalling that $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$ is saturated and that $p^{-1}(\mathscr{E})$ is $\hat{\mathscr{N}}$-generating (because $\mathscr{E}$ is $\mathscr{N}$-generating and p is supremum preserving), one infers from the Distributivity Lemma 3.4 that $f^{-1}\{c_Z\} \setminus p^{-1}(Z) \in \hat{\mathscr{N}}$.

Thus $p^{-1}(Z) \ominus f^{-1}\{c_Z\} \in \hat{\mathscr{N}}$. As $\mathbf{p}$ is a local isomorphism, according to Propositions 4.16(B) and 5.3(A), $\big(p^{-1}(Z), \hat{\mathscr{A}}_{p^{-1}(Z)}, \hat{\mathscr{N}}_{p^{-1}(Z)}\big)$ is ccc. By what precedes, so is $\big(f^{-1}\{c_Z\}, \hat{\mathscr{A}}_{f^{-1}\{c_Z\}}, \hat{\mathscr{N}}_{f^{-1}\{c_Z\}}\big)$. Therefore, the MSN

$$(Y, \mathscr{B}, \mathscr{M}) := \coprod_{Z\in\mathscr{E}} \big(f^{-1}\{c_Z\}, \hat{\mathscr{A}}_{f^{-1}\{c_Z\}}, \hat{\mathscr{N}}_{f^{-1}\{c_Z\}}\big)$$

is 4c, by definition. It remains to establish that $(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}})$ and $(Y, \mathscr{B}, \mathscr{M})$ are isomorphic in $\mathsf{MSN}_{\mathsf{sp}}$. This is a consequence of Proposition 5.1 applied to the measurable partition $\mathscr{F} = \{f^{-1}\{c_Z\} : Z \in \mathscr{E}\}$. Recalling that $p^{-1}(\mathscr{E})$ is $\hat{\mathscr{N}}$-generating, it ensues from the preceding paragraph that so is $\mathscr{F}$. Since $(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}})$ is locally determined (whence, has locally determined negligible sets, recall 5.2), $\mathscr{F}$ satisfies hypotheses (1) and (2) of Proposition 5.1.

(B) Apply Lemma 7.3 to $\mathscr{C}$ and let $\mathscr{E}$ be the family thus obtained. By Theorem 7.1, the MSN $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big) := \coprod_{Z\in\mathscr{E}}(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ and the morphism $\mathbf{p}$ induced by $p = \coprod_{Z\in\mathscr{E}} \iota_Z$ constitute the 4c version of $(X, \mathscr{A}, \mathscr{N})$. Furthermore, $\big(\hat{X}, \hat{\mathscr{A}}\big)$ is countably separated, by Proposition 6.8. In order to prove that $\big[\big(\hat{X}, \hat{A}, \hat{\mathscr{N}}\big), \mathbf{p}\big]$ is an lld version, we need to adapt the end of the proof of 7.1.

Let $(Y, \mathscr{B}, \mathscr{M})$ be an lld MSN and $\mathbf{q}\colon (Y, \mathscr{B}, \mathscr{M}) \to (X, \mathscr{A}, \mathscr{N})$ a supremum preserving morphism represented by $q \in \mathbf{q}$. As before, we let $\hat{\iota}_Z\colon Z \to \hat{X}$ be the inclusion map and $q_Z := \hat{\iota}_Z \circ \big(q \llcorner q^{-1}(Z)\big)$ for all $Z \in \mathscr{E}$. The family $\langle q_Z \rangle_{Z\in\mathscr{E}}$ subordinated to $\big\langle q^{-1}(Z)\big\rangle_{Z\in\mathscr{E}}$ is compatible. This time we use the gluing result 6.10 instead, that provides a gluing $r\colon Y \to X$ of $\langle q_Z \rangle_{Z\in\mathscr{E}}$. We argue as before to show that r induces the unique supremum preserving morphism $\mathbf{r}\colon (Y, \mathscr{B}, \mathscr{M}) \to (X, \mathscr{A}, \mathscr{N})$ such that $\mathbf{p} \circ \mathbf{r} = \mathbf{q}$. ∎

8. Strictly localizable version of a measure space

Lemma 8.1. *Let $(X, \mathscr{A}, \mu)$ be a measure space and $\mathscr{E} \subseteq \mathscr{A}$ an $\mathscr{N}_\mu$-generating collection that is closed under finite union. Then, for every $A \in \mathscr{A}$, we have $\mu(A) = \sup\{\mu(A \cap Z) : Z \in \mathscr{E}\}$.*

Proof. If $\alpha := \sup\{\mu(A \cap Z) : Z \in \mathscr{E}\}$ is infinite, there is nothing to prove. Otherwise, select an increasing sequence $\langle Z_n \rangle_{n\in\mathbb{N}}$ such that $\lim_n \mu(A \cap Z_n) = \alpha$. Set $A' := A \cap \bigcup_{n\in\mathbb{N}} Z_n$. Suppose that $\mu((A \setminus A') \cap Z) > 0$ for some $Z \in \mathscr{E}$.

Then

$$\alpha \geqslant \mu(A \cap (Z_n \cup Z)) \geqslant \mu(A \cap Z_n) + \mu((A \setminus A') \cap Z).$$

Letting $n \to \infty$ gives a contradiction. So we conclude that $(A \setminus A') \cap Z$ is negligible for all $Z \in \mathscr{E}$. With the help of Lemma 3.4, we obtain that $\mu(A \setminus A') = 0$. Consequently, $\mu(A) = \mu(A') = \lim_{n\to\infty} \mu(A \cap Z_n) = \alpha$. ∎

Definition 8.2. (Pushforward of a measure by a morphism) Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN. A measure $\mu \colon \mathscr{A} \to [0, \infty]$ is *absolutely continuous with respect to* $\mathscr{N}$ whenever $N \in \mathscr{N}$ implies $\mu(N) = 0$. Let $\mathbf{q} \colon (X, \mathscr{A}, \mathscr{N}) \to (Y, \mathscr{B}, \mathscr{M})$ a morphism of saturated MSNs. We define the *pushforward* measure $\mathbf{q}_{\#}\mu := q_{\#}\mu$, where q is any representative of $\mathbf{q}$. This definition makes sense, because, for all $q' \in \mathbf{q}$ and $A \in \mathscr{A}$, we have $\mu(q^{-1}(A) \ominus (q')^{-1}(A)) = 0$, owing to the absolute continuity of μ. Trivially, $\mathbf{q}_{\#}\mu$ is absolutely continuous with respect to $\mathscr{M}$.

Definition 8.3. (Pre-image Measure) Let $(X, \mathscr{A}, \mu)$ be a complete semi-finite measure space. To simplify the notations, we abbreviate $\mathscr{N}_\mu$ to $\mathscr{N}$. Following the discussion in Paragraph 4.15, $(X, \mathscr{A}, \mathscr{N})$ is locally ccc. Recording Corollary 7.4, $(X, \mathscr{A}, \mathscr{N})$ has a 4c version $\big[\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big), \mathbf{p}\big]$ and we shall show that there is a unique measure $\hat{\mu}$ on $(\hat{X}, \hat{\mathscr{A}})$ such that

(1) $\mathscr{N}_{\hat{\mu}} = \hat{\mathscr{N}}$;

(2) $\mathbf{p}_{\#}\hat{\mu} = \mu$.

Such a measure $\hat{\mu}$ is referred to as the *pre-image measure* of μ. Moreover, we will show that the measure space $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mu}\big)$ is strictly localizable; we say that $\big[\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mu}\big), \mathbf{p}\big]$ is the *strictly localizable version* of the measure space $(X, \mathscr{A}, \mu)$.

We start to prove the uniqueness of $\hat{\mu}$. Fix a representative $p \in \mathbf{p}$. For any $F \in \mathscr{A}$ we define $\hat{F} := p^{-1}(F)$ and call $p_F \colon \hat{F} \to F$ the restriction of p, which induces, as usual, a morphism $\mathbf{p}_F \colon \big(\hat{F}, \hat{\mathscr{A}}_{\hat{F}}, \hat{\mathscr{N}}_{\hat{F}}\big) \to \big(F, \mathscr{A}_F, \mathscr{N}_F\big)$. Call $\mathscr{A}^f := \mathscr{A} \cap \{F : \mu(F) < \infty\}$. A pre-image measure $\hat{\mu}$ must satisfy $\mathbf{p}_{F\#}(\hat{\mu} \llcorner \hat{F}) = \mu \llcorner F$ for every $F \in \mathscr{A}^f$. But $(F, \mathscr{A}_F, \mathscr{N}_F)$ is ccc, so by Proposition 4.16, $\mathbf{p}_F$ is an isomorphism, forcing $\hat{\mu} \llcorner \hat{F} = (\mathbf{p}_F^{-1})_{\#}(\mu \llcorner F)$ to hold. Since p is supremum preserving, the collection $\big\{\hat{F} : F \in \mathscr{A}^f\big\}$ admits $\hat{X}$ as an $\hat{\mathscr{N}}$ essential supremum. By (1) and Lemma 8.1 we infer that

$$\hat{\mu}(A) = \sup\Big\{\hat{\mu}(A \cap \hat{F}) : F \in \mathscr{A}^f\Big\} = \sup\Big\{(\mathbf{p}_F^{-1})_{\#}(\mu \llcorner F)(A \cap \hat{F}) : F \in \mathscr{A}^f\Big\}$$

for all $A \in \hat{\mathscr{A}}$, from which the uniqueness of the pre-image measure follows straightforwardly.

8.4. To deal with the existence of pre-image measures, we will fix a 4c version, obtained by an application Theorem 7.1 to the family $\mathscr{A}^f$ defined above. As all 4c versions of $(X, \mathscr{A}, \mathscr{N})$ are isomorphic, there is no restriction in considering this special case.

Henceforth we suppose that $\left(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right) = \coprod_{Z \in \mathscr{E}}(Z, \mathscr{A}_Z, \mathscr{N}_Z)$, where $\mathscr{E} \subseteq \mathscr{A}^f \setminus \mathscr{N}$ is a collection such that (A), (B) and (C) of Theorem 7.1 hold. We now define $\hat{\mu}$ on $(\hat{X}, \hat{\mathscr{A}})$ by

$$\hat{\mu}\left(\coprod_{Z \in \mathscr{E}} A_Z\right) := \sum_{Z \in \mathscr{E}} \mu(A_Z)$$

each A_Z being an arbitrary measurable subset of Z. We choose the representative $p = \coprod_{Z \in \mathscr{E}} \iota_Z$ of $\mathbf{p}$, each $\iota_Z \colon Z \to X$ being the inclusion map.

Obviously, $(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$ is a strictly localizable measure space and $\mathscr{N}_{\hat{\mu}} = \hat{\mathscr{N}}$, which is condition (1) of Paragraph 8.3. The next result gathers some facts about the measure $\hat{\mu}$. In particular, condition (2) of Paragraph 8.3 is proven in Proposition 8.5(B).

Proposition 8.5. *With the notations of paragraph* 8.4*:*

(A) *For all $A \in \mathscr{A}$, one has $\mu(A) = \sum_{Z \in \mathscr{E}} \mu(A \cap Z)$.*

(B) $\mathbf{p}_{\#}\hat{\mu} = \mu$.

(C) *For every set $A \in \hat{\mathscr{A}}$ with σ-finite $\hat{\mu}$-measure, there is $B \in \mathscr{A}$ with σ-finite μ-measure such that $\hat{\mu}(A \ominus \hat{B}) = 0$, where $\hat{B} := p^{-1}(B)$.*

Proof. (A) When $\mathscr{E} \cap \{Z : \mu(A \cap Z) > 0\}$ is uncountable, the result follows easily, for there is $\alpha > 0$ such that $\mathscr{E}_\alpha := \mathscr{E} \cap \{Z : \mu(A \cap Z) > \alpha\}$ is infinite. Taking a countable subset $\mathscr{E}'_\alpha \subseteq \mathscr{E}_\alpha$, then

$$\mu(A) \geqslant \mu\left(A \cap \bigcup \mathscr{E}'_\alpha\right) = \sum_{Z \in \mathscr{E}'_\alpha} \mu(A \cap Z) = \infty$$

because $\mathscr{E}'_\alpha$ is almost disjointed.

On the other hand, suppose $\mathscr{E}' := \mathscr{E} \cap \{Z : \mu(A \cap Z) > 0\}$ is countable and set $A' := A \setminus \bigcup \mathscr{E}'$. Then $\mu(A' \cap Z) = 0$ for every $Z \in \mathscr{E}$. By Lemma 3.4, A' is an $\mathscr{N}$ essential supremum of $\{A' \cap Z : Z \in \mathscr{E}\}$, which forces A' to be negligible. Consequently,

$$\mu(A) = \mu\left(A \cap \bigcup \mathscr{E}'\right) = \sum_{Z \in \mathscr{E}'} \mu(A \cap Z) = \sum_{Z \in \mathscr{E}} \mu(A \cap Z)$$

(B) For any $A \in \mathscr{A}$, one has

$$\mathbf{p}_{\#}\hat{\mu}(A) = \hat{\mu}(p^{-1}(A)) = \hat{\mu}\left(\coprod_{Z\in\mathscr{E}} A \cap Z\right) = \sum_{Z\in\mathscr{E}} \mu(A\cap Z).$$

We conclude by means of (A).

(C) Let $A \in \hat{\mathscr{A}}$ a set of σ-finite $\hat{\mu}$ measure. Writing $A = \coprod_{Z\in\mathscr{E}} A_Z$, each A_Z being a measurable subset of Z, the set $\mathscr{E}' := \mathscr{E} \cap \{Z : \mu(A_Z) > 0\}$ must be countable. Define $B := \bigcup\{A_Z : Z \in \mathscr{E}'\}$. We claim that the set $A \ominus \hat{B} = \coprod_{Z\in\mathscr{E}} \big(A_Z \ominus (B\cap Z)\big)$ is negligible, or, equivalently, all $A_Z \ominus (B \cap Z)$ are negligible. Indeed, for $Z \in \mathscr{E}'$, one has

$$A_Z \ominus (B \cap Z) \subseteq \bigcup \big\{A_{Z'} \cap Z : Z' \in \mathscr{E}' \text{ and } Z' \neq Z\big\} \in \mathscr{N}.$$

If $Z \notin \mathscr{E}'$, then both A_Z and $B \cap Z$ are negligible. ∎

Proposition 8.6. *The Banach spaces $\mathbf{L}_1(X, \mathscr{A}, \mu)$ and $\mathbf{L}_1(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$ are isometrically isomorphic.*

Proof. For any $f, f' \in \mathbf{f} \in \mathbf{L}_1(X, \mathscr{A}, \mu)$ we check that $f\circ p$ and $f'\circ p$ coincide almost everywhere. Thus, the linear map $\varphi \colon \mathbf{L}_1(X, \mathscr{A}, \mu) \to \mathbf{L}_1(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$ which assigns to $\mathbf{f}$ the equivalence class of $f \circ p$ is well-defined. Furthermore, we have

$$\begin{aligned}\int_X |f|\,d\mu &= \int_0^\infty \mu\big(\{|f| > t\}\big)\,dt = \int_0^\infty \mathbf{p}_{\#}\hat{\mu}\big(\{|f| > t\}\big)\,dt \\ &= \int_0^\infty \hat{\mu}\big(\{|f\circ p| > t\}\big)\,dt = \int_{\hat{X}} |f \circ p|\,d\hat{\mu}\,,\end{aligned}$$

showing that φ is an isometry.

Let us show that φ is onto. Let $\hat{f}$ be an integrable function on $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mu}\big)$. As $\{\hat{f} \neq 0\}$ has σ-finite $\hat{\mu}$ measure, Proposition 8.5(B) provides a set $B \in \mathscr{A}$ of σ-finite μ measure such that $\hat{\mu}(\{\hat{f} \neq 0\} \ominus \hat{B}) = 0$. But $\big(B, \mathscr{A}_B, \mathscr{N}_B\big)$ is strictly localizable, and by Proposition 4.16, the morphism $\mathbf{p}_B \colon \big(\hat{B}, \hat{\mathscr{A}}_{\hat{B}}, \hat{\mathscr{N}}_{\hat{B}}\big) \to (B, \mathscr{A}_B, \mathscr{N}_B)$ induced by the restriction $p_B \colon \hat{B} \to B$ of p is an isomorphism. We choose $q_B \colon B \to \hat{B}$ a representative of $\mathbf{p}_B^{-1}$ and define the map $f \colon X \to \mathbb{R}$ by $f(x) := \hat{f}(q_B(x))$ for $x \in B$ and $f(x) := 0$ otherwise. Finally, because $\{\hat{f} \neq f \circ p\} \subseteq \big(\hat{B} \cap \{x : q_B(p(x)) \neq x\}\big) \cup (\{\hat{f} \neq 0\} \setminus \hat{B})$, the maps $\hat{f}$ and $f \circ p$ coincide almost everywhere. ∎

COROLLARY 8.7. *The dual of* $\mathbf{L}_1(X, \mathscr{A}, \mu)$ *is* $\mathbf{L}_\infty(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$.

Proof. It follows from Proposition 8.6 and the (strict) localizability of $(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$, see *e.g.* [6, 243G]. ∎

DEFINITION 8.8. (SEMI-FINITE VERSION) We report on [6, 213X(c)]. Let $(X, \mathscr{A}, \mu)$ be a measure space. We define a measure $\breve{\mu}$ on $\mathscr{A}$ by the formula

$$\breve{\mu}(A) = \sup \left\{ \mu(A \cap F) : F \in \mathscr{A}^f \right\},$$

$A \in \mathscr{A}$. As usual, $\mathscr{A}^f = \mathscr{A} \cap \{A : \mu(A) < \infty\}$. The following hold.

(1) $(X, \mathscr{A}, \breve{\mu})$ is semi-finite.

(2) If $A \in \mathscr{A}$ and $\mu \llcorner A$ is σ-finite, then $\mu \llcorner A = \breve{\mu} \llcorner A$.

(3) If $A \in \mathscr{A}$ and $\breve{\mu}(A) < \infty$, then there are $F \in \mathscr{A}^f$ and $N \in \mathscr{N}_{\breve{\mu}}$ such that $A = F \cup N$.

(4) The Banach space $\mathbf{L}_1(X, \mathscr{A}, \mu)$ and $\mathbf{L}_1(X, \mathscr{A}, \breve{\mu})$ are isometrically isomorphic.

These all straightforwardly follow from the definition.

If we let $(X, \tilde{A}, \tilde{\mu})$ be the completion of $(X, \mathscr{A}, \breve{\mu})$, it follows from (4) that $\mathbf{L}_1(X, \mathscr{A}, \mu)$ is isometrically isomorphic to $\mathbf{L}_1(X, \tilde{A}, \tilde{\mu})$ and, in turn, to $\mathbf{L}_1(\hat{X}, \hat{\mathscr{A}}, \hat{\mu})$, according to Proposition 8.6. In other words, we have associated with *each* measure space $(X, \mathscr{A}, \mu)$ a strictly localizable "version", and we have identified the dual of $\mathbf{L}_1(X, \mathscr{A}, \mu)$. However, reference to Zorn's Lemma in Section 7 (by means of Lemma 4.9) makes it difficult to understand the corresponding space $\hat{X}$. This is why we determine $\hat{X}$ explicitly in Sections 10 and 11, in some special cases of interest.

9. A DIRECTIONAL RADON-NIKODÝM THEOREM

In this section, we prove an extension of the Radon-Nikodým theorem for measure spaces that are not necessarily localizable, in connection with the duality outlined in Corollary 8.7. So to speak, it involves a generalized Radon-Nikodým density that also depends on the direction: as a function, it is defined on the strictly localizable version.

This result is a slight extension of the Radon-Nikodým theorem that was discovered independently by McShane [12, Theorem 7.1] and Zaanen [20]. Using Fremlin's version of the Radon-Nikodým theorem [6, 232E] in the proof below instead of the standard one (between measure spaces of finite measure), we are able to weaken one of the hypotheses in [12] and ask (2) instead. But the main difference with [12] and [20] is in terms of formulation. In their work, the Radon-Nikodým density takes the form of a "quasi-function" or a "cross-section", a notion that is very close to that of a compatible family of measurable functions.

THEOREM 9.1. *Let $(X, \mathscr{A}, \mu)$ be a complete semi-finite measure space and ν a semi-finite measure on $(X, \mathscr{A})$. We let $\big[(\hat{X}, \hat{\mathscr{A}}, \hat{\mu}), \mathbf{p}\big]$ be the strictly localizable version of $(X, \mathscr{A}, \mu)$. Suppose that*

(1) *ν is absolutely continuous with respect to μ.*
(2) *For all $A \in \mathscr{A}$ such that $\nu(A) > 0$, there is an $\mathscr{A}$-measurable subset $F \subseteq A$ such that $\mu(F) < \infty$ and $\nu(F) > 0$.*

Then there is a $\hat{\mathscr{A}}$-measurable function $f \colon \hat{X} \to \mathbb{R}^+$, unique up to equality $\hat{\mu}$-almost everywhere, such that $\nu = \mathbf{p}_\#(f\hat{\mu})$.

Proof. We let $\mathscr{A}^f := \{F : \nu(F) < \infty\}$. We claim that this family is $\mathscr{N}_\mu$-generating. Let $U \in \mathscr{A}$ be an $\mathscr{N}_\mu$-essential upper bound of $\mathscr{A}^f$. Then $\mu(F \setminus U) = 0$ for all $F \in \mathscr{A}^f$. By absolute continuity, it follows that $\nu((X \setminus U) \cap F) = \nu(F \setminus U) = 0$ for all $F \in \mathscr{A}^f$. However, $\mathscr{A}^f$ is $\mathscr{N}_\nu$-generating, by semi-finiteness of ν, and a routine application of the Distributivity Lemma 3.4 shows that $\nu(X \setminus U) = 0$. Hence $X \setminus U \in \mathscr{A}^f$ and $\mu(X \setminus U) = \mu((X \setminus U) \setminus U) = 0$.

Now, the measure $\nu \llcorner F$ is truly continuous with respect to $\mu \llcorner F$, for $F \in \mathscr{A}^f$. Indeed, the hypotheses of [6, 232B(b)] are all satisfied. Thus we can apply Fremlin's version of the Radon-Nikodým theorem. It says that $\nu \llcorner F$ has a Radon-Nikodým density $g_F \colon F \to \mathbb{R}^+$ with respect to $\mu \llcorner F$. It is easy to show that, for any $F, F' \in \mathscr{A}^f$, one has $\mu\big(F \cap F' \cap \{g_F \neq g'_F\}\big) = 0$. Hence $\langle g_F \rangle_{F \in \mathscr{A}^f}$ is a compatible family subordinated to $\mathscr{A}^f$.

Fix a representative $p \in \mathbf{p}$ and set $\hat{F} := p^{-1}(F)$ and $f_F := g_F \circ p_F$ for each $F \in \mathscr{A}^f$. where $p_F \colon \hat{F} \to F$ is the restriction of p. We claim that $\langle f_F \rangle_{F \in \mathscr{A}^f}$ is a compatible family subordinated to $\langle \hat{F} \rangle_{F \in \mathscr{A}^f}$. Indeed, for distinct $F, F' \in \mathscr{A}^f$, we have $\hat{F} \cap \hat{F}' \cap \{f_F \neq f_{F'}\} = p^{-1}(F \cap F' \cap \{g_F \neq g_{F'}\})$. Since p is $[(\hat{\mathscr{A}}, \mathscr{N}_{\hat{\mu}}), (\mathscr{A}, \mathscr{N}_\mu)]$-measurable, $\hat{F} \cap \hat{F}' \cap \{f_F \neq f_{F'}\} \in \mathscr{N}_{\hat{\mu}}$. Owing to the supremum preserving character of p, the family $\{\hat{F} : F \in \mathscr{A}^f\}$ is $\mathscr{N}_{\hat{\mu}}$-generating. By Proposition 6.4, the family $\langle f_F \rangle_{F \in \mathscr{A}^f}$ has a gluing

$f\colon \hat{X} \to \mathbb{R}^+$. For every $A \in \mathscr{A}$ and $F \in \mathscr{A}^f$, we have

$$
\begin{aligned}
\nu(A \cap F) &= \int \mathbb{1}_{A\cap F} g_F d\mu && \text{Radon-Nikodým Theorem}\\
&= \int \mathbb{1}_{A\cap F} g_F d\mathbf{p}_\# \hat{\mu} && \hat{\mu} \text{ is a pre-image measure}\\
&= \int \mathbb{1}_{p^{-1}(A\cap F)} f_F d\hat{\mu} &&\\
&= \int_{p^{-1}(A)\cap \hat{F}} f d\hat{\mu} && f = f_F \text{ a.e on } \hat{F}\\
&= (f\hat{\mu})(p^{-1}(A) \cap \hat{F})
\end{aligned}
$$

Applying Lemma 8.1, we obtain $\nu(A) = \sup\left\{\nu(A\cap F) : F \in \mathscr{A}^f\right\}$. Also, if we set $Z := \hat{X} \cap \{x : f(x) > 0\}$, then in the subMSN $\left(Z, \hat{\mathscr{A}}_Z, (\mathscr{N}_{\hat{\mu}})_Z\right)$ the family $\{Z \cap \hat{F} : F \in \mathscr{A}^f\}$ admits Z as an $(\mathscr{N}_{\hat{\mu}})_Z$-essential supremum, because $p \circ \iota_Z$ is supremum preserving ($\iota_Z\colon Z \to \hat{X}$, being the inclusion map, is supremum preserving, and the composition of supremum preserving maps is supremum preserving). Since $(\mathscr{N}_{\hat{\mu}})_Z = \mathscr{N}_{f\hat{\mu} \llcorner Z}$, we can apply Lemma 8.1 again and deduce

$$
\begin{aligned}
(f\hat{\mu})\big(p^{-1}(A)\big) &= (f\hat{\mu} \llcorner Z)\big(p^{-1}(A) \cap Z\big)\\
&= \sup\left\{(f\hat{\mu} \llcorner Z)(p^{-1}(A) \cap Z \cap \hat{F}) \,:\, F \in \mathscr{A}^f\right\}\\
&= \sup\left\{(f\hat{\mu})(p^{-1}(A) \cap \hat{F}) \,:\, F \in \mathscr{A}^f\right\}.
\end{aligned}
$$

Hence $\nu(A) = \mathbf{p}_\#(f\hat{\mu})(A)$.

Now we prove the uniqueness of f. Let f' be another density, and suppose $\hat{\mu}(\{f' > f\}) > 0$. By semi-finiteness of $\hat{\mu}$ there is a set $A \in \hat{\mathscr{A}}$ such that $A \subseteq \{f' > f\}$ and $0 < \hat{\mu}(A) < \infty$. By Proposition 8.5(C), there is $B \in \mathscr{A}$ such that $\hat{\mu}(A \ominus p^{-1}(B)) = 0$. However, we have

$$\mathbf{p}_\#(f'\hat{\mu})(B) = (f'\hat{\mu})(A) > (f\hat{\mu})(A) = \mathbf{p}_\#(f\hat{\mu})(B),$$

which is a contradiction. It follows that $f' \leqslant f$ almost everywhere. Similarly, we prove the reverse inequality. ∎

10. 4c version deduced from a compatible family of lower densities

We devote this section to an explicit construction of the 4c and lld version under some extra assumptions. It will be applied in the next section.

DEFINITION 10.1. Let $(X, \mathscr{A}, \mathscr{N})$ be an MSN. A *lower density* for $(X, \mathscr{A}, \mathscr{N})$ is a function $\Theta\colon \mathscr{A} \to \mathscr{A}$ such that:

(1) $\Theta(A) = \Theta(B)$ for all $A, B \in \mathscr{A}$ such that $A \ominus B \in \mathscr{N}$;

(2) $A \ominus \Theta(A) \in \mathscr{N}$ for all $A \in \mathscr{A}$;

(3) $\Theta(\emptyset) = \emptyset$;

(4) $\Theta(A \cap B) = \Theta(A) \cap \Theta(B)$ for all $A, B \in \mathscr{A}$.

PROPOSITION 10.2. *Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN, $\mathscr{E} \subseteq \mathscr{A}$, and $\Theta\colon \mathscr{A} \to \mathscr{A}$ a lower density. Assume that*

(A) *for all $Z \in \mathscr{E}$, the subMSN $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is ccc;*

(B) *$\mathscr{E}$ is $\mathscr{N}$-generating;*

(C) *One has*

(i) $\forall A \subseteq X : \left[\forall Z \in \mathscr{E} : A \cap Z \in \mathscr{A}\right] \Rightarrow A \in \mathscr{A}$;

(ii) $\forall N \subseteq X : \left[\forall Z \in \mathscr{E} : N \cap Z \in \mathscr{N}\right] \Rightarrow N \in \mathscr{N}$.

Then $(X, \mathscr{A}, \mathscr{N})$ is 4c.

Proof. Let $\mathscr{E}_1$ be associated with $\mathscr{E}$ in Lemma 4.9. Thus, $\mathscr{E}_1$ is almost disjointed and $\mathscr{N}$-generating, and $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is ccc for all $Z \in \mathscr{E}_1$.

We claim that $\mathscr{E}$ may be replaced by $\mathscr{E}_1$ in hypothesis (C). Let $A \in \mathscr{P}(X)$ be such that $A \cap Z \in \mathscr{A}$ for every $Z \in \mathscr{E}_1$. Let $Z' \in \mathscr{E}$.

Define $\mathscr{Z} := \{Z \cap Z' : Z \in \mathscr{E}_1 \text{ and } Z \cap Z' \notin \mathscr{N}\}$. Notice that $(Z', \mathscr{A}_{Z'}, \mathscr{N}_{Z'})$ is ccc and $\mathscr{Z}$ is almost disjointed. Thus $\mathscr{Z}$ is countable and $\bigcup \mathscr{Z}$ is an $\mathscr{N}$-essential supremum of $\mathscr{Z}$. Besides, by the Distributivity Lemma 3.4, the family $\{Z \cap Z' : Z \in \mathscr{E}_1\}$ admits Z' as an $\mathscr{N}$-essential supremum. This family differs from $\mathscr{Z}$ only by negligible sets. Therefore, $Z' \ominus \bigcup \mathscr{Z} \in \mathscr{N}$. Since $(X, \mathscr{A}, \mathscr{N})$ is saturated, we deduce that $(A \cap Z') \ominus \left(A \cap \bigcup \mathscr{Z}\right) \in \mathscr{N}$. Thus, one needs to establish that $A \cap \bigcup \mathscr{Z} \in \mathscr{A}$ in order to show that $A \cap Z' \in \mathscr{A}$. This is readily done by observing that

$$A \cap \bigcup \mathscr{Z} = \bigcup \left\{A \cap Z \cap Z' \,:\, Z \in \mathscr{E}_1 \text{ and } Z \cap Z' \notin \mathscr{N}\right\}$$

is a countable union of measurable sets. We just proved that $A \cap Z' \in \mathscr{A}$ for all $Z' \in \mathscr{E}$. By hypothesis (C)(i), $A \in \mathscr{A}$. Now assume that $A \cap Z \in \mathscr{N}$, for each $Z \in \mathscr{E}_1$, and let Z' and $\mathscr{Z}$ be as above. Since Z' is an $\mathscr{N}$-essential supremum of $\mathscr{Z}$, it follows from Lemma 3.4 that $A \cap Z'$ is an $\mathscr{N}$-essential supremum of $\{A \cap Z \cap Z' : Z \in \mathscr{E}_1 \text{ and } Z \cap Z' \notin \mathscr{N}\}$. Therefore, $A \cap Z' \in \mathscr{N}$. Since Z' is arbitrary, it follows that $A \in \mathscr{N}$, by hypothesis (C)(ii).

Next we define $\mathscr{E}_2 = \{\Theta(Z) : Z \in \mathscr{E}_1\}$. The family $\mathscr{E}_2$ is disjointed, for $\Theta(Z) \cap \Theta(Z') = \Theta(Z \cap Z') = \emptyset$ for any distinct $Z, Z' \in \mathscr{E}_1$, since $Z \cap Z' \in \mathscr{N}$. We next claim that $\mathscr{E}$ (or, for that matter, $\mathscr{E}_1$) may be replaced by $\mathscr{E}_2$ in hypothesis (C). Indeed, for every $A \subseteq X$ and every $Z \in \mathscr{E}_1$,

$$(A \cap Z) \ominus (A \cap \Theta(Z)) \subseteq Z \ominus \Theta(Z) \in \mathscr{N},$$

therefore (i) $A \cap Z \in \mathscr{A}$ if and only if $A \cap \Theta(Z) \in \mathscr{A}$, and (ii) $A \cap Z \in \mathscr{N}$ if and only if $A \cap \Theta(Z) \in \mathscr{N}$, since $(X, \mathscr{A}, \mathscr{N})$ is saturated. In particular, letting $N := X \setminus \cup \mathscr{E}_2$, we infer from (C)(ii) with $\mathscr{E}$ replaced by $\mathscr{E}_2$ that $N \in \mathscr{N}$. Finally, the conclusion follows from Proposition 5.1 applied to $\mathscr{F} = \mathscr{E}_2 \cup \{N\}$. ∎

Definition 10.3. Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN and $\mathscr{E} \subseteq \mathscr{A}$. A *compatible family of lower densities* is a family $\langle \Theta_Z \rangle_{Z \in \mathscr{E}}$ such that

(1) For all $Z \in \mathscr{E}$, the map $\Theta_Z \colon \mathscr{A}_Z \to \mathscr{A}_Z$ is a lower density for $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$;
(2) For all $Z, Z' \in \mathscr{E}$ and $A \subseteq Z \cap Z'$ a measurable set, $\Theta_Z(A) = \Theta_{Z'}(A)$;
(3) $\Theta_Z(Z) = Z$ for all $Z \in \mathscr{E}$.

Condition (3) is merely of technical nature. If a family satisfies only (1) and (2), we can enforce (3) by replacing $\mathscr{E}$ with $\{\Theta_Z(Z) : Z \in \mathscr{E}\}$ and observing that $\Theta_Z \colon \mathscr{A}_Z \to \mathscr{A}_Z$ restricts to $\mathscr{A}_{\Theta_Z(Z)} \to \mathscr{A}_{\Theta_Z(Z)}$. This, indeed, follows from the fact that $\Theta_Z(A) \subseteq \Theta_Z(B)$, whenever $A, B \in \mathscr{A}_Z$ and $A \subseteq B$, and $\Theta_Z \circ \Theta_Z = \Theta_Z$, as one easily checks from the definition of lower density.

Definition 10.4. (Germ space) In the sequel, we consider a saturated MSN $(X, \mathscr{A}, \mathscr{N})$ that has a compatible family of lower densities $\langle \Theta_Z \rangle_{Z \in \mathscr{E}}$, where $\mathscr{E}$ is a family such that

(1) $\mathscr{E}$ is $\mathscr{N}$-generating;
(2) $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is ccc for each $Z \in \mathscr{E}$.

Under these assumptions, we will now construct a new MSN $\left(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right)$ that we call the *germ space of* $(X, \mathscr{A}, \mathscr{N})$ *associated with* $\mathscr{E}$ *and* $\langle \Theta_Z \rangle_{Z \in \mathscr{E}}$.

For every $x \in X$, we set $\mathscr{E}_x := \mathscr{E} \cap \{Z : x \in Z\}$ and we define the relation $\sim_x$ on $\mathscr{E}_x$ by $Z \sim_x Z' \iff x \in \Theta_Z(Z \cap Z')$. We claim that it is an equivalence relation. Indeed, it is reflexive because of 10.3(3); it is symmetric because of the set equality $\Theta_Z(Z \cap Z') = \Theta_{Z'}(Z \cap Z')$ implied by 10.3(2). Let us check that it is transitive. For $Z, Z', Z'' \in \mathscr{E}_x$ such that $Z \sim_x Z' \sim_x Z''$, we have

$$\begin{aligned} x \in \Theta_{Z'}(Z \cap Z') \cap \Theta_{Z'}(Z' \cap Z'') &= \Theta_{Z'}(Z \cap Z' \cap Z'') && 10.1(4) \\ &= \Theta_Z(Z \cap Z' \cap Z'') && 10.3(2) \\ &\subseteq \Theta_Z(Z \cap Z''), \end{aligned}$$

hence $Z \sim_x Z''$.

We define the quotient set $\Gamma_x := \mathscr{E}_x / \sim_x$. The equivalence class of $Z \in \mathscr{E}_x$ is denoted $[Z]_x \in \Gamma_x$. Next we define the set $\hat{X} := \big\{(x, [Z]_x) : x \in X$ and $[Z]_x \in \Gamma_x\big\}$ and the projection map $p\colon \hat{X} \to X$ which assigns $(x, \mathbf{Z})$ to x. For each $Z \in \mathscr{E}$, we define the map $\gamma_Z\colon Z \to \hat{X}$ by $\gamma_Z(x) = (x, [Z]_x)$ for $x \in Z$. We define a σ-algebra $\hat{\mathscr{A}}$ and a σ-ideal $\hat{\mathscr{N}}$ on $\hat{X}$ by

$$\begin{aligned} \hat{\mathscr{A}} &:= \mathscr{P}(\hat{X}) \cap \big\{A : \gamma_Z^{-1}(A) \in \mathscr{A}_Z, \forall Z \in \mathscr{E}\big\}, \\ \hat{\mathscr{N}} &:= \mathscr{P}(\hat{X}) \cap \big\{N : \gamma_Z^{-1}(N) \in \mathscr{N}_Z, \forall Z \in \mathscr{E}\big\}. \end{aligned}$$

Actually, $\hat{\mathscr{A}}$ and $\hat{\mathscr{N}}$ are the finest σ-algebra and σ-ideal such that the maps γ_Z become $[(\mathscr{A}_Z, \mathscr{N}_Z), (\hat{\mathscr{A}}, \hat{\mathscr{N}})]$-measurable. Clearly, $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$ is saturated.

Let us check that the projection map $p\colon \hat{X} \to X$ is $[(\hat{\mathscr{A}}, \hat{\mathscr{N}}), (\mathscr{A}, \mathscr{N})]$-measurable. If $A \in \mathscr{A}$ then for any $Z \in \mathscr{E}$ we have

$$\gamma_Z^{-1}\big(p^{-1}(A)\big) = (p \circ \gamma_Z)^{-1}(A) = Z \cap A \in \mathscr{A}_Z,$$

so by definition $p^{-1}(A) \in \hat{\mathscr{A}}$. One proves similarly that $p^{-1}(N) \in \hat{\mathscr{N}}$ for all $N \in \mathscr{N}$.

THEOREM 10.5. *Let $(X, \mathscr{A}, \mathscr{N})$ be a saturated MSN that has a compatible family of lower density $\langle \Theta_Z \rangle_{Z \in \mathscr{E}}$, where $\mathscr{E} \subseteq \mathscr{A}$ is a family such that conditions* (1) *and* (2) *of definition* 10.4 *hold. Then the germ space $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$ constructed in definition* 10.4 *together with* $\mathbf{p}$ *is the 4c version of $(X, \mathscr{A}, \mathscr{N})$. It is also its lld version in case* $\operatorname{card} \mathscr{E} \leqslant \mathfrak{c}$ *and $(Z, \mathscr{A}_Z)$ is countably separated for all $Z \in \mathscr{E}$.*

Proof. The second conclusion is a consequence of the first and of Theorem 7.6.

STEP 1: *we prove that $(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}})$ possesses a lower density Θ, obtained by "patching together" the lower densities Θ_Z for $Z \in \mathscr{E}$.* For every $A \in \hat{\mathscr{A}}$, we set

$$\Theta(A) := \hat{X} \cap \big\{(x, [Z]_x) : x \in \Theta_Z\big(\gamma_Z^{-1}(A)\big)\big\}.$$

The condition $x \in \Theta_Z\big(\gamma_Z^{-1}(A)\big)$ does not depend on the representative Z of $[Z]_x$. Indeed, if $Z' \sim_x Z$ for some $Z' \in \mathscr{E}_x$, then

$$x \in \Theta_Z\big(\gamma_Z^{-1}(A)\big) \cap \Theta_Z(Z \cap Z') = \Theta_Z\big(\gamma_Z^{-1}(A) \cap Z'\big).$$

Note that the sets $\gamma_Z^{-1}(A) \cap Z'$ and $\gamma_{Z'}^{-1}(A) \cap Z$ coincide $\mathscr{N}$-almost everywhere, as

$$\begin{aligned}\big(\gamma_Z^{-1}(A) \cap Z'\big) \ominus \big(\gamma_{Z'}^{-1}(A) \cap Z\big) &\subseteq Z \cap Z' \cap \big\{y : [Z]_y \neq [Z']_y\big\} \\ &\subseteq Z \cap Z' \setminus \Theta_Z(Z \cap Z')\end{aligned}$$

is negligible by 10.1(2). Consequently,

$$\begin{aligned}\Theta_Z\big(\gamma_Z^{-1}(A) \cap Z'\big) &= \Theta_Z\big(\gamma_{Z'}^{-1}(A) \cap Z\big) && 10.1(2) \\ &= \Theta_{Z'}\big(\gamma_{Z'}^{-1}(A) \cap Z\big) && 10.3(2) \\ &\subseteq \Theta_{Z'}\big(\gamma_{Z'}^{-1}(A)\big)\end{aligned}$$

and in turn $x \in \Theta_{Z'}\big(\gamma_{Z'}^{-1}(A)\big)$, as expected.

Next we show that Θ satisfies the four properties required to be a lower density:

- Let $A, B \in \hat{\mathscr{A}}$ such that $A \ominus B \in \hat{\mathscr{N}}$. Then $\gamma_Z^{-1}(A) \ominus \gamma_Z^{-1}(B) \in \mathscr{N}$ for all $Z \in \mathscr{E}$, which implies

$$\begin{aligned}(x, [Z]_x) \in \Theta(A) &\iff x \in \Theta_Z(\gamma_Z^{-1}(A)) \\ &\iff x \in \Theta_Z(\gamma_Z^{-1}(B)) && 10.1(1) \\ &\iff (x, [Z]_x) \in \Theta(B)\end{aligned}$$

  and we conclude that $\Theta(A) = \Theta(B)$.
- Let $A \in \hat{\mathscr{A}}$. By construction, $\gamma_Z^{-1}(\Theta(A)) = \Theta_Z\big(\gamma_Z^{-1}(A)\big)$, for all $Z \in \mathscr{E}$. This gives that $\gamma_Z^{-1}(A \ominus \Theta(A)) = \gamma_Z^{-1}(A) \ominus \gamma_Z^{-1}(\Theta(A)) \in \mathscr{N}$. By definition of the σ-ideal $\hat{\mathscr{N}}$, we infer that $A \ominus \Theta(A) \in \hat{\mathscr{N}}$.
- That $\Theta(\emptyset) = \emptyset$ is straightforward.
- Let $A, B \in \hat{\mathscr{A}}$. We have

$$\begin{aligned}(x, [Z]_x) \in \Theta(A \cap B) &\iff x \in \Theta_Z\big(\gamma_Z^{-1}(A \cap B)\big) \\ &\iff x \in \Theta_Z\big(\gamma_Z^{-1}(A) \cap \gamma_Z^{-1}(B)\big) \\ &\iff x \in \Theta_Z\big(\gamma_Z^{-1}(A)\big) \cap \Theta_Z\big(\gamma_Z^{-1}(B)\big) \\ &\iff (x, [Z]_x) \in \Theta(A) \cap \Theta(B).\end{aligned}$$

Step 2: *we establish that* $\mathbf{p}$ *is a "local isomorphism".* Set $\hat{Z} := p^{-1}(Z)$ for all $Z \in \mathscr{E}$. We also call p_Z and s_Z the respective restrictions of p and γ_Z to $\hat{Z} \to Z$ and $Z \to \hat{Z}$. First, we remark that $p_Z \circ s_Z = \mathrm{id}_Z$.

Let us show that $\hat{Z} \setminus s_Z(Z) \in \hat{\mathscr{N}}$. For $Z' \in \mathscr{E}$, we find that

$$\begin{aligned} x \in \gamma_{Z'}^{-1}(\hat{Z} \setminus s_Z(Z)) &\iff x \in Z' \text{ and } (x, [Z']_x) \in \hat{Z} \setminus s_Z(Z) \\ &\iff x \in Z \cap Z' \text{ and } [Z']_x \neq [Z]_x \\ &\iff x \in Z \cap Z' \setminus \Theta_{Z'}(Z \cap Z'). \end{aligned}$$

So $\gamma_{Z'}^{-1}\big(\hat{Z} \setminus s_Z(Z)\big) \in \mathscr{N}_{Z'}$. As this holds for all $Z' \in \mathscr{E}$, we deduce that $\hat{Z} \setminus s_Z(Z)$ is $\hat{\mathscr{N}}$-negligible.

Since $\hat{Z} \cap \{\xi : (s_Z \circ p_Z)(\xi) \neq \xi\} = \hat{Z} \setminus s_Z(Z)$, this shows that $s_Z \circ p_Z$ and $\mathrm{id}_{\hat{Z}}$ coincide $\hat{\mathscr{N}}$-almost everywhere. As a consequence, the morphisms $\mathbf{p}_Z$ and $\mathbf{s}_Z$ induced by p_Z and s_Z are reciprocal isomorphisms of MSN between $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ and $\big(\hat{Z}, \hat{\mathscr{A}}_{\hat{Z}}, \hat{\mathscr{N}}_{\hat{Z}}\big)$. They are supremum preserving, according to Proposition 3.7(A).

Step 3: $\big(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\big)$ *is "locally determined" (in the sense of Proposition 10.2(C)) by the family* $\hat{\mathscr{E}} := \big\{\hat{Z} : Z \in \mathscr{E}\big\}$. Let A a subset of $\hat{X}$. By definition of $\hat{\mathscr{A}}$, we have

$$\begin{aligned} A \in \hat{\mathscr{A}} &\iff \forall Z \in \mathscr{E} : \gamma_Z^{-1}(A) \in \mathscr{A}_Z \\ &\iff \forall Z \in \mathscr{E} : s_Z^{-1}(A \cap \hat{Z}) \in \mathscr{A}_Z \\ &\iff \forall Z \in \mathscr{E} : A \cap \hat{Z} \in \hat{\mathscr{A}}_{\hat{Z}}. \end{aligned}$$

The direct implication of the last equivalence is justified as follows: if $s_Z^{-1}\big(A \cap \hat{Z}\big)$ is measurable, then so is $p_Z^{-1}\big(s_Z^{-1}(A \cap \hat{Z})\big)$, from which $A \cap \hat{Z}$ differs only by an $\hat{\mathscr{N}}$ negligible set. We prove analogously that a set $N \subseteq \hat{X}$ is negligible if and only if $N \cap \hat{Z} \in \hat{\mathscr{N}}$ for all $Z \in \mathscr{E}$.

Step 4: *p is supremum preserving.* Let $\mathscr{F} \subseteq \mathscr{A}$ be a collection which has an $\mathscr{N}$-essential supremum denoted S. Clearly, $p^{-1}(S)$ is an $\hat{\mathscr{N}}$-essential upper bound of $p^{-1}(\mathscr{F}) := \big\{p^{-1}(F) : F \in \mathscr{F}\big\}$. Let U be an arbitrary $\hat{\mathscr{N}}$-essential upper bound of $p^{-1}(\mathscr{F})$. We need to prove that $p^{-1}(S) \setminus U \in \hat{\mathscr{N}}$, that is, $\gamma_Z^{-1}\big(p^{-1}(S) \setminus U\big) \in \hat{\mathscr{N}}$ for all $Z \in \mathscr{E}$. But

$$\gamma_Z^{-1}\big(p^{-1}(S) \setminus U\big) = (p \circ \gamma_Z)^{-1}(S) \setminus \gamma_Z^{-1}(U) = Z \cap S \setminus \gamma_Z^{-1}(U).$$

By Lemma 3.4 we recognize $Z \cap S$ as an $\mathscr{N}$-essential supremum of $\{Z \cap F : F \in \mathscr{F}\}$. This last collection can be also written $\big\{\gamma_Z^{-1}(p^{-1}(F)) : F \in \mathscr{F}\big\}$, of which $\gamma_Z^{-1}(U)$ is an $\mathscr{N}$-essential upper bound, leading to $Z \cap S \setminus \gamma_Z^{-1}(U) \in \mathscr{N}$.

STEP 5: $(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}})$ *is 4c.* This is an application of Proposition 10.2 with the collection $\hat{\mathscr{E}}$. We check that all the hypotheses are satisfied. For $Z \in \mathscr{E}$, the subMSN $(\hat{Z}, \hat{\mathscr{A}}_{\hat{Z}}, \hat{\mathscr{N}}_{\hat{Z}})$ is ccc because of the isomorphism $\mathbf{p}_Z \colon (\hat{Z}, \hat{\mathscr{A}}_{\hat{Z}}, \hat{\mathscr{N}}_{\hat{Z}}) \to (Z, \mathscr{A}_Z, \mathscr{N}_Z)$. Since p is supremum preserving, $\hat{X}$ is an $\hat{\mathscr{N}}$-essential supremum of $\hat{\mathscr{E}}$. The "local determination" property was established in step 3.

STEP 6: *The pair* $((\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}), \mathbf{p})$ *satisfies the universal property of Definition* 4.11. We finish the proof in a way similar to the proof of Theorem 7.1. Let $(Y, \mathscr{B}, \mathscr{M})$ a 4c MSN and $\mathbf{q} \colon (Y, \mathscr{B}, \mathscr{M}) \to (X, \mathscr{A}, \mathscr{N})$ a supremum preserving morphism, represented by a map $q \in \mathbf{q}$. For every $Z \in \mathscr{E}$, we define $q_Z = \gamma_Z \circ (q \llcorner q^{-1}(Z)) \colon q^{-1}(Z) \to \hat{X}$. We claim that $\langle q_Z \rangle_{Z \in \mathscr{E}}$ is a compatible family subordinated to $\langle q^{-1}(Z) \rangle_{Z \in \mathscr{E}}$. Indeed, for distinct $Z, Z' \in \mathscr{E}$,

$$\begin{aligned} q^{-1}(Z) \cap q^{-1}(Z') \cap \{q_Z \neq q_{Z'}\} &= q^{-1}\big(Z \cap Z' \cap \{\gamma_Z \neq \gamma_{Z'}\}\big) \\ &= q^{-1}\big(Z \cap Z' \setminus \Theta_Z(Z \cap Z')\big) \end{aligned}$$

is negligible, using that Θ_Z is a lower density and q is $[(\mathscr{B}, \mathscr{M}), (\mathscr{A}, \mathscr{N})]$-measurable. Then, by Proposition 6.4, the family $\langle q_Z \rangle_{Z \in \mathscr{E}}$ has a gluing that we denote $r \colon Y \to \hat{X}$. That r is $[(\mathscr{B}, \mathscr{M}), (\hat{\mathscr{A}}, \hat{\mathscr{N}})]$-measurable and supremum preserving follows from Lemma 6.2. Indeed, each γ_Z is supremum preserving. This follows from the same property of s_Z, proved in Step 2, and the Distributivity Lemma 3.4.

We need to show that $\{p \circ r \neq q\}$ is $\mathscr{M}$-negligible. In fact, for any $Z \in \mathscr{E}$ and $y \in q^{-1}(Z)$, we note that $p(q_Z(y)) = p(\gamma_Z(q(y)) = q(y)$, so $q^{-1}(Z) \cap \{p \circ r \neq q\} \subseteq \big(q^{-1}(Z) \cap \{r \neq q_Z\}\big)$ is $\mathscr{M}$-negligible. We then use that $(Y, \mathscr{B}, \mathscr{M})$ has locally determined negligible sets (see Proposition 5.3(E) and the preceding Paragraph 5.2) to conclude that $p \circ r = q$ almost everywhere. We have found a supremum preserving morphism $\mathbf{r} \colon (Y, \mathscr{B}, \mathscr{M}) \to (\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}})$, namely the one induced from r, such that $\mathbf{p} \circ \mathbf{r} = \mathbf{q}$.

We now prove that this factorization is unique. Let $\mathbf{r}$ be a supremum preserving morphism such that $\mathbf{p} \circ \mathbf{r} = \mathbf{q}$ and $r \in \mathbf{r}$. For $Z \in \mathscr{E}$ and almost every $y \in q^{-1}(Z)$, we have $p(r(y)) = q(y) \in Z$. Therefore $r(y) \in \hat{Z}$ for almost all $y \in q^{-1}(Z)$. For such a y, we have $q(y) = p(r(y)) = p_Z(r(y))$, which implies that $q_Z(y) = \gamma_Z(q(y)) = s_Z(q(y)) = s_Z(p_Z(r(y)))$. But $s_Z \circ p_Z$ and $\mathrm{id}_{\hat{Z}}$ coincide almost everywhere on $\hat{Z}$ as we saw in Step 2. This implies that $r(y) = q_Z(y)$ for almost all $y \in q^{-1}(Z)$. The map r must be a gluing of $\langle q_Z \rangle_{Z \in \mathscr{E}}$, so it is unique up to equality almost everywhere according to Proposition 6.4. ∎

11. Applications

Here, we apply Theorem 10.5 to two different situations. For this result to apply to an MSN $(X, \mathscr{A}, \mathscr{N})$, the following conditions need to be met:

(i) $(X, \mathscr{A}, \mathscr{N})$ is saturated.

(ii) An $\mathscr{N}$-generating family $\mathscr{E} \subseteq \mathscr{A}$ is given.

(iii) For every $Z \in \mathscr{E}$, the MSN $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$ is ccc.

(iv) For every $Z \in \mathscr{E}$, the measurable space $(Z, \mathscr{A}_Z)$ is countably separated.

(v) $\operatorname{card} \mathscr{E} \leqslant \mathfrak{c}$.

(vi) For every $Z \in \mathscr{E}$, a lower density Θ_Z is given for $(Z, \mathscr{A}_Z, \mathscr{N}_Z)$, so that $\Theta_Z(Z) = Z$.

(vii) For every $Z, Z' \in \mathscr{E}$ and $A \in \mathscr{A}$ such that $A \subseteq Z \cap Z'$, one has $\Theta_Z(A) = \Theta_{Z'}(A)$.

In that case, the corresponding germ space $\left(\hat{X}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right)$ constructed in 10.4 is the 4c version and the lld version of $(X, \mathscr{A}, \mathscr{N})$.

11.1. (Purely unrectifiable negligibles) Fix integers $1 \leqslant k \leqslant m-1$. Recall [4, 3.2.14] that a subset $N \subseteq \mathbb{R}^m$ is called *purely* $(\mathscr{H}^k, k)$-*unrectifiable* whenever $\mathscr{H}^k(N \cap M) = 0$ for every k-rectifiable set $M \subseteq \mathbb{R}^m$. This is equivalent to $\mathscr{H}^k(N \cap M) = 0$ for every k-dimensional embedded submanifold $M \subseteq \mathbb{R}^m$ of class C^1 with $\mathscr{H}^k(M) < \infty$, by [4, 3.1.15]. We denote by $\mathscr{N}_{\mathrm{pu},k}$ the collection of purely $(\mathscr{H}^k, k)$-unrectifiable subsets of $\mathbb{R}^m$. It is a σ-ideal of $\mathscr{P}(\mathbb{R}^m)$. We also introduce the Borel σ-algebra $\mathscr{B}(\mathbb{R}^m)$ of $\mathbb{R}^m$ and its completion $\overline{\mathscr{B}(\mathbb{R}^m)} := \left\{B \ominus N : B \in \mathscr{B}(\mathbb{R}^m), N \in \mathscr{N}_{\mathrm{pu},k}\right\}$. We shall show that the MSN $(\mathbb{R}^m, \overline{\mathscr{B}(\mathbb{R}^m)}, \mathscr{N}_{\mathrm{pu},k})$ can be associated with a germ space, as in 10.4. We notice that, by definition, this MSN is saturated. We let $\mathscr{E}$ be the collection of all k-dimensional (embedded) submanifolds $M \subseteq \mathbb{R}^m$ of class C^1, [4, 3.1.19], such that $\mathscr{H}^k \llcorner M$ is locally finite (that is $\mathscr{H}^k(M \cap B) < \infty$ for every bounded Borel set $B \subseteq \mathbb{R}^m$). Clearly, each member of $\mathscr{E}$ is Borel.

(ii) We now show that $\mathscr{E}$ is $\mathscr{N}_{\mathrm{pu},k}$-generating. Let $U \in \overline{\mathscr{B}(\mathbb{R}^m)}$ be such that $\mathbb{R}^m \setminus U \notin \mathscr{N}_{\mathrm{pu},k}$. By definition of this σ-ideal, there exists $M \in \mathscr{E}$ such that $\mathscr{H}^k((\mathbb{R}^m \setminus U) \cap M) > 0$. In other words, $M \setminus U \notin \mathscr{N}_{\mathrm{pu},k}$, i.e. U is not an $\mathscr{N}_{\mathrm{pu},k}$-essential upper bound of $\mathscr{E}$.

(iii) We next claim that $(M, \overline{\mathscr{B}(\mathbb{R}^m)}_M, (\mathscr{N}_{\mathrm{pu},k})_M)$ is ccc, for every $M \in \mathscr{E}$. To this end, we notice that for every $M \in \mathscr{E}$ the following holds:

$$\text{For every } S \subseteq M : S \in \mathscr{N}_{\mathrm{pu},k} \text{ if and only if } \mathscr{H}^k(S) = 0. \qquad (\bigstar)$$

In other words, $\big(M, \overline{\mathscr{B}(\mathbb{R}^m)}_M, (\mathscr{N}_{\mathrm{pu},k})_M\big)$ is the saturation of the MSN associated with the measure space $\big(M, \mathscr{B}(M), \mathscr{H}^k \llcorner M\big)$. Since the latter is σ-finite, the claim follows from Proposition 4.5.

We also record the following useful consequence of ($\bigstar$), for $M \in \mathscr{E}$:

$$\begin{gathered}\text{If } S \in \overline{\mathscr{B}(\mathbb{R}^m)}_M \text{ then } S = B \ominus N \text{ for some}\\ B \in \mathscr{B}(M) \text{ and } N \subseteq M \text{ with } \mathscr{H}^k(N) = 0.\end{gathered} \tag{$\blacklozenge$}$$

Indeed, $S = B' \ominus N'$, $B' \in \mathscr{B}(\mathbb{R}^m)$, $N' \in \mathscr{N}_{\mathrm{pu},k}$.

Thus $S = M \cap S = (M \cap B') \ominus (M \cap N')$, which proves ($\blacklozenge$). In particular, S is $\mathscr{H}^k$-measurable, even though some $S \in \overline{\mathscr{B}(\mathbb{R}^m)}$ may not be $\mathscr{H}^k$-measurable.

(iv) Consider $M \in \mathscr{E}$. We observe that the canonical embedding $\big(M, \overline{\mathscr{B}(\mathbb{R}^m)}_M\big) \to \big(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m)\big)$ is, indeed, injective and measurable. Therefore, $\big(M, \overline{\mathscr{B}(\mathbb{R}^m)}_M\big)$ is countably separated, according to Proposition 6.7.

(v) Since $\mathscr{E} \subseteq \mathscr{B}(\mathbb{R}^m)$ we infer that $\operatorname{card} \mathscr{E} \leqslant \mathfrak{c}$, according to [17, 3.3.18].

(vi) In order to define lower densities, we recall [4, 2.10.19] the density numbers $\Theta^k_*(\phi, x)$ and $\Theta^{*\,k}(\phi, x)$, defined by means of closed Euclidean balls, associated with an outer measure ϕ on $\mathbb{R}^m$ and $x \in \mathbb{R}^m$. Given $M \in \mathscr{E}$ we abbreviate $\phi_M = \mathscr{H}^k \llcorner M$ and we define

$$\Theta_M(A) = M \cap \big\{x : \Theta^k_*(\phi_M, x) = 1\big\},$$

whenever $A \in \overline{\mathscr{B}(\mathbb{R}^m)}_M$. Given $x \in \mathbb{R}^m$, the function $r \mapsto \phi_M(\mathbf{B}(x,r))$ is right continuous, since ϕ_M is locally finite. It easily follows that $x \mapsto \Theta^k_*(\phi_M, x)$ is Borel measurable and, in turn, that $\Theta_M(A) \in \mathscr{B}(\mathbb{R}^m)$. In particular, Θ_M maps $\overline{\mathscr{B}(\mathbb{R}^m)}_M$ to itself. The following is the main point of the construction:

$$\text{For every } x \in M : \Theta^m_*(\phi_M, x) = 1. \tag{$\clubsuit$}$$

See for instance the proof of [3, 3.6.1]. For instance, it follows that $\Theta_M(M) = M$. We now turn to checking that Θ_M is a lower density. If $A, B \subseteq M$ are such that $A \ominus B \in \mathscr{N}_{\mathrm{pu},k}$ then $\mathscr{H}^k(A \ominus B) = 0$, recall (iii). Consequently, $\phi_M(A \cap \mathbf{B}(x,r)) = \phi_M(B \cap \mathbf{B}(x,r))$ for all $x \in \mathbb{R}^m$ and $r > 0$. Thus, $\Theta^k_*(A, x) = \Theta^k_*(B, x)$. Since x is arbitrary, $\Theta_M(A) = \Theta_M(B)$. This proves condition of 10.1. Condition (3) of 10.1 is trivial. In view of proving 10.1(3) we let $A \in \overline{\mathscr{B}(\mathbb{R}^m)}_M$. According to condition (1) just proved and ($\blacklozenge$), there is no restriction to assume that A is Borel. We ought to show that the equation $\Theta^k_*(\phi_M \llcorner A, x) = \mathbb{1}_A(x)$ holds for $\mathscr{H}^k$-almost every $x \in M$. Letting $\psi = \phi_M \llcorner A$, we infer from the Besicovitch Covering Theorem as in [11, 2.12] that $\lim_{r\to 0^+}\frac{\psi(\mathbf{B}(x,r))}{\phi_M(\mathbf{B}(x,r))} = \mathbb{1}_A(x)$ for ϕ_M-almost every every $x \in \mathbb{R}^n$. In view of (♣), it ensues that the sought for equation holds $\mathscr{H}^k$-almost everywhere on M. To establish that Θ_M is a lower density, it remains to proves 10.1(4). Let $A, B \in \overline{\mathscr{B}(\mathbb{R}^m)}_M$. We observe that

$$\begin{aligned}\Theta^k_*\big(\phi_M \llcorner (A\cap B), x\big) \geqslant \Theta^k_*(\phi_M, x) &- \Theta^{*\,k}\big(\phi_M \llcorner (M\setminus A), x\big)\\ &- \Theta^{*\,k}\big(\phi_M \llcorner (M\setminus B), x\big)\end{aligned}$$

for all $x \in M$. Now, as A and B are ϕ_M-measurable, according to (♦), if $x \in \Theta_M(A) \cap \Theta_M(B)$, then it follows from (♣) that

$$\Theta^{*\,k}\big(\phi_M \llcorner (M\setminus A), x\big) = \Theta^{*\,k}\big(\phi_M \llcorner (M\setminus B), x\big) = 0$$

and, in turn, referring to (♣) again, that $\Theta^k_*\big(\phi_M \llcorner (A\cap B), x\big) = 1$. Thus, $x \in \Theta_M(A\cap B)$. We have shown that $\Theta_M(A)\cap\Theta_M(B) \subseteq \Theta_M(A\cap B)$. The other inclusion is trivial, so that Θ_M is, indeed, a lower density.

(vii) Let $M, M' \in \mathscr{E}$ and $A \in \overline{\mathscr{B}(\mathbb{R}^m)}$ be such that $A \subseteq M \cap M'$. Notice that $A = A\cap M' = A \cap M$ and $\phi_M \llcorner A\cap M' = \phi_{M'} \llcorner A \cap M$. Therefore, if $x \in \Theta_M(A)$, then

$$\begin{aligned}1 &= \Theta^k_*\big(\phi_M \llcorner A, x\big) = \Theta^k_*\big(\phi_M \llcorner A\cap M, x\big)\\ &= \Theta^k_*\big(\phi_{M'} \llcorner A\cap M, x\big) = \Theta^k_*\big(\phi_{M'} \llcorner A, x\big).\end{aligned}$$

Since also $x \in M'$, we conclude that $x \in \Theta_{M'}(A)$. Switching the rôles of M and M' we conclude that $\Theta_M(A) = \Theta_{M'}(A)$.

It is interesting to try to understand the corresponding germ space. Each $(x, \mathbf{M}) \in \widehat{\mathbb{R}^m}$ consists of a pair where $x \in \mathbb{R}^m$ belongs to the base space $\mathbb{R}^m$ and $\mathbf{M}$ is an equivalence class of a k-dimensional submanifolds passing through x. If $M \ni x \in M'$ are two such submanifolds, then $M \sim_x M'$ if and only if

$$\lim_{r\to 0^+} \frac{\mathscr{H}^k\big(M\cap M'\cap \mathbf{B}(x,r)\big)}{\boldsymbol{\alpha}(k)r^k} = 1.$$

This relation is finer than the usual notion of a germ of a k-dimensional submanifold passing through x. Of course if M and M' belong to the same, classically defined, germ, i.e. if there exists a neighborhood V of x in $\mathbb{R}^m$ such that $M\cap V = M'\cap V$, then $M \sim_x M'$. Notwithstanding, the following example illustrates the difference. Let $x \in \mathbb{R}^m$, let $W \subseteq \mathbb{R}^m$ be a k-dimensional affine subspace containing x, and let $C \subseteq W$ be closed with empty interior and such that $\Theta^k_*(\phi_W \llcorner C, x) = 1$. Choose a k-dimensional submanifold $M \subseteq \mathbb{R}^m$ of class C^1 "that sticks to W exactly along C", that is $W \cap M = C$. It follows that $W \sim_x M$, yet $M\cap V \neq W \cap V$, for every neighborhood V of x. We note, however, that if $M \sim_x M'$, then $\operatorname{Tan}(M,x) = \operatorname{Tan}(M',x)$.

The construction here could be repeated by replacing $\mathscr{E}$ by $\mathscr{E}'$, the collection of all Borel measurable, countably $(\mathscr{H}^k,k)$-rectifiable subsets M of X such that $\mathscr{H}^k \llcorner M$ is locally finite, *and* $\Theta^k_*(\phi_M, x) = 1$ for every $x \in M$. The latter does not hold in general for rectifiable sets, unlike the case of (embedded) submanifolds. It is critical when establishing that condition 10.1(4) holds.

11.2. (Integral geometric measure) Here, we show that the methods of 11.1 apply, in fact, to a special measure space. We keep the same notations as in 11.1 and we let $\mathscr{I}^k_\infty$ be the integral geometric outer measure on $\mathbb{R}^m$ defined in [4, 2.10.5(1)] or [11, 5.14]. The measure space $\left(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m), \mathscr{I}^k_\infty\right)$ is not semi-finite (for the case $1 = k = m-1$, see [4, 3.3.20]). Thus, recalling 8.8, we introduce the following:

$$\check{\mathscr{I}}^k_\infty(A) = \sup\left\{\mathscr{I}^k_\infty(A \cap B) \,:\, B \in \mathscr{B}(\mathbb{R}^m), B \subseteq A \text{ and } \mathscr{I}^k_\infty(B) < \infty\right\},$$

for $A \in \mathscr{B}(\mathbb{R}^m)$. The measure space $\left(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m), \check{\mathscr{I}}^k_\infty\right)$ is semi-finite, and $\check{\mathscr{I}}^k_\infty(A) = 0$ whenever $A \in \mathscr{B}(\mathbb{R}^m)$ is purely $\mathscr{I}^k_\infty$-infinite, i.e. A itself and all its Borel subsets of nonzero measure have infinite measure. We denote by $\left(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \tilde{\mathscr{I}}^k_\infty\right)$ its completion. Our goal is to describe its 4c, lld, and strictly localizable version. The corresponding MSN $\left(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}\right)$ is readily saturated. We will check conditions (ii) through (vii) at the beginning of this section.

(ii) We claim that $\mathscr{E}$ is $\mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}$-generating in the MSN $\left(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}\right)$, where $\mathscr{E}$ is as in 11.1.

We know that the collection $\mathscr{A} := \widetilde{\mathscr{B}(\mathbb{R}^m)} \cap \left\{A \,:\, \tilde{\mathscr{I}}^k_\infty(A) < \infty\right\}$ is $\mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}$-generating, by 4.2. It is easy to check that it suffices to establish the following: For every $A \in \mathscr{A}$ there is a sequence $\langle M_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{E}$ such that $A \setminus \bigcup_{n\in\mathbb{N}} M_n \in \mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}$. Let $A \in \mathscr{A}$. By definition of completion of a measure space, there are $B \in \mathscr{B}(\mathbb{R}^m)$, $N \in \mathscr{N}_{\check{\mathscr{I}}^k_\infty}$, and $N' \subseteq N$ such that $A = B \ominus N'$. Since $\tilde{\mathscr{I}}^k_\infty(N') = 0$, it suffices to prove the existence of a sequence $\langle M_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{E}$ such that $B \setminus \bigcup_{n\in\mathbb{N}} M_n \in \mathscr{N}_{\check{\mathscr{I}}^k_\infty}$. Since $\check{\mathscr{I}}^k_\infty(B) = \tilde{\mathscr{I}}^k_\infty(B) = \tilde{\mathscr{I}}^k_\infty(A) < \infty$, there are Borel sets F and N such that $B = F \cup N$, $\mathscr{I}^k_\infty(F) < \infty$, and $\check{\mathscr{I}}^k_\infty(N) = 0$, by 8.8(3). It follows from the Besicovitch Structure Theorem [4, 3.3.14] that F is $(\mathscr{I}^k_\infty, k)$-rectifiable. In particular, there is a sequence $\langle M_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{E}$ such that $F \setminus \bigcup_{n\in\mathbb{N}} M_n \in \mathscr{N}_{\mathscr{H}^k} \subseteq \mathscr{N}_{\mathscr{I}^k_\infty}$. Since $B \setminus F \in \mathscr{N}_{\check{\mathscr{I}}^k_\infty} \subseteq \mathscr{N}_{\tilde{\mathscr{I}}^k_\infty}$, the proof is complete.

In order to establish (iii) through (vii), it suffices to observe that for each $M \in \mathscr{E}$ the MSNs $\left(M, \overline{\mathscr{B}(\mathbb{R}^m)}_M, (\mathscr{N}_{\mathrm{pu},k})_M\right)$ and $\left(M, \widetilde{\mathscr{B}(\mathbb{R}^m)}_M, (\mathscr{N}_{\check{\mathscr{I}}^k_\infty})_M\right)$ are the same. We recall from 11.1(iii) that the former is the saturation of $(M, \mathscr{B}(\mathbb{R}^m)_M, \mathscr{H}^k \llcorner M)$. Let us prove that the latter has the same property. Let $S \in \widetilde{\mathscr{B}(\mathbb{R}^m)}_M$. There are $B \in \mathscr{B}(\mathbb{R}^m)$ and $N \in \mathscr{N}_{\check{\mathscr{I}}^k_\infty}$ such that $S = B \ominus N$. Since $S = S \cap M = (B \cap M) \ominus (M \cap N)$, there is no restriction to assume that both B and N are contained in M. Therefore, we ought to show that $\mathscr{H}^k(N) = 0$. There exists a Borel set $N' \subseteq M$ containing N and such that $\check{\mathscr{I}}^k_\infty(N') = 0$. We observe that $\check{\mathscr{I}}^k_\infty \llcorner M = \mathscr{I}^k_\infty \llcorner M = \mathscr{H}^k \llcorner M$, where the second equality follows from [4, 3.2.26], and the first follows from 8.8(2) and the fact that M has σ-finite $\mathscr{I}^k_\infty$ measure. Thus, $\mathscr{H}^k(N') = 0$ and we are done.

It follows that the germ space $\left(\widehat{\mathbb{R}^m}, \hat{\mathscr{A}}, \hat{\mathscr{N}}\right)$ constructed in 11.1 is, in fact, also the 4c and lld version of the MSN $\left(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \mathscr{N}_{\check{\mathscr{I}}^k_\infty}\right)$. Furthermore, if $\hat{\mathscr{I}}^k_\infty$ denotes the pre-image measure of $\tilde{\mathscr{I}}^k_\infty$ along the projection map $p \colon \widehat{\mathbb{R}^m} \to \mathbb{R}^m$, then $\left(\widehat{\mathbb{R}^m}, \widehat{\widetilde{\mathscr{B}(\mathbb{R}^m)}}, \hat{\mathscr{I}}^k_\infty\right)$ is the strictly localizable version of $\left(\mathbb{R}^m, \widetilde{\mathscr{B}(\mathbb{R}^m)}, \tilde{\mathscr{I}}^k_\infty\right)$.

11.3. (Hausdorff measures) Here, we briefly comment on why the lower densities set up so far in this section do not help to describe explicitly the 4c and lld version of the saturation of the MSN $\left(\mathbb{R}^m, \mathscr{B}(\mathbb{R}^m), \mathscr{N}_{\mathscr{H}^k}\right)$. The main reason is that we would need to enlarge the collection $\mathscr{E}$ for it to be generating, since there are (much) less $\mathscr{H}^k$-negligible sets than there are purely k-unrectifiable sets. In doing so we loose (♣), which was critical for implementing the techniques of the previous section. In fact, if $M \subseteq \mathbb{R}^m$ is Borel, $\phi_M = \mathscr{H}^k \llcorner M$ is locally finite, and $\Theta^k(\phi_M, x) = 1$ for $\mathscr{H}^k$-almost every $x \in M$, then M is countably $(\mathscr{H}^k, k)$-rectifiable, see *e.g.* [11, 17.6(1)]. Since we ought to include non $\mathscr{H}^k$-negligible, purely k-unrectifiable sets in an $\mathscr{N}_{\mathscr{H}^k}$-generating family, our only choice is, if possible, to change the definition of the lower densities Θ_M. So far, we do not know how to construct, in this case, a compatible family of lower densities.

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