Generalized representations of 3-Hom-Lie algebras

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Abstract: The propose of this paper is to extend generalized representations of 3-Lie algebras to Hom-type algebras. We introduce the concept of generalized representation of multiplicative 3-Hom-Lie algebras, develop the corresponding cohomology theory and study semi-direct products. We provide a key construction, various examples and computation of 2-cocycles of the new cohomology. Also, we give a connection between a split abelian extension of a 3-Hom-Lie algebra and a generalized semidirect product 3-Hom-Lie algebra.

Key words: 3-Hom-Lie algebra, representation, generalized representation, cohomology, abelian extension.

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Introduction

The first instances of ternary Lie algebras appeared first in Nambu's generalization of Hamiltonian mechanics [23], which was formulated algebraically by Takhtajan [29]. The structure of n-Lie algebras was studied by Filippov [15] then completed by Kasymov in [21].

The representation theory of n-Lie algebras was first introduced by Kasymov in [21]. The adjoint representation is defined by the ternary bracket in which two elements are fixed. Through fundamental objects one may also represent a 3-Lie algebra and more generally an n-Lie algebra by a Leibniz algebra [11]. The cohomology of n-Lie algebras, generalizing the Chevalley-Eilenberg Lie algebras cohomology, was introduced by Takhtajan [30] in its simplest form, later a complex adapted to the study of formal deformations was introduced by Gautheron [17], then reformulated by Daletskii and Takhtajan [11] using the notion of base Leibniz algebra of an n-Lie algebra. In [2, 3], the structure and cohomology of 3-Lie algebras induced by Lie algebras has been investigated.

The concept of generalized representation of a 3-Lie algebra was introduced by Liu, Makhlouf and Sheng in [19]. They study the corresponding generalized semidirect product 3-Lie algebra and cohomology theory. Furthermore, they describe general abelian extensions of 3-Lie algebras using Maurer-Cartan elements. Non-abelian extensions were explored in [26].

The aim of this paper is to extend the concept of generalized representation of 3-Lie algebras to Hom-type algebras. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [18] as part of a study of deformations of the Witt and the Virasoro algebras. The n-Hom-Lie algebras and various generalizations of n-ary algebras were considered in [4]. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. In particular, representations and cohomologies of Hom-Lie algebras were studied in [25], while the representations and cohomology of n-Hom-Lie algebras were first studied in [1].

The paper is organized as follows. In Section 1, we provide some basics about 3-Hom-Lie algebras, representations and cohomology. The second Section includes the new concept of generalized representation of a 3-Hom-Lie algebra, extending to Hom-type algebras the notion and results obtained in [19]. We define a corresponding semi-direct product and provide a twist procedure leading to generalized representations of 3-Hom-Lie algebras starting from generalized representations of 3-Hom-Lie algebras and algebra maps. In Section 3, we construct a new cohomology corresponding to generalized representations and show examples. In the last section we discuss abelian extensions of multiplicative 3-Hom-Lie algebras. One recovers the results in [19] when the twist map is the identity.

1. REPRESENTATIONS OF 3-HOM-LIE ALGEBRAS

The aim of this section is to recall some basics about 3-Lie algebras and 3-Hom-Lie algebras. We refer mainly to [15] and [4]. In this paper, all vector spaces are considered over a field $\mathbb{K}$ of characteristic 0.

DEFINITION 1.1. A 3-Lie algebra is a pair $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ consisting of a $\mathbb{K}$-vector space $\mathfrak{g}$ and a trilinear skew-symmetric multiplication $[\cdot, \cdot, \cdot]$ satisfying the Filippov-Jacobi identity: for x, y, z, u, v in $\mathfrak{g}$

$$[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]].$$

In this paper, we are dealing with 3-Hom-Lie algebras corresponding to the following definition.

DEFINITION 1.2. A 3-Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ consisting of a $\mathbb{K}$-vector space $\mathfrak{g}$, a trilinear skew-symmetric multiplication $[\cdot, \cdot, \cdot]$ and an algebra map $\alpha : \mathfrak{g} \to \mathfrak{g}$ satisfying the Hom-Filippov-Jacobi identity: for x, y, z, u, v in $\mathfrak{g}$

$$[\alpha(u), \alpha(v), [x, y, z]] = [[u, v, x], \alpha(y), \alpha(z)] + [\alpha(x), [u, v, y], \alpha(z)]$$
$$+ [\alpha(x), \alpha(y), [u, v, z]].$$

Remark 1.3. There is more general definition of 3-Hom-Lie algebras which are given by a quadruple $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha_1, \alpha_2)$ consisting of a $\mathbb{K}$-vector space $\mathfrak{g}$, two linear maps $\alpha_1, \alpha_2 : \mathfrak{g} \to \mathfrak{g}$ and a trilinear skew-symmetric multiplication $[\cdot, \cdot, \cdot]$ satisfying the following generalized Hom-Filippov-Jacobi identity: for x, y, z, u, v in $\mathfrak{g}$

$$[\alpha_1(u), \alpha_2(v), [x, y, z]] = [[u, v, x], \alpha_1(y), \alpha_2(z)] + [\alpha_1(x), [u, v, y], \alpha_2(z)]$$
$$+ [\alpha_1(x), \alpha_2(y), [u, v, z]].$$

We get our class of 3-Hom-Lie algebras when $\alpha_1 = \alpha_2 = \alpha$ and where α is an algebra morphism. This kind of algebras are usually called multiplicative 3-Hom-Lie algebras.

PROPOSITION 1.4. *Let* $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ *be a 3-Lie algebra and* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *be a 3-Lie algebra morphism. Then* $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\alpha := \alpha \circ [\cdot, \cdot, \cdot], \alpha)$ *is a 3-Hom-Lie algebra.*

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ be a 3-Hom-Lie algebra, elements in $\wedge^2 \mathfrak{g}$ are called *fundamental objects* of the 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$. There is a bilinear operation $[\cdot, \cdot]_{\mathcal{L}}$ on $\wedge^2 \mathfrak{g}$, which is given by

$$[X, Y]_{\mathcal{L}} = [x_1, x_2, y_1] \wedge \alpha(y_2) + \alpha(y_1) \wedge [x_1, x_2, y_2]$$

for all $X = x_1 \wedge x_2$ and $Y = y_1 \wedge y_2$, and a linear map $\overline{\alpha}$ on $\wedge^2 \mathfrak{g}$ defined by $\overline{\alpha}(X) = \alpha(x_1) \wedge \alpha(x_2)$, for simplicity, we will write $\overline{\alpha}(X) = \alpha(X)$. It is well-known that $(\wedge^2 \mathfrak{g}, [\cdot, \cdot]_{\mathcal{L}}, \overline{\alpha})$ is a Hom-Leibniz algebra [1, 31].

DEFINITION 1.5. A representation of a 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ on a vector space V with respect to $A \in gl(V)$ is a skew-symmetric linear map $\rho : \wedge^2 \mathfrak{g} \to End(V)$ such that

$$\rho(\alpha(x_1), \alpha(x_2)) \circ A = A \circ \rho(x_1, x_2), \tag{1.1}$$
$$\rho(\alpha(x_1), \alpha(x_2))\rho(x_3, x_4) - \rho(\alpha(x_3), \alpha(x_4))\rho(x_1, x_2) \tag{1.2}$$
$$= \big(\rho([x_1, x_2, x_3], \alpha(x_4)) - \rho([x_1, x_2, x_4], \alpha(x_3))\big) \circ A,$$

$$\rho([x_1, x_2, x_3], \alpha(x_4)) \circ A - \rho(\alpha(x_2), \alpha(x_3))\rho(x_1, x_4) \qquad (1.3)$$
$$= \rho(\alpha(x_3), \alpha(x_1))\rho(x_2, x_4) + \rho(\alpha(x_1), \alpha(x_2))\rho(x_3, x_4),$$

for x_1, x_2, x_3 and x_4 in $\mathfrak{g}$.

THEOREM 1.6. *Let $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra, (V, ρ) be a representation, $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a 3-Lie algebra morphism and $A : V \to V$ be a linear map such that $A \circ \rho(x_1, x_2) = \rho(\alpha(x_1), \alpha(x_2)) \circ A$. Then $(V, \widetilde{\rho} := A \circ \rho, A)$ is a representation of the 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\alpha := \alpha \circ [\cdot, \cdot, \cdot], \alpha)$.*

Proof. Let $x_i \in \mathfrak{g}$, where $1 \leq i \leq 5$. Then we have

$$\widetilde{\rho}([x_3, x_4, x_5]_\alpha, \alpha(x_1)) \circ A - \widetilde{\rho}(\alpha(x_3), \alpha(x_4))\widetilde{\rho}(x_5, x_1) - \widetilde{\rho}(\alpha(x_4), \alpha(x_5))\widetilde{\rho}(x_3, x_1)$$
$$- \widetilde{\rho}(\alpha(x_5), \alpha(x_3))\widetilde{\rho}(x_4, x_1)$$
$$= A^2 \circ (\rho([x_3, x_4, x_5], x_1) - \rho(x_3, x_4)\rho(x_5, x_1)$$
$$- \rho(x_4, x_5)\rho(x_3, x_1) - \rho(x_5, x_3)\rho(x_4, x_1)) = 0.$$

The second condition (1.2) is obtained similarly. ∎

The previous result allows to twist along morphisms a 3-Lie algebra with a representation to a 3-Hom-Lie algebra with a corresponding representation.

PROPOSITION 1.7. *Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ be a 3-Hom-Lie algebra, V be a vector space, $A \in gl(V)$ and $\rho : \wedge^2\mathfrak{g} \to gl(V)$ be a skew-symmetric linear map. Then $(V; \rho, A)$ is a representation of 3-Hom-Lie algebra $\mathfrak{g}$ if and only if there is a 3-Hom-Lie algebra structure $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_\rho, \alpha_{\mathfrak{g} \oplus V})$ on the direct sum of vector spaces $\mathfrak{g} \oplus V$, defined by*

$$[x_1 + v_1, x_2 + v_2, x_3 + v_3]_\rho = [x_1, x_2, x_3] + \rho(x_1, x_2)v_3$$
$$+ \rho(x_3, x_1)v_2 + \rho(x_2, x_3)v_1,$$

and $\alpha_{\mathfrak{g} \oplus V} = \alpha + A$, for all $x_i \in \mathfrak{g}$, $v_i \in V$, $1 \leq i \leq 3$. The obtained 3-Hom-Lie algebra is denoted by $\mathfrak{g} \ltimes_\rho V$ and called semidirect product.

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ be a 3-Hom-Lie algebra and (V, ρ, A) be a representation of $\mathfrak{g}$. We denote by $C^p_{\alpha, A}(\mathfrak{g}, V)$ the space of all linear maps

$$\varphi : \underbrace{\wedge^2\mathfrak{g} \otimes \cdots \otimes \wedge^2\mathfrak{g}}_{(p-1)} \wedge \mathfrak{g} \to V$$

satisfying:

$$A \circ \varphi(X_1 \otimes \cdots \otimes X_{p-1}, y) = \varphi(\alpha(X_1) \otimes \cdots \otimes \alpha(X_{p-1}), \alpha(y)),$$

for all $X_1, \ldots, X_{p-1} \in \wedge^2 \mathfrak{g}$, $y \in \mathfrak{g}$. It is called the space of p-cochains.

Let φ be a $(p-1)$-cochain, the coboundary operator $\delta_\rho : C_{\alpha,A}^{p-1}(\mathfrak{g}, V) \to C_{\alpha,A}^p(\mathfrak{g}, V)$ is given by

$$
\begin{aligned}
(\delta_\rho \varphi)&(X_1, \ldots, X_p, z) \\
&= \sum_{1 \leq j < k} (-1)^j \varphi(\alpha(X_1), \ldots, \hat{X}_j, \ldots, \alpha(X_{k-1}), [X_j, X_k]_{\mathcal{L}}, \alpha(X_{k+1}), \\
&\qquad\qquad\qquad\qquad \ldots, \alpha(X_p), \alpha(z)) \\
&\quad + \sum_{j=1}^p (-1)^j \varphi(\alpha(X_1), \ldots, \hat{X}_j, \ldots, \alpha(X_p), [X_j, z]) \\
&\quad + \sum_{j=1}^p (-1)^{j+1} \rho(\alpha^p(X_j)) \varphi(X_1, \ldots, \hat{X}_j, \ldots, X_p, z) \\
&\quad + (-1)^{p+1} \Big(\rho(\alpha^p(y_p), \alpha^p(z)) \varphi(X_1, \ldots, X_{p-1}, x_p) \\
&\quad + \rho(\alpha^p(z), \alpha^p(x_p)) \alpha(X_1, \ldots, X_{p-1}, y_p) \Big),
\end{aligned}
\tag{1.4}
$$

for all $X_i = (x_i, y_i) \in \wedge^2 \mathfrak{g}$, $z \in \mathfrak{g}$ and where $[X_i, z] = [x_i, y_i, z]$. An element $\varphi \in C_{\alpha,A}^{p-1}(\mathfrak{g}, V)$ is called a p-cocycle if $\delta_\rho \varphi = 0$. It is called a p-coboundary if there exists some $f \in C_{\alpha,A}^{p-2}(\mathfrak{g}, V)$ such that $\varphi = \delta_\rho f$. Denote by $Z_{3HL}^p(\mathfrak{g}; V)$ and $B_{3HL}^p(\mathfrak{g}; V)$ the sets of p-cocycles and p-coboundaries respectively. Then the p-th cohomology group is

$$H_{3HL}^p(\mathfrak{g}; V) = Z_{3HL}^p(\mathfrak{g}; V) / B_{3HL}^p(\mathfrak{g}; V). \tag{1.5}$$

In [28], the author constructed a graded Lie algebra structure by which one can describe an n-Leibniz algebra structure as a canonical structure. Here, we give the precise formulas for the 3-Hom-Lie algebra case, generalizing the result in [19].

Set $C_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g}) = \oplus_{p \geq 0} C_{\alpha,\alpha}^p(\mathfrak{g}, \mathfrak{g})$. Let $\varphi \in C_{\alpha,\alpha}^q(\mathfrak{g}, \mathfrak{g})$, $\psi \in C_{\alpha,\alpha}^p(\mathfrak{g}, \mathfrak{g})$, $p, q \geq 0$, $X_i = x_i \wedge y_i \in \wedge^2 \mathfrak{g}$ for $i = 1, 2, \ldots, p + q$ and $x \in \mathfrak{g}$. For each subset $J = \{j_1, \ldots, j_p\}_{j_1 < \cdots < j_p} \subset N \triangleq \{1, 2, \ldots, p + q\}$, let $I = \{i_1, \ldots, i_q\}_{i_1 < \cdots < i_q} = N/J$. Define on the graded vector space $C_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g})$ the graded commutator bracket

$$[\varphi, \psi]^{3HL} = (-1)^{pq} j_\psi^\alpha(\varphi) - j_\varphi^\alpha(\psi) = (-1)^{pq} \varphi \circ_\alpha \psi - \psi \circ_\alpha \varphi,$$

where $\varphi \circ_\alpha \psi \in C_{\alpha,\alpha}^{p+q}(\mathfrak{g}, \mathfrak{g})$ is defined by

$$j_\psi^\alpha(\varphi)(X_1, \ldots, X_{p+q}, x) = \varphi \circ_\alpha \psi(X_1, \ldots, X_{p+q}, x)$$

$$= \sum_{J, j_q < i_{k+1} \leq p+q} (-1)^{(J,I)} \varphi\Big(\alpha^p(X_{i_1}), \ldots, \alpha^p(X_{i_k}),$$

$$\psi(X_{j_1}, \ldots, X_{j_p},) \bullet_\alpha X_{i_{k+1}}, \ldots, \alpha^p(X_{i_q}), \alpha^p(x)\Big)$$

$$+ \sum_J (-1)^{(J,I)}(-1)^q \varphi\Big(\alpha^p(X_{i_1}), \ldots, \alpha^p(X_{i_q}), \psi(X_{j_1}, \ldots, X_{j_p}, x)\Big),$$

with

$$\psi(X_{j_1}, \ldots, X_{j_q},) \bullet_\alpha X_{i_{k+1}} = \psi(X_{j_1}, \ldots, X_{j_p}, x_{i_{k+1}}) \wedge \alpha^p(y_{i_{k+1}})$$
$$+ \alpha^p(x_{i_{k+1}}) \wedge \psi(X_{j_1}, \ldots, X_{j_p}, y_{i_{k+1}})$$

where k is uniquely determined by the condition $j_p \leq i_{k+1}$ and if $j_p \leq i_1$ then $j_p = p$, $i_1 = p+1$ and $(-1)^{(J,I)}$ is the sign of the permutation $(J, I) = (j_1, \ldots, j_p, i_1, \ldots, i_q)$ of N.

We need the following lemma to establish a structure of graded Lie algebra on $C_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g})$.

LEMMA 1.8. *We have* $j_{[\varphi,\psi]^{3HL}} = -[j_\varphi^\alpha, j_\psi^\alpha]$ *for all* $\varphi, \psi \in C_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g})$, *where* $[\cdot, \cdot]$ *is the graded commutator on* $End(C_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g}))$.

Proof. Let $\varphi \in C_{\alpha,\alpha}^q(\mathfrak{g}, \mathfrak{g}), \psi \in C_{\alpha,\alpha}^p(\mathfrak{g}, \mathfrak{g}), \xi \in C_{\alpha,\alpha}^r(\mathfrak{g}, \mathfrak{g}), X_1, \ldots, X_{p+q+r} \in \wedge^2 \mathfrak{g}$ and $x \in \mathfrak{g}$

$$[j_\varphi^\alpha, j_\psi^\alpha](\xi)(X_1, X_2, \ldots, X_{p+q+r}, x)$$

$$= (j_\varphi^\alpha(j_\psi^\alpha \xi) - (-1)^{pq} j_\psi^\alpha(j_\varphi^\alpha \xi))(X_1, X_2, \ldots, X_{p+q+r}, x)$$

$$= D_1 - (-1)^{pq} D_2,$$

where

$$D_1 = j_\varphi^\alpha(j_\psi^\alpha \xi)(X_1, X_2, \ldots, X_{p+q+r}, x),$$

$$D_2 = j_\psi^\alpha(j_\varphi^\alpha \xi))(X_1, X_2, \ldots, X_{p+q+r}, x).$$

For each subset $J = \{j_1, \ldots, j_q\}_{j_1 < \cdots < j_q} \subset N \triangleq \{1, 2, \ldots, p+q+r\}$, let $I = \{i_1, \ldots, i_{p+r}\}_{i_1 < \cdots < i_{p+r}} = N \setminus J$ and $L = \{l_1, \ldots, l_p\}_{l_1 < \cdots < l_p} \subset I \triangleq \{i_1, i_2, \ldots, i_{p+r}\}$, let $H = \{h_1, \ldots, h_r\}_{h_1 < \cdots < h_r} = I \setminus L$.

We have for D_1:

$$j_\varphi^\alpha(j_\psi^\alpha\xi)(X_1, X_2, \ldots, X_{p+q+r}, x) = (j_\psi^\alpha(\xi)) \circ_\alpha \varphi(X_1, X_2, \ldots, X_{p+q+r}, x)$$

$$= \sum_{J, j_q < i_{k+1} \le p+q+r} (-1)^{(J,I)} (j_\psi^\alpha(\xi))(\alpha^q(X_{i_1}), \ldots, \alpha^q(X_{i_k}), \varphi(X_{j_1}, \ldots, X_{j_q},) \bullet_\alpha X_{i_{k+1}}, \alpha^q(X_{i_{k+2}}), \ldots, \alpha^q(X_{i_{p+r}}), \alpha^q(x))$$

$$+ \sum_J (-1)^{(J,I)} (-1)^{p+r} (j_\psi^\alpha(\xi))(\alpha^q(X_{i_1}), \ldots, \alpha^q(X_{i_{p+r}}), \varphi(X_{j_1}, \ldots, X_{j_q}, x))$$

$$= \sum_{\substack{J, j_q < i_{k+1} \le p+q+r}} \sum_{\substack{L, l_p < h_{m+1} \le i_{p+r}, \\ i_{k+1}=h_n, n \le m}} (-1)^{(J,I)} (-1)^{(L,H)} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_{n-1}}), \alpha^p(\varphi(X_{j_1}, \ldots, X_{j_q},) \bullet_\alpha X_{i_{k+1}}),$$

$$\ldots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_p}),) \bullet_\alpha \alpha^q(X_{h_{m+1}}), \alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$+ \sum_{\substack{J, j_q < i_{k+1} \le p+q+r}} \sum_{\substack{L, l_p < h_{m+1} \le i_{p+r}, \\ i_{k+1}=h_{m+1}}} (-1)^{(J,I)} (-1)^{(L,H)} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_p}),)$$

$$\bullet_\alpha (\varphi(X_{j_1}, \ldots, X_{j_q},) \bullet_\alpha X_{i_{k+1}}), \alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$+ \sum_{\substack{J, j_q < i_{k+1} \le p+q+r}} \sum_{\substack{L, l_p < h_{m+1} \le i_{p+r}, \\ i_{k+1}=h_n, n > m+1}} (-1)^{(J,I)} (-1)^{(L,H)} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_p}),)$$

$$\bullet_\alpha \alpha^{p+q}(X_{h_{m+1}}), \alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_{n-1}}), \alpha^p(\varphi(X_{j_1}, \ldots, X_{j_q},)$$

$$\bullet_\alpha X_{i_{k+1}}), \alpha^{p+q}(X_{h_{n+1}}, \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$+ \sum_{\substack{J, j_q < i_{k+1} \leq p+q+r}} \sum_{\substack{L, l_p < h_{m+1} \leq i_{p+r}, \\ i_{k+1} = l_s, s \leq p}} (-1)^{(J,I)} (-1)^{(L,H)} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_{s-1}}),$$

$$\varphi(X_{j_1}, \ldots, X_{j_q}, X_{j_q}, \,) \bullet_\alpha X_{i_{k+1}}, \alpha^q(X_{l_{s+1}}), \ldots, \alpha^q(X_{l_p}), \,) \bullet_\alpha \alpha^q(X_{h_{m+1}}), \alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$+ \sum_{\substack{J, j_q < i_{k+1} \leq p+q+r}} \sum_{\substack{L, i_{k+1} = h_n}} (-1)^{(J,I)} (-1)^{(L,H)} (-1)^r \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_{n-1}}), \alpha^p(\varphi(X_{j_1}, \ldots, X_{j_q}, \,) \bullet_\alpha X_{i_{k+1}}),$$

$$\alpha^{p+q}(X_{h_{n+1}}), \ldots, \alpha^{p+q}(X_{h_r}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_p}), \alpha^q(x)))$$

$$+ \sum_{\substack{J, j_q < i_{k+1} \leq p+q+r}} \sum_{\substack{L, i_{k+1} = l_s}} (-1)^{(J,I)} (-1)^{(L,H)} (-1)^r \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_r}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_{s-1}}),$$

$$\varphi(X_{j_1}, \ldots, X_{j_q}, \,) \bullet_\alpha X_{i_{k+1}}, \alpha^q(X_{l_{s+1}}), \ldots, \alpha^q(X_{l_p}), \alpha^q(x)))$$

$$+ \sum_J \sum_{\substack{L, l_p < h_{m+1} \leq i_{p+r}}} (-1)^{(J,I)} (-1)^{p+r} (-1)^{(L,H)} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_p}), \,) \bullet_\alpha (\alpha^q(X_{h_{m+1}}),$$

$$\alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^p(\varphi(X_{j_1}, \ldots, X_{j_q}, x)))$$

$$+ \sum_J \sum_L (-1)^{(J,I)} (-1)^{p+r} (-1)^{(L,H)} (-1)^r \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_r}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^{p+q}(X_{l_p}), \varphi(X_{j_1}, \ldots, X_{j_q}, x))).$$

Similarly one can compute D_2.

Let $A = \{a_1, a_2, \ldots, a_{p+q}\}_{a_1 < a_2 < \cdots < a_{p+q}} \subseteq N = \{1, \ldots, p+q+r\}$, $H = \{h_1, \ldots, h_r\}_{h_1 < h_2 < \cdots < h_r} = N \setminus A$,

and $J = \{j_1, \ldots, j_q\}_{j_1 < \cdots < j_q} \subseteq A$ and $L = \{l_1, \ldots, l_p\}_{l_1 < \cdots < l_p} = A \setminus J$. We have:

$$j_{[\varphi,\psi]^{3HL}}(\xi)(X_1, \ldots, X_{p+q+r}, x)$$

$$= \sum_{A, a_{p+q} < h_{m+1} \leq p+q+r} (-1)^{(A,H)} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), [\varphi,\psi]_\alpha^{3HL}(X_{a_1}, \ldots, X_{a_{p+q}},)$$

$$\bullet_\alpha X_{h_{m+1}}, \alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$+ \sum_A (-1)^{(A,H)}(-1)^r \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_r}), [\varphi,\psi]_\alpha^{3HL}(X_{a_1}, \ldots, X_{a_{p+q}}, x))$$

$$= \sum_{A, a_{p+q} < h_{m+1} \leq p+q+r} \sum_{L, lp < j_{t+1} \leq a_{p+q}} (-1)^{pq}(-1)^{(A,H)}(-1)^{(L,J)} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), \varphi(\alpha^p(X_{j_1}), \ldots, \alpha^p(X_{j_t}),$$

$$\psi(X_{l_1}, \ldots, X_{l_p},) \bullet_\alpha X_{j_{t+1}}, \alpha^p(X_{j_{t+2}}) \ldots, \alpha^p(X_{j_q}),) \bullet_\alpha \alpha^p(X_{h_{m+1}}), \alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$- \sum_{A, a_{p+q} < h_{m+1} \leq p+q+r} \sum_{J, jq < l_{s+1} \leq a_{p+q}} (-1)^{(A,H)}(-1)^{(J,L)} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_s}),$$

$$\varphi(X_{j_1}, \ldots, X_{j_q},) \bullet_\alpha X_{l_{s+1}}, \alpha^q(X_{l_{s+2}}) \ldots, \alpha^q(X_{l_p}),) \bullet_\alpha \alpha^q(X_{h_{m+1}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$+ \sum_{A, a_{p+q} < h_{m+1} \leq p+q+r} \sum_L (-1)^{(A,H)}(-1)^{(L,J)}(-1)^{pq}(-1)^q \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), \varphi(\alpha^p(X_{j_1}), \ldots, \alpha^p(X_{j_q}),)$$

$$\bullet_\alpha (\psi(X_{l_1}, \ldots, X_{l_p},) \bullet_\alpha X_{h_{m+1}}), \alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$- \sum_{A, a_{p+q} < h_{m+1} \leq p+q+r} \sum_{J} (-1)^{(A,H)} (-1)^{(J,L)} (-1)^{pq} (-1)^{P} \xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_m}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_p},)$$

$$\bullet_\alpha (\varphi(X_{j_1}, \ldots, X_{j_q},) \bullet_\alpha X_{h_{m+1}}), \alpha^{p+q}(X_{h_{m+2}}), \ldots, \alpha^{p+q}(X_{h_r}), \alpha^{p+q}(x))$$

$$+ \sum_{A} \sum_{L, l_p < j_{t+1} \leq a_{p+q}} (-1)^r (-1)^{pq} (-1)^{(A,H)} (-1)^{(L,J)} (\xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_r}), \varphi(\alpha^p(X_{j_1}), \ldots, \alpha^p(X_{j_t}),$$

$$\psi(X_{l_1}, \ldots, X_{l_p},) \bullet_\alpha X_{j_{t+1}}, \alpha^p(X_{j_{t+2}}), \ldots, \alpha^p(X_{j_q}), \alpha^p(x)))$$

$$+ \sum_{A} \sum_{L} (-1)^r (-1)^q (-1)^{pq} (-1)^{(A,H)} (-1)^{(L,J)} (\xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_r}), \varphi(\alpha^p(X_{j_1}), \ldots, \alpha^p(X_{j_q}), \psi(X_{l_1}, \ldots, X_{l_p}, x))$$

$$- \sum_{A} \sum_{J, J_q < l_{s+1} \leq a_{p+q}} (-1)^r (-1)^{(A,H)} (-1)^{(J,L)} (\xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_r}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_s}),$$

$$\varphi(X_{j_1}, \ldots, X_{j_q},) \bullet_\alpha X_{l_{s+1}}, \alpha^q(X_{l_{s+2}}), \ldots, \alpha^q(X_{l_p}), \alpha^q(x)))$$

$$- \sum_{A} \sum_{J} (-1)^p (-1)^r (-1)^{(A,H)} (-1)^{(J,L)} (\xi(\alpha^{p+q}(X_{h_1}), \ldots, \alpha^{p+q}(X_{h_r}), \psi(\alpha^q(X_{l_1}), \ldots, \alpha^q(X_{l_p}), \varphi(X_{j_1}, \ldots, X_{j_q}, x)).$$

By a straightforward verification, we obtain $D_1 - (-1)^{pq} D_2 = j^{\alpha}_{[\varphi, \psi]^{3HL}}(\xi)(X_1, \ldots, X_{p+q+r}, x)$. Hence the proof. $\blacksquare$

THEOREM 1.9. *The pair* $(C_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g}), [\cdot,\cdot]^{3HL})$ *is a graded Lie algebra.*

Proof. Let $\varphi \in C_{\alpha,\alpha}^q(\mathfrak{g},\mathfrak{g})$, $\psi \in C_{\alpha,\alpha}^p(\mathfrak{g},\mathfrak{g})$ and $\phi \in C_{\alpha,\alpha}^r(\mathfrak{g},\mathfrak{g})$.

(1) Skew-symmetry:

$$[\varphi,\psi]^{3HL} = (-1)^{pq} j_\varphi^\alpha(\psi) - j_\psi^\alpha(\varphi)$$
$$= (-1)^{pq+1}\big((-1)^{pq} j_\psi^\alpha(\varphi) - j_\varphi^\alpha(\psi)\big)$$
$$= -(-1)^{pq}[\psi,\varphi]^{3HL}.$$

(2) Graded Jacobi identity:

$$\circlearrowleft_{\varphi,\psi,\phi} (-1)^{qr}\big[\varphi, [\psi,\phi]^{3HL}\big]^{3HL}$$
$$= (-1)^{qp} j_{[\psi,\phi]^{3HL}}(\varphi) - (-1)^{qr} j_\varphi^\alpha\big([\psi,\phi]^{3HL}\big)$$
$$+ (-1)^{pr} j_{[\phi,\varphi]^{3HL}}(\psi) - (-1)^{pq} j_\psi^\alpha\big([\phi,\varphi]^{3HL}\big)$$
$$+ (-1)^{rq} j_{[\varphi,\psi]^{3HL}}(\phi) - (-1)^{rp} j_\phi^\alpha\big([\varphi,\psi]^{3HL}\big)$$
$$= (-1)^{qp} j_{[\psi,\phi]^{3HL}}(\varphi) - (-1)^{qr} j_\varphi^\alpha\big((-1)^{rp} j_\phi^\alpha(\psi) - j_\psi^\alpha(\phi)\big)$$
$$+ (-1)^{pr} j_{[\phi,\varphi]^{3HL}}(\psi) - (-1)^{pq} j_\psi^\alpha\big((-1)^{qr} j_\varphi^\alpha(\phi) - j_\phi^\alpha(\varphi)\big)$$
$$+ (-1)^{rq} j_{[\varphi,\psi]^{3HL}}(\phi) - (-1)^{rp} j_\phi^\alpha\big((-1)^{qp} j_\psi^\alpha(\varphi) - j_\varphi^\alpha(\psi)\big).$$

Organizing these terms leads to

$$\circlearrowleft_{\varphi,\psi,\phi}(-1)^{qr}\big[\varphi, [\psi,\phi]^{3HL}\big]^{3HL}$$
$$= (-1)^{pq} j_{[\psi,\phi]^{3HL}}(\varphi) + (-1)^{pq}\big(j_\psi^\alpha(j_\phi^\alpha(\varphi)) - (-1)^{rp} j_\phi^\alpha(j_\psi^\alpha(\varphi))\big)$$
$$+ (-1)^{rp} j_{[\phi,\varphi]^{3HL}}(\psi) + (-1)^{rp}\big(j_\phi^\alpha(j_\varphi^\alpha(\psi)) - (-1)^{qr} j_\varphi^\alpha(j_\phi^\alpha(\psi))\big)$$
$$+ (-1)^{qr} j_{[\varphi,\psi]^{3HL}}(\phi) + (-1)^{qr}\big(j_\varphi^\alpha(j_\psi^\alpha(\phi)) - (-1)^{qp} j_\psi^\alpha(j_\varphi^\alpha(\phi))\big)$$
$$= (-1)^{pq}\big([j_\psi^\alpha, j_\phi^\alpha] + j_{[\psi,\phi]^{3HL}}\big)(\varphi)$$
$$+ (-1)^{rp}\big([j_\phi^\alpha, j_\varphi^\alpha] + j_{[\phi,\varphi]^{3HL}}\big)(\psi)$$
$$+ (-1)^{qr}\big([j_\varphi^\alpha, j_\psi^\alpha] + j_{[\varphi,\psi]^{3HL}}\big)(\phi).$$

Using the previous lemma, we get

$$\circlearrowleft_{\varphi,\psi,\phi} (-1)^{qr}\big[\varphi, [\psi,\phi]^{3HL}\big]^{3HL} = 0.$$

∎

Remark 1.10.   The pair $(C_{\alpha,\alpha}(\mathfrak{g},\mathfrak{g}), \circ_\alpha)$ is a right symmetric graded algebra.

The previous structure of graded Lie algebra is useful to describe 3-Hom-Lie algebra structures as well as coboundary operators.

COROLLARY 1.11. *The maps* $\pi : \wedge^3\mathfrak{g} \to \mathfrak{g}$ *and* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *define a 3-Hom-Lie structure if and only if* $[\pi, \pi]^{3HL} = 0$.

Let $\mathfrak{g}$ be a 3-Hom-Lie algebra. Given $x_1, x_2 \in \mathfrak{g}$, define $ad : \wedge^2\mathfrak{g} \to gl(\mathfrak{g})$ by

$$ad_{x_1,x_2} y = [x_1, x_2, y].$$

Then, the pair (ad, α) defines a representation of the 3-Hom-Lie algebra $\mathfrak{g}$ on itself, which we call adjoint representation of $\mathfrak{g}$. The coboundary operator associated to this representation is denoted by $\delta_{\mathfrak{g}}$.

COROLLARY 1.12. *If* $\pi : \wedge^3\mathfrak{g} \to \mathfrak{g}$ *is a 3-Hom-Lie bracket, then we have*

$$[\pi, \varphi]^{3HL} = \delta_g(\varphi) \qquad \text{for all } \varphi \in C^p_{\alpha,\alpha}(\mathfrak{g}, \mathfrak{g}), \ p \geq 0.$$

2. GENERALIZED REPRESENTATIONS OF 3-HOM-LIE ALGEBRAS

In this section, we provide the Hom-type version of generalized representations of a 3-Lie algebras introduced in [19]. First, we show that a representation of a 3-Hom-Lie algebra will give rise to a canonical structure.

Let $\mathfrak{g}$ be a 3-Hom-Lie algebra and V be a vector space. Let $\rho : \wedge^2\mathfrak{g} \to gl(V)$ be a linear map. Then, it induces a linear map $\overline{\rho} : \wedge^3(\mathfrak{g} \oplus V) \to \mathfrak{g} \oplus V$ defined by

$$\overline{\rho}(x + u, y + v, z + w) = \rho(x, y)(w) + \rho(y, z)(u) + \rho(z, x)(v)$$

for all $x, y, z \in \mathfrak{g}$, $u, v, w \in V$. Consider the graded Lie algebra given in Theorem 1.9 associated to the vector space $\mathfrak{g} \oplus V$.

PROPOSITION 2.1. *A linear map* $\rho : \wedge^2\mathfrak{g} \to gl(V)$ *is a representation on a vector space* V *of the 3-Hom-Lie algebra* $\mathfrak{g}$ *with respect to* $A \in gl(V)$ *if and only if* $\pi + \overline{\rho}$ *is a canonical structure in the graded Lie algebra associated to* $\mathfrak{g} \oplus V$, *i.e.*

$$[\pi + \overline{\rho}, \pi + \overline{\rho}]^{3HL} = 0.$$

Proof. By Proposition 1.7, $\rho : \wedge^2 \mathfrak{g} \to gl(V)$ is a representation of $\mathfrak{g}$ if and only if $\mathfrak{g} \oplus V$ is a 3-Hom-Lie algebra, where the 3-Hom-Lie structure is exactly given by

$$[x + u, y + v, z + w]_\rho = [x, y, z] + \rho(x, y)(w) + \rho(y, z)(u) + \rho(z, x)(v)$$
$$= (\pi + \bar{\rho})(x + u, y + v, z + w),$$

and $\alpha_{\mathfrak{g} \oplus V} = \alpha + A$. Thus, by Lemma 1.11, $\rho : \wedge^2 \mathfrak{g} \to gl(V)$ is a representation of $\mathfrak{g}$ if and only if $\pi + \bar{\rho}$ is a canonical structure. ∎

The concept of representation of 3-Lie algebras introduced by Liu, Makhlouf and Sheng [19] is generalized to Hom-type algebras as follows.

DEFINITION 2.2. A *generalized representation* of a 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ with respect to $A \in gl(V)$ consists of linear maps $\rho : \wedge^2 \mathfrak{g} \to gl(V)$, $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$, such that

$$[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3HL} = 0,$$

where $\bar{\nu} : \wedge^3(\mathfrak{g} \oplus V) \to (\mathfrak{g} \oplus V)$ is induced by ν via

$$\bar{\nu}(x + u, y + v, z + w) = \nu(x)(v \wedge w) + \nu(y)(w \wedge u) + \nu(z)(u \wedge v)$$

for all $x, y, z \in \mathfrak{g}$, $u, v, w \in V$. We will refer to a generalized representation by $(V; \rho, \nu, A)$.

Remark 2.3. If $\nu = 0$, then we recover the usual definition of a representation of a 3-Hom-Lie algebra on a vector space V. If the dimension of the vector space V is 1, then ν must be zero. In this case, we only have the usual representation.

Given linear maps $\rho : \wedge^2 \mathfrak{g} \to End(V)$, $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$, and $A : V \to V$, define a trilinear bracket operation on $\mathfrak{g} \oplus V$ by

$$\begin{aligned}
[x + u, y + v, z + w]_{(\rho, \nu)} = {}& [x, y, z] + \rho(x, y)(w) + \rho(y, z)(u) \\
& + \rho(z, x)(v) + \nu(x)(v \wedge w) \\
& + \nu(y)(w \wedge u) + \nu(z)(u \wedge v).
\end{aligned} \tag{2.1}$$

THEOREM 2.4. *Let* $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ *be a 3-Hom-Lie algebra and* $(V; \rho, \nu, A)$ *a generalized representation of* $\mathfrak{g}$ *with respect to* A. *Then* $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu)}$, $\alpha_{\mathfrak{g} \oplus V} = \alpha + A)$ *is 3-Hom-Lie algebra, where* $[\cdot, \cdot, \cdot]_{(\rho, \nu)}$ *is given by* (2.1). *We call the 3-Hom-Lie algebra* $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu)}, \alpha_{\mathfrak{g} \oplus V} = \alpha + A)$ *the generalized semidirect product of* $\mathfrak{g}$ *and* V.

Proof. It follows from $[x+u, y+v, z+w](\rho, \nu) = (\pi + \overline{\rho} + \overline{\nu})(x+u, y+v, z+w)$ and Lemma 1.11. ∎

In the following, we give a characterization of a generalized representation of a 3-Hom-Lie algebra.

PROPOSITION 2.5. *Let* $\rho : \wedge^2 \mathfrak{g} \to End(V)$, $\nu : \mathfrak{g} \to Hom(\wedge^2 V, V)$ *and* $A : V \to V$ *be linear maps. They give rise to a generalized representation of a 3-Hom-Lie algebra* $\mathfrak{g}$ *with respect to* A *if and only if for all* $x_i \in \mathfrak{g}$, $v_j \in V$, *the following equalities hold:*

$$\rho(\alpha(x_1), \alpha(x_2))\rho(x_3, x_4) = \rho([x_1, x_2, x_3], \alpha(x_4)) \circ A \qquad (2.2)$$
$$- \rho([x_1, x_2, x_4], \alpha(x_3)) \circ A$$
$$+ \rho(\alpha(x_3), \alpha(x_4))\rho(x_1, x_2),$$

$$\rho([x_1, x_2, x_3], \alpha(x_4)) \circ A = \rho(\alpha(x_2), \alpha(x_3))\rho(x_1, x_4) \qquad (2.3)$$
$$+ \rho(\alpha(x_3), \alpha(x_1))\rho(x_2, x_4)$$
$$+ \rho(\alpha(x_1), \alpha(x_2))\rho(x_3, x_4),$$

$$\rho(\alpha(x_1), \alpha(x_2))\nu(x_3)(v_1, v_2) = \nu([x_1, x_2, x_3])(A(v_1), A(v_2)) \qquad (2.4)$$
$$+ \nu(\alpha(x_3))(\rho(x_1, x_2)v_1, A(v_2))$$
$$+ \nu(\alpha(x_3))(A(v_2), \rho(x_1, x_2)v_1),$$

$$\nu(\alpha(x_1))(A(v_1), \rho(x_2, x_3)v_2) = \nu(\alpha(x_3))(A(v_2), \rho(x_2, x_1)v_1) \qquad (2.5)$$
$$+ \nu(\alpha(x_2))(\rho(x_3, x_1)v_1, A(v_2))$$
$$+ \rho(\alpha(x_2), \alpha(x_3))\nu(x_1)(v_1, v_2),$$

$$\nu(\alpha(x_1))(A(v_1), \nu(x_2)(v_2, v_3)) = \nu(\alpha(x_2))(\nu(x_1)(v_1, v_2), A(v_3)) \qquad (2.6)$$
$$+ \nu(\alpha(x_2))(A(v_2), \nu(x_1)(v_1, v_3)),$$

$$\nu(\alpha(x_1))(\nu(x_2)(v_1, v_2), A(v_3)) = \nu(\alpha(x_2))(\nu(x_1)(v_1, v_2), A(v_3)). \qquad (2.7)$$

Proof. The quadruple $(V; \rho, \nu, A)$ is a generalized representation if and only if $[\pi + \overline{\rho} + \overline{\nu}, \pi + \overline{\rho} + \overline{\nu}]^{3HL} = 0$. By straightforward computations,

$$[\pi + \overline{\rho} + \overline{\nu}, \pi + \overline{\rho} + \overline{\nu}]^{3HL}(x_1, x_2, x_3, x_4, v) = 0$$

is equivalent to (2.2); and

$$[\pi + \overline{\rho} + \overline{\nu}, \pi + \overline{\rho} + \overline{\nu}]^{3HL}(x_1, v, x_2, x_3, x_4) = 0$$

is equivalent to (2.3). Other identities can be proved similarly. The details are omitted. ∎

Remark 2.6. By (2.2) and (2.3), the map ρ in a generalized representation $(V; \rho, \nu, A)$ gives rise to a usual representation in the sense of Definition 1.5. Conversely, for any representation ρ, $(V; \rho, \nu = 0, A)$ is a generalized representation.

DEFINITION 2.7. Let $(V_1; \rho_1, \nu_1, A_1)$ and $(V_2; \rho_2, \nu_2, A_2)$ be two generalized representations of a 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$. They are said to be *equivalent* if there exists an isomorphism of vector spaces $T : V_1 \to V_2$ such that

$$T\rho_1(x,y)(u) = \rho_2(x,y)(Tu), \ \ T\nu_1(x)(u,v) = \nu_2(x)(Tu, Tv), \ \ T \circ A_1 = A_2 \circ T$$

for all $x, y \in \mathfrak{g}$, $u, v \in V_1$. In terms of diagrams, we have

$$
\begin{array}{ccc}
\wedge^2\mathfrak{g} \times V_1 \xrightarrow{\rho_1} V_1 & \wedge^2 V_1 \times \mathfrak{g} & V_1 \xrightarrow{A_1} V_1 \\
\end{array}
$$

In the following, we provide a series of examples to illustrate the new concept of generalized representation and also a procedure to twist a generalized representation along linear maps.

EXAMPLE 2.8. Let $\mathfrak{g}$ be an abelian 3-Hom-Lie algebra. Define $\rho = 0$, and $\nu = \xi \otimes \pi$, where $\xi \in \mathfrak{g}^*$ and $\pi \in \mathrm{Hom}(\wedge^2 V \otimes V)$ is a Hom-Lie algebra structure on (V, π, A). Then $(V; \rho, \nu, A)$ is a generalized representation. In fact, since $\mathfrak{g}$ is abelian and $\rho = 0$, (2.2)-(2.5) hold naturally. Since π satisfies the Hom-Jacobi identity, (2.6) and (2.7) also hold.

PROPOSITION 2.9. *Let* $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ *be a 3-Hom-Lie algebra,* (V, ρ, ν, A) *be a generalized representation,* $\beta : \mathfrak{g} \to \mathfrak{g}$ *be an algebra morphism and* $B : V \to V$ *a linear map such that*

$$B \circ \rho(x_1, x_2) = \rho(\beta(x_1), \beta(x_2)) \circ B,$$

$$B \circ \nu(x) = \nu(\beta(x)) \circ (B \otimes B),$$

$$B \circ A = A \circ B.$$

Then $(V, \widetilde{\rho}, \widetilde{\nu}, B)$ is a generalized representation of 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\beta, \beta \circ \alpha)$ where

$$[\cdot, \cdot, \cdot]_\beta = [\cdot, \cdot, \cdot] \circ \beta^{\otimes 3}, \qquad \widetilde{\rho} = B \circ \rho, \ \widetilde{\nu}(x) = B \circ \nu(x).$$

Proof. We have to show that $\widetilde{\rho}$ and $\widetilde{\nu}$ satisfy Eqs. (2.2)-(2.7).

Let $x_1, x_2, x_3 \in \mathfrak{g}$ and $v_1, v_2 \in V$,

$\widetilde{\rho}(\beta \circ \alpha(x_1), \beta \circ \alpha(x_2))\widetilde{\nu}(x_3)(v_1, v_2) - \widetilde{\nu}([x_1, x_2, x_3]_\beta)(B \circ A(v_1), B \circ A(v_2))$

$\qquad - \widetilde{\nu}(\beta \circ \alpha(x_3))(\widetilde{\rho}(x_1, x_2)v_1, B \circ A(v_2))$

$\qquad - \widetilde{\nu}(\beta \circ \alpha(x_3))(B \circ A(v_2), \ widetilde\rho(x_1, x_2)v_1)$

$= B \circ \rho(\beta \circ \alpha(x_1), \beta \circ \alpha(x_2)) \circ B \circ \nu(x_3)(v_1, v_2)$

$\qquad - B \circ \nu(\beta \circ [x_1, x_2, x_3]) \circ (B \otimes B)(A(v_1), A(v_2))$

$\qquad - B \circ \nu(\beta \circ \alpha(x_3))(B \circ \rho(x_1, x_2)v_1, B \circ A(v_2))$

$\qquad - B \circ \nu(\beta\alpha(x_3))(B \circ A(v_2), B \circ \rho(x_1, x_2)v_1))$

$= B^2 \circ (\rho(\alpha(x_1), \alpha(x_2)) \circ \nu(x_3)(v_1, v_2) - \nu([x_1, x_2, x_3])(A(v_1), A(v_2))$

$\qquad - \nu(\alpha(x))(\rho(x_1, x_2)v_1, A(v_2)) - \nu(\alpha(x_3))[2pt](A(v_2), \rho(x_1, x_2)v_1)) = 0,$

$\widetilde{\nu}(\beta \circ \alpha(x_1))(B \circ A(v_1), \widetilde{\rho}(x_2, x_3)v_2) - \widetilde{\nu}(\beta \circ \alpha(x_3))(B \circ A(v_2), \widetilde{\rho}(x_2, x_1)v_1)$

$\qquad - \widetilde{\nu}(\beta \circ \alpha(x_2))(\widetilde{\rho}(x_3, x_1)v_1, B \circ A(v_2))$

$\qquad - \widetilde{\rho}(\beta \circ \alpha(x_2), \beta \circ \alpha(x_3))\widetilde{\nu}(x_1)(v_1, v_2)$

$= B \circ \nu(\beta \circ \alpha(x_1))(B \circ A(v_1), B \circ \rho(x_2, x_3)v_2)$

$\qquad - B \circ \nu(\beta \circ \alpha(x_3))(B \circ A(v_2), B \circ \rho(x_2, x_1)v_1)$

$\qquad - B \circ \nu(\beta \circ \alpha(x_2))(B \circ \rho(x_3, x_1)v_1, B \circ A(v_2))$

$\qquad - B \circ \rho(\beta \circ \alpha(x_2), \beta \circ \alpha(x_3)) \circ B \circ \nu(x_1)(v_1, v_2)$

$= B^2 \circ (\nu(\alpha(x_1))(A(v_1), \rho(x_2, x_3)v_2) - \nu(\alpha(x_3))(v_2, \rho(x_2, x_1)v_1)$

$\qquad - \nu(\alpha(x_2))(\rho(x_3, x_1)v_1, A(v_2)) - \rho(\alpha(x_2), \alpha(x_3)) \circ \nu(x_1)(v_1, v_2)) = 0.$

Then identities (2.4) and (2.5) are proved. One similarly proves identities (2.6) and (2.7). ∎

COROLLARY 2.10. *Let* $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ *be a 3-Lie algebra,* (V, ρ, ν) *be a generalized representation,* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *be an algebra morphism and* $A : V \to V$ *be a*

linear map such that for all $x_1, x_2 \in \mathfrak{g}$ *and* $v_1, v_2 \in V$,

$$A \circ \rho(x_1, x_2) = \rho(\alpha(x_1), \alpha(x_2)) \circ A,$$
$$A \circ \nu(x)(v_1, v_2) = \nu(\alpha(x)) \circ (A \otimes A)(v_1, v_2).$$

Then $(V, \widetilde{\rho} := A \circ \rho, \widetilde{\nu} := A \circ \nu, A)$ *is a generalized representation of the 3-Hom-Lie algebra* $(L, [\cdot, \cdot, \cdot]_\alpha := \alpha \circ [\cdot, \cdot, \cdot], \alpha)$.

EXAMPLE 2.11. Let $\mathfrak{g}$ be the 3-dimensional 3-Lie algebra defined with respect to a basis $\{e_1, e_2, e_3\}$ by the skew-symmetric bracket $[e_1, e_2, e_3] = e_1$. Let V be a 2-dimensional vector space and $\{v_1, v_2\}$ its basis. We have a representation defined by the following maps (ρ, μ), given with respect to previous bases by

$$\rho(e_1, e_2)(v_1) = 0, \qquad \rho(e_1, e_2)(v_2) = v_1, \qquad \rho(e_1, e_3)(v_1) = 0,$$
$$\rho(e_1, e_3)(v_2) = r_1 v_1, \qquad \rho(e_2, e_3)(v_1) = v_1, \qquad \rho(e_2, e_3)(v_2) = r_2 v_1,$$
$$\nu(e_1)(v_1, v_2) = 0, \qquad \nu(e_2)(v_1, v_2) = s v_1, \qquad \nu(e_3)(v_1, v_2) = s r_1 v_1,$$

where r_1, r_2, s are parameters in $\mathbb{K}$.

Let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be an algebra morphism and $A \in gl(V)$ defined respectively by:

$$\alpha(e_1) = \lambda e_1, \qquad \alpha(e_2) = e_2, \qquad \alpha(e_3) = e_3,$$
$$A(v_1) = \lambda v_1, \qquad A(v_2) = r_2 v_1 + v_2,$$

where λ is a parameter in $\mathbb{K}$. They satisfy

$$A \circ \rho(x_1, x_2) = \rho(\alpha(x_1), \alpha(x_2)) \circ A,$$
$$A \circ \nu(x)(u_1, u_2) = \nu(\alpha(x)) \circ (A \otimes A)(u_1, u_2),$$

where x, x_1, x_2 are in $\mathfrak{g}$ and u_1, u_2 in V.

Then, using the Twist procedure, $(V; \widetilde{\rho}, \widetilde{\nu}, A)$ is a generalized representation of the 3-Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\alpha, \alpha)$. More precisely, we have

$$[e_1, e_2, e_3]_\alpha = [\alpha(e_1), \alpha(e_2), \alpha(e_3)] = \lambda e_1,$$
$$\widetilde{\rho}(e_1, e_2)(v_1) = 0, \qquad \widetilde{\rho}(e_1, e_2)(v_2) = \lambda v_1,$$
$$\widetilde{\rho}(e_1, e_3)(v_1) = 0, \qquad \widetilde{\rho}(e_1, e_3)(v_2) = r_1 r_2(\lambda - 1)v_1 - r_1 v_2,$$
$$\widetilde{\rho}(e_2, e_3)(v_1) = \lambda v_1, \qquad \widetilde{\rho}(e_2, e_3)(v_2) = r_2 \lambda v_1,$$
$$\widetilde{\nu}(e_1)(v_1, v_2) = 0, \qquad \widetilde{\nu}(e_2)(v_1, v_2) = s \lambda v_1, \qquad \widetilde{\nu}(e_3)(v_1, v_2) = s r_1 \lambda v_1.$$

EXAMPLE 2.12. Let $\mathfrak{g}$ be the 4-dimensional 3-Lie algebra defined, with respect to a basis $\{e_1, e_2, e_3, e_4\}$, by the skew-symmetric brackets

$$[e_1, e_2, e_4] = e_3, \qquad [e_1, e_3, e_4] = e_2, \qquad [e_2, e_3, e_4] = e_1.$$

Every generalized representation $(V; \rho, \nu)$, on a 2-dimensional vector space V with trivial ρ, of $\mathfrak{g}$ is given by one of the following maps ν defined, with respect to a basis $\{v_1, v_2\}$ of V, by

$$\nu(e_1)(v_1, v_2) = 0, \qquad \nu(e_2)(v_1, v_2) = 0,$$
$$\nu(e_3)(v_1, v_2) = 0, \qquad \nu(e_4)(v_1, v_2) = s_1 v_1 + s_2 v_2,$$

where s_1, s_2 are parameters in $\mathbb{K}$, and $s_1 s_2 \neq 0$.

Let $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a 3-Lie algebra morphism and $A : V \to V$ be a linear map, defined respectively by

$$\alpha(e_1) = a_1 e_1, \quad \alpha(e_2) = -a_1 e_2, \quad \alpha(e_3) = a_1 e_3, \quad \alpha(e_4) = \frac{-1}{a_1} e_4,$$

$$A(v_1) = -a_1 v_1, \qquad A(v_2) = a_2 v_1 + \frac{a_2 s_2 - a_1 s_1}{s_1} v_2,$$

where a_1, a_2, s_1, s_2 are parameters in $\mathbb{K}$ such that, $a_1 s_1 s_2 \neq 0$.

They satisfy $A \circ \nu(x) = \nu(\alpha(x)) \circ (A \otimes A)$. Therefore, using the Twist procedure, $(V; \widetilde{\rho}, \widetilde{\nu}, A)$ is a generalized representation of the 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot]_\alpha, \alpha)$ with trivial $\widetilde{\rho}$. Namely, we have

$$[e_1, e_2, e_4]_\alpha = a_1 e_3, \qquad [e_1, e_3, e_4]_\alpha = -a_1 e_2, \qquad [e_2, e_3, e_4]_\alpha = a_1 e_1,$$

and

$$\widetilde{\nu}(e_1)(v_1, v_2) = 0, \qquad \widetilde{\nu}(e_2)(v_1, v_2) = 0, \qquad \widetilde{\nu}(e_3)(v_1, v_2) = 0,$$

$$\widetilde{\nu}(e_4)(v_1, v_2) = (a_2 s_2 - a_1 s_1) v_1 + \left(\frac{a_2 s_2^2}{s_1} - a_1 s_2 \right) v_2.$$

3. NEW COHOMOLOGY COMPLEX OF 3-HOM-LIE ALGEBRAS

Based on the generalized representations defined in the previous section, we introduce a new type of cohomology for 3-Hom-Lie algebras.

Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ be a 3-Hom-Lie algebra and $(V; \rho, \nu, A)$ be a generalized representation of $\mathfrak{g}$. We set $\widetilde{C}^p_{\alpha+A, A}(\mathfrak{g} \oplus V, V)$ to be the set of $(p+1)$-Hom-cochains, which are defined as a subset of $C^p_{\alpha+A, A}(\mathfrak{g} \oplus V, V)$ such that

$$C^p_{\alpha+A, A}(\mathfrak{g} \oplus V, V) = \widetilde{C}^p_{\alpha+A, A}(\mathfrak{g} \oplus V, V) \oplus C^p_A(V, V).$$

Elements of $C_{\alpha+A,A}^p(\mathfrak{g} \oplus V, V)$ are of the form

$$\varphi : \wedge^2(\mathfrak{g} \oplus V) \otimes \overset{(p \text{ times})}{\cdots} \otimes \wedge^2 (\mathfrak{g} \oplus V) \wedge (\mathfrak{g} \oplus V) \to V.$$

By direct calculation, we have

$$[\pi + \bar{\rho} + \bar{\nu}, \widetilde{C}_{\alpha+A,A}^\bullet(\mathfrak{g} \oplus V, V)] \subseteq \widetilde{C}_{\alpha+A,A}^{\bullet+1}(\mathfrak{g} \oplus V, V).$$

Define $d : \widetilde{C}_{\alpha+A,A}^p(\mathfrak{g} \oplus V, V) \to \widetilde{C}_{\alpha+A,A}^{p+1}(\mathfrak{g} \oplus V, V)$ by

$$d(\varphi) := [\pi + \bar{\rho} + \bar{\nu}, \varphi]^{3HL}, \qquad \varphi \in \widetilde{C}_{\alpha+A,A}^p(\mathfrak{g} \oplus V, V).$$

THEOREM 3.1. *Let* $(V; \rho, \nu, A)$ *be a generalized representation of a 3-Hom-Lie algebra* $\mathfrak{g}$. *Then* $d \circ d = 0$. *Thus, we obtain a new cohomology complex, where the space of p-Hom-cochains is given by* $\widetilde{C}_{\alpha+A,A}^{p-1}(\mathfrak{g} \oplus V, V)$.

Proof. By the graded Jacobi identity, for any $\varphi \in \widetilde{C}_{\alpha+A,A}^{p-1}(\mathfrak{g} \oplus V, V)$, one obtains

$$d \circ d(\varphi) := [\pi + \bar{\rho} + \bar{\nu}, [\pi + \bar{\rho} + \bar{\nu}, \varphi]^{3HL}]^{3HL}$$

$$= \frac{1}{2}[[\pi + \bar{\rho} + \bar{\nu}, \pi + \bar{\rho} + \bar{\nu}]^{3HL}, \varphi]^{3HL} = 0. \qquad \blacksquare$$

An element $\varphi \in \widetilde{C}_{\alpha+A,A}^{p-1}(\mathfrak{g} \oplus V, V)$ is called a p-cocycle if $d(\varphi) = 0$; it is called a p-coboundary if there exists $f \in \widetilde{C}_{\alpha+A,A}^{p-2}(\mathfrak{g} \oplus V, V)$ such that $\varphi = d(f)$.

Denote by $\mathcal{Z}_{3HL}^p(\mathfrak{g}; V)$ and $\mathcal{B}_{3HL}^p(\mathfrak{g}; V)$ the sets of p-cocycles and p-coboundaries respectively. By Theorem 3.1, we have $\mathcal{B}_{3HL}^p(\mathfrak{g}; V) \subset \mathcal{Z}^p(\mathfrak{g}; V)$. We define the p-th cohomolgy group $\mathcal{H}_{3HL}^p(\mathfrak{g}; V)$ to be $\mathcal{Z}_{3HL}^p(\mathfrak{g}; V)/\mathcal{B}_{3HL}^p(\mathfrak{g}; V)$.

The following proposition provides a relationship between this new cohomology and the one given by (1.5).

PROPOSITION 3.2. *There is a forgetful map from* $\mathcal{H}_{3HL}^p(\mathfrak{g}; V)$ *to* $H_{3HL}^p(\mathfrak{g}; V)$.

Proof. It is obvious that $C_{\alpha,A}^p(\mathfrak{g}, V) \subseteq \widetilde{C}_{\alpha+A,A}^p(\mathfrak{g} \oplus V, V)$. By direct calculation, for $X_i \in \wedge^2\mathfrak{g}$, $z \in \mathfrak{g}$, we have

$$d(\varphi)(X_1, \ldots, X_{p+1}, z) = \delta_\rho(\varphi)(X_1, \ldots, X_{p+1}, z), \qquad \varphi \in C_{\alpha,A}^p(\mathfrak{g}, V),$$

where δ_ρ is the coboundary operator given by (1.4). Thus, the natural projection from $\widetilde{C}_{\alpha+A,A}^p(\mathfrak{g} \oplus V, V)$ to $C_{\alpha,A}^p(\mathfrak{g}, V)$ induces a forgetful map from $\mathcal{H}_{3HL}^p(\mathfrak{g}; V)$ to $H_{3HL}^p(\mathfrak{g}; V)$. $\blacksquare$

In the sequel, we give some characterization of low dimensional cocycles.

PROPOSITION 3.3. *A linear map $\varphi \in \operatorname{Hom}(\mathfrak{g}, V)$ is a 1-cocycle if only if for all $x_1, x_2, x_3 \in \mathfrak{g}$, $v \in V$, the following identities hold:*

$$\varphi \circ \alpha = A \circ \varphi,$$

$$\nu(x_1)(\varphi(x_2), v) - \nu(x_2)(\varphi(x_1), v) = 0,$$

$$\varphi([x_1, x_2, x_3]) - \rho(\alpha(x_1), \alpha(x_2))(\varphi(x_3))$$

$$-\rho(\alpha(x_2), \alpha(x_3))(\varphi(x_1)) - \rho(\alpha(x_3), \alpha(x_1))(\varphi(x_2)) = 0.$$

Proof. For $\varphi \in \operatorname{Hom}(\mathfrak{g}, V)$ satisfying $\varphi \circ \alpha = A \circ \varphi$, we have

$$d(\varphi)(x_1, x_2, v) = \nu(x_1)(\varphi(x_2), v) - \nu(x_2)(\varphi(x_1), v),$$

and

$$d(\varphi)(x_1, x_2, x_3) = \delta_\rho(\varphi)(x_1, x_2, x_3)$$
$$= \rho(\alpha(x_1), \alpha(x_2))(\varphi(x_3)) + \rho(\alpha(x_2), \alpha(x_3))(\varphi(x_1))$$
$$+ \rho(\alpha(x_3), \alpha(x_1))(\varphi(x_2)) - \varphi([x_1, x_2, x_3]).$$ ∎

PROPOSITION 3.4. *A 2-cochain $\varphi_1 + \varphi_2 + \varphi_3 \in \widetilde{C}^1_{\alpha,A}(\mathfrak{g} \oplus V, V)$, where $\varphi_1 \in \operatorname{Hom}(\wedge^2 V \wedge \mathfrak{g}, V)$, $\varphi_2 \in \operatorname{Hom}(\wedge^2 \mathfrak{g} \wedge V, V)$, $\varphi_3 \in \operatorname{Hom}(\wedge^3 \mathfrak{g}, V)$, is a 2-cocycle if and only if for all $x_i \in \mathfrak{g}, v_j \in V$ and $v \in V$, the following identities hold:*

$$0 = -\rho(\alpha(x_1), \alpha(x_2))(\varphi_3(x_3, x_4, x_5)) - \varphi_3(\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]) \quad (3.1)$$
$$+ \rho(\alpha(x_4), \alpha(x_5))(\varphi_3(x_1, x_2, x_3)) + \varphi_3([x_1, x_2, x_3], \alpha(x_4), \alpha(x_5))$$
$$+ \rho(\alpha(x_5), \alpha(x_3))(\varphi_3(x_1, x_2, x_4)) + \varphi_3(\alpha(x_3), [x_1, x_2, x_4], \alpha(x_5))$$
$$+ \rho(\alpha(x_3), \alpha(x_4))(\varphi_3(x_1, x_2, x_5)) + \varphi_3(\alpha(x_3), \alpha(x_4), [x_1, x_2, x_5]),$$
$$0 = \nu(\alpha(x_4))(\varphi_3(x_1, x_2, x_3), A(v)) + \nu(\alpha(x_3))(A(v), \varphi_3(x_1, x_2, x_4)) \quad (3.2)$$
$$+ \rho(\alpha(x_1), \alpha(x_2))(\varphi_2(x_3, x_4, v)) - \rho(\alpha(x_3), \alpha(x_4))(\varphi_2(x_1, x_2, v))$$
$$- \varphi_2([x_1, x_2, x_3], \alpha(x_4), A(v)) - \varphi_2(\alpha(x_3), [x_1, x_2, x_4], A(v)),$$
$$0 = \nu(\alpha(x_1))(A(v), \varphi_3(x_2, x_3, x_4)) + \rho(\alpha(x_3), \alpha(x_4))(\varphi_2(x_1, x_2, v)) \quad (3.3)$$
$$- \rho(\alpha(x_2), \alpha(x_4))(\varphi_2(x_1, x_3, v)) + \rho(\alpha(x_2), \alpha(x_3))(\varphi_2(x_1, x_4, v))$$
$$+ \varphi_2(\alpha(x_3), \alpha(x_4), \rho(x_1, x_2)(v)) - \varphi_2(\alpha(x_2), \alpha(x_4), \rho(x_1, x_3)(v))$$
$$+ \varphi_2(\alpha(x_2), \alpha(x_3), \rho(x_1, x_4)(v)) - \varphi_2(\alpha(x_1), [x_2, x_3, x_4], A(v)),$$

$$0 = \nu(\alpha(x_3))(A(v_2), \varphi_2(x_1, x_2, v_1)) + \nu(\alpha(x_3))(\varphi_2(x_1, x_2, A(v_2)), v_1) \quad (3.4)$$
$$+ \varphi_2(\alpha(x_1), \alpha(x_2), \nu(x_3)(v_1, v_2)) + \rho(\alpha(x_1), \alpha(x_2))(\varphi_1(v_1, v_2, x_3))$$
$$- \varphi_1(\rho(x_1, x_2)(v_1), A(v_2), \alpha(x_3)) - \varphi_1(A(v_1), \rho(x_1, x_2)(v_2), \alpha(x_3))$$
$$- \varphi_1(A(v_1), A(v_2), [x_1, x_2, x_3]),$$

$$0 = \nu(\alpha(x_3))(A(v_2), \varphi_2(x_2, x_1, v_1)) + \nu(\alpha(x_2))(\varphi_2(x_3, x_1, v_1), A(v_2)) \quad (3.5)$$
$$- \nu(\alpha(x_1))(A(v_1), \varphi_2(x_2, x_3, v_2)) + \varphi_2(\alpha(x_2), \alpha(x_3), \nu(x_1)(v_1, v_2))$$
$$+ \rho(\alpha(x_2), \alpha(x_3))(\varphi_1(v_1, v_2, x_1)) + \varphi_1(\rho(x_1, x_2)(v_1), A(v_2), \alpha(x_3))$$
$$- \varphi_1(A(v_1), \rho(x_2, x_3)(v_2), \alpha(x_1)) + \varphi_1(A(v_2), \rho(x_1, x_3)(v_1), \alpha(x_2)),$$

$$0 = \varphi_2(\alpha(x_1), \alpha(x_3), \nu(x_2)(v_1, v_2)) - \varphi_2(\alpha(x_2), \alpha(x_3), \nu(x_1)(v_1, v_2)) \quad (3.6)$$
$$- \varphi_2(\alpha(x_1), \alpha(x_2), \nu(x_3)(v_1, v_2)) + \rho(\alpha(x_1), \alpha(x_3))(\varphi_1(v_1, v_2, x_2))$$
$$- \rho(\alpha(x_1), \alpha(x_2))(\varphi_1(v_1, v_2, x_3)) - \rho(\alpha(x_2), \alpha(x_3))(\varphi_1(v_1, v_2, x_1))$$
$$+ \varphi_1(A(v_1), A(v_2), [x_1, x_2, x_3]),$$

$$0 = -\nu(\alpha(x_2))(\varphi_1(v_1, v_2, x_1), A(v_3)) - \nu(\alpha(x_2))(A(v_2), \varphi_1(v_1, v_3, x_1)) \quad (3.7)$$
$$+ \nu(\alpha(x_1))(A(v_1), \varphi_1(v_2, v_3, x_2)) - \varphi_1(\nu(x_1)(v_1, v_2), A(v_3), \alpha(x_2))$$
$$- \varphi_1(A(v_2), \nu(x_1)(v_1, v_3), \alpha(x_2)) + \varphi_1(A(v_1), \nu(x_2)(v_2, v_3), \alpha(x_1)),$$

$$0 = \nu(\alpha(x_2))(\varphi_1(v_1, v_2, x_1), A(v_3)) - \nu(\alpha(x_1))(A(v_3), \varphi_1(v_1, v_2, x_2)) \quad (3.8)$$
$$+ \varphi_1(\nu(x_1)(v_1, v_2), A(v_3), \alpha(x_2)) - \varphi_1(\nu(x_2)(v_1, v_2), A(v_3), \alpha(x_1)).$$

Proof. For $\varphi_3 \in \mathrm{Hom}(\wedge^3 \mathfrak{g}, V)$, we have

$$d(\varphi_3)(x_1, x_2, x_3, x_4, x_5) = \rho(\alpha(x_1), \alpha(x_2))(\varphi_3(x_3, x_4, x_5))$$
$$+ \varphi_3(\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5])$$
$$- \rho(\alpha(x_4), \alpha(x_5))(\varphi_3(x_1, x_2, x_3))$$
$$- \varphi_3([x_1, x_2, x_3], \alpha(x_4), \alpha(x_5))$$
$$- \rho(\alpha(x_5), \alpha(x_3))(\varphi_3(x_1, x_2, x_4))$$
$$- \varphi_3(\alpha(x_3), [x_1, x_2, x_4], \alpha(x_5))$$
$$- \rho(\alpha(x_3), \alpha(x_4))(\varphi_3(x_1, x_2, x_5))$$
$$- \varphi_3(\alpha(x_3), \alpha(x_4), [x_1, x_2, x_5]),$$
$$d(\varphi_3)(x_1, x_2, x_3, x_4, v) = \nu(\alpha(x_4))(\varphi_3(x_1, x_2, x_3), A(v))$$
$$+ \nu(\alpha(x_3))(A(v), \varphi_3(x_1, x_2, x_4)),$$
$$d(\varphi_3)(x_1, v, x_2, x_3, x_4) = \nu(\alpha(x_1))(A(v), \varphi_3(x_2, x_3, x_4)).$$

For $\varphi_2 \in \text{Hom}(\wedge^2 \mathfrak{g} \wedge V, V)$, we have

$$
\begin{aligned}
d(\varphi_2)(x_1, x_2, x_3, x_4, v) = {} & \rho(\alpha(x_1), \alpha(x_2))(\varphi_2(x_3, x_4, v)) \\
& - \rho(\alpha(x_3), \alpha(x_4))(\varphi_2(x_1, x_2, v)) \\
& - \varphi_2([x_1, x_2, x_3], \alpha(x_4), A(v)) \\
& - \varphi_2(\alpha(x_3), [x_1, x_2, x_4], A(v)),
\end{aligned}
$$

$$
\begin{aligned}
d(\varphi_2)(x_1, v, x_2, x_3, x_4) = {} & \rho(\alpha(x_3), \alpha(x_4))(\varphi_2(x_1, x_2, v)) \\
& - \rho(\alpha(x_2), \alpha(x_4))(\varphi_2(x_1, x_3, v)) \\
& + \rho(\alpha(x_2), \alpha(x_3))(\varphi_2(x_1, x_4, v)) \\
& + \varphi_2(\alpha(x_3), \alpha(x_4), \rho(x_1, x_2)(v)) \\
& - \varphi_2(\alpha(x_2), \alpha(x_4), \rho(x_1, x_3)(v)) \\
& + \varphi_2(\alpha(x_2), \alpha(x_3), \rho(x_1, x_4)(v)) \\
& - \alpha_2(x_1, [x_2, x_3, x_4], v),
\end{aligned}
$$

$$
\begin{aligned}
d(\varphi_2)(x_1, x_2, v_1, v_2, x_3) = {} & \nu(\alpha(x_3))(A(v_2), \varphi_2(x_1, x_2, v_1)) \\
& + \nu(\alpha(x_3))(\varphi_2(x_1, x_2, v_2), A(v_1)) \\
& + \varphi_2(\alpha(x_1), \alpha(x_2), \nu(x_3)(v_1, v_2)),
\end{aligned}
$$

$$
\begin{aligned}
d(\varphi_2)(x_1, v_1, x_2, v_2, x_3) = {} & \nu(\alpha(x_3))(A(v_2), \varphi_2(x_2, x_1, v_1)) \\
& + \nu(\alpha(x_2))(\varphi_2(x_3, x_1, v_1), A(v_2)) \\
& - \nu(\alpha(x_1))(A(v_1), \varphi_2(x_2, x_3, v_2)) \\
& + \varphi_2(\alpha(x_2), \alpha(x_3), \nu(x_1)(v_1, v_2)),
\end{aligned}
$$

$$
\begin{aligned}
d(\varphi_2)(v_1, v_2, x_1, x_2, x_3) = {} & \varphi_2(\alpha(x_1), \alpha(x_3), \nu(x_2)(v_1, v_2)) \\
& - \varphi_2(\alpha(x_2), \alpha(x_3), \nu(x_1)(v_1, v_2)) \\
& - \varphi_2(\alpha(x_1), \alpha(x_2), \nu(x_3)(v_1, v_2)).
\end{aligned}
$$

For $\varphi_1 \in \text{Hom}(\wedge^2 V \wedge \mathfrak{g}, V)$, we have

$$
\begin{aligned}
d(\varphi_1)(x_1, x_2, v_1, v_2, x_3) = {} & \rho(\alpha(x_1), \alpha(x_2))(\varphi_1(v_1, v_2, x_3)) \\
& - \varphi_1(\rho(x_1, x_2)(v_1), A(v_2), \alpha(x_3)) \\
& - \varphi_1(A(v_1), \rho(x_1, x_2)(v_2), \alpha(x_3)) \\
& - \varphi_1(A(v_1), A(v_2), [x_1, x_2, x_3]),
\end{aligned}
$$

$$
\begin{aligned}
d(\varphi_1)(x_1, v_1, x_2, v_2, x_3) = {} & \rho(\alpha(x_2), \alpha(x_3))(\varphi_1(v_1, v_2, x_1)) \\
& + \varphi_1(\rho(x_1, x_2)(v_1), A(v_2), \alpha(x_3)) \\
& - \varphi_1(A(v_1), \rho(x_2, x_3)(v_2), \alpha(x_1)) \\
& + \varphi_1(A(v_2), \rho(x_1, x_3)(v_1), \alpha(x_2)),
\end{aligned}
$$

$$d(\varphi_1)(v_1, v_2, x_1, x_2, x_3) = \rho(\alpha(x_1), \alpha(x_3))(\varphi_1(v_1, v_2, x_2))$$
$$- \rho(\alpha(x_1), \alpha(x_2))(\varphi_1(v_1, v_2, x_3))$$
$$- \rho(\alpha(x_2), \alpha(x_3))(\varphi_1(v_1, v_2, x_1))$$
$$+ \varphi_1(A(v_1), A(v_2), [x_1, x_2, x_3]),$$

$$d(\varphi_1)(x_1, v_1, v_2, v_3, x_2) = -\nu(\alpha(x_2))(\varphi_1(v_1, v_2, x_1), A(v_3))$$
$$- \nu(\alpha(x_2))(A(v_2), \varphi_1(v_1, v_3, x_1))$$
$$+ \nu(\alpha(x_1))(A(v_1), \varphi_1(v_2, v_3, x_2))$$
$$- \varphi_1(\nu(x_1)(v_1, v_2), A(v_3), \alpha(x_2))$$
$$- \varphi_1(A(v_2), \nu(x_1)(v_1, v_3), \alpha(x_2))$$
$$+ \varphi_1(A(v_1), \nu(x_2)(v_2, v_3), \alpha(x_1)),$$

$$d(\varphi_1)(v_1, v_2, x_1, x_2, v_3) = \nu(\alpha(x_2))(\varphi_1(v_1, v_2, x_1), A(v_3))$$
$$- \nu(\alpha(x_1))(A(v_3), \varphi_1(v_1, v_2, x_2))$$
$$+ \varphi_1(\nu(x_1)(v_1, v_2), A(v_3), \alpha(x_2))$$
$$- \varphi_1(\nu(x_2)(v_1, v_2), A(v_3), \alpha(x_1)).$$

Thus, $d(\varphi_1 + \varphi_2 + \varphi_3) = 0$ if and only if Eqs. (3.1)-(3.8) hold. ∎

In the following we provide an example of computation of 2-cocycles of a 3-dimensional 3-Hom-Lie algebra.

EXAMPLE 3.5. Let $(\mathfrak{g}, [\cdot, \cdot, \cdot], \alpha)$ be the 3-dimensional 3-Hom-Lie algebra defined, with respect to a basis $\{e_1, e_2, e_3\}$, by $[e_1, e_2, e_3] = a_1 e_1$, $\alpha(e_1) = a_1 e_1$, $\alpha(e_2) = a_2 e_2$, $\alpha(e_3) = \frac{1}{a_2} e_3$. Let V be a 2-dimensional vector space, $\{v_1, v_2\}$ its basis and $A \in gl(V)$ defined by: $A(v_1) = a_1 v_1$, $A(v_2) = \frac{a_2 a_3}{a_1} v_2$, where a_1, a_2, a_3 are parameters in $\mathbb{K}$.

We consider the generalized representation (V, ρ, ν, A), where ρ and ν are defined with respect to the basis by

$$\rho(e_1, e_2)(v_1) = 0, \qquad \rho(e_1, e_2)(v_2) = 0, \qquad \rho(e_1, e_3)(v_1) = 0,$$
$$\rho(e_1, e_3)(v_2) = r_2 a_1 v_1, \qquad \rho(e_2, e_3)(v_1) = a_1 v_1, \qquad \rho(e_2, e_3)(v_2) = \frac{r_1 a_2 a_3}{a_1} v_2,$$
$$\nu(e_1)(v_1, v_2) = 0, \qquad \nu(e_2)(v_1, v_2) = 0, \qquad \nu(e_3)(v_1, v_2) = s_1 a_1 v_1,$$

with s_1, r_1, r_2 parameters in $\mathbb{K}$ and $a_1 a_2 s_1 \neq 0$.

We have the following 2-cocycles: $\varphi_1 = 0$, $\varphi_3 = 0$ and φ_2 defined as

$$\varphi_2(e_1, e_2, v_1) = 0, \qquad \varphi_2(e_1, e_2, v_2) = 0,$$
$$\varphi_2(e_1, e_3, v_1) = c_1 v_1, \qquad \varphi_2(e_1, e_3, v_2) = 0,$$
$$\varphi_2(e_2, e_3, v_1) = c_2 v_1, \qquad \varphi_2(e_2, e_3, v_2) = c_3 v_2,$$

where c_1, c_2, c_3 are parameters in $\mathbb{K}$.

4. ABELIAN EXTENSIONS OF 3-HOM-LIE ALGEBRAS

In this section, we show that associated to any abelian extension, there is a generalized representation and a 2-cocycle.

DEFINITION 4.1. Let $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\mathfrak{g}, \alpha)$, $(V, [\cdot, \cdot, \cdot]_V, A)$ and $(\hat{\mathfrak{g}}, [\cdot, \cdot, \cdot]_{\hat{\mathfrak{g}}}, \alpha_{\hat{\mathfrak{g}}})$ be 3-Hom-Lie algebras and $i : V \to \hat{\mathfrak{g}}$, $p : \hat{\mathfrak{g}} \to \mathfrak{g}$ be morphisms of 3-Hom-Lie algebras. The following sequence of 3-Hom-Lie algebras is a short exact sequence if $\mathrm{Im}(i) = \mathrm{Ker}(p)$, $\mathrm{Ker}(i) = 0$ and $\mathrm{Im}(p) = \mathfrak{g}$:

$$0 \to V \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \to 0,$$

where $A(V) = \alpha_{\hat{\mathfrak{g}}}(V)$. In this case, we call $\hat{\mathfrak{g}}$ an extension of $\mathfrak{g}$ by V, and denote it by $E_{\hat{\mathfrak{g}}}$. It is called an abelian extension if V is an abelian ideal of $\hat{\mathfrak{g}}$, i.e., $[u, v, w]_V = 0$ for all $u, v, w \in V$. A section σ of $p : \hat{\mathfrak{g}} \to \mathfrak{g}$ consists of a linear maps $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ such that $p \circ \sigma = id_\mathfrak{g}$ and $\sigma \circ \alpha = \alpha_{\hat{\mathfrak{g}}} \circ \sigma$.

DEFINITION 4.2. Two extensions of 3-Hom-Lie algebras,

$$E_{\hat{\mathfrak{g}}} : 0 \to V \xrightarrow{i_1} \hat{\mathfrak{g}} \xrightarrow{p_1} \mathfrak{g} \to 0 \qquad \text{and} \qquad E_{\tilde{\mathfrak{g}}} : 0 \to V \xrightarrow{i_2} \tilde{\mathfrak{g}} \xrightarrow{p_2} \mathfrak{g} \to 0,$$

are equivalent if there exists a morphism of 3-Hom-Lie algebras $\phi : \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{i_1} & \hat{\mathfrak{g}} & \xrightarrow{p_1} & \mathfrak{g} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{Id}_V} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{Id}_\mathfrak{g}} & & \\
0 & \longrightarrow & V & \xrightarrow{i_2} & \tilde{\mathfrak{g}} & \xrightarrow{p_2} & \mathfrak{g} & \longrightarrow & 0
\end{array}
$$

A linear map $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ is called a splitting of $\mathfrak{g}$ if it satisfies $p \circ \sigma = id_\mathfrak{g}$. If there exists a splitting which is also a homomorphism between 3-Hom-Lie

algebras, we say that the abelian extension is split. Let $\hat{\mathfrak{g}}$ be a split abelian extension and $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ the corresponding splitting. Define $\rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(V)$ and $\nu : \mathfrak{g} \to \mathrm{Hom}(\wedge^2 V, V)$ by

$$\rho(x,y)(u) = [\sigma(x), \sigma(y), u]_{\hat{\mathfrak{g}}},$$
$$\nu(x)(u,v) = [\sigma(x), u, v]_{\hat{\mathfrak{g}}}.$$

Then, we can transfer the 3-Hom-Lie algebra structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus V$ in terms of ρ and ν.

Note that the Hom-Filippov-Jacobi identity gives the character of ρ and ν:

$$[x+u, y+v, z+w]_{(\rho,\nu)} = [x,y,z] + \rho(x,y)(w) + \rho(y,z)(u) + \rho(z,x)(v)$$
$$+ \nu(x)(v \wedge w) + \nu(y)(w \wedge u) + \nu(z)(u \wedge v).$$

However, by Theorem 2.4, it is straightforward to obtain the following proposition.

PROPOSITION 4.3. *Any split abelian extension of 3-Hom-Lie algebras is isomorphic to a generalized semidirect of product 3-Hom-Lie algebra.*

Now, for non-split abelian extensions, we can further define $\omega : \wedge^3 \mathfrak{g} \to V$ by

$$\omega(x,y,z) = [\sigma(x), \sigma(y), \sigma(z)]_{\hat{\mathfrak{g}}} - \sigma[x,y,z]_{\mathfrak{g}}.$$

Then, we also transfer the 3-Hom-Lie algebra structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus V$ in terms of ρ, ν and ω:

$$[x_1+v_1, x_2+v_2, x_3+v_3]_{(\rho,\nu,\omega)} = [x_1, x_2, x_3]_{\mathfrak{g}} + \rho(x_1, x_2)(v_3)$$
$$+ \rho(x_3, x_1)(v_2) + \rho(x_2, x_3)(v_1)$$
$$+ \nu(x_1)(v_2, v_3) + \nu(x_2)(v_3, v_1)$$
$$+ \nu(x_3)(v_1, v_2) + \omega(x_1, x_2, x_3).$$

THEOREM 4.4. *With above notations, $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho,\nu,\omega)}, \alpha_{\mathfrak{g} \oplus V})$ is a 3-Hom-Lie algebra if and only if for all $x_1, x_2, x_3, x_4, x_5 \in \mathfrak{g}$ and $v, v_1, v_2, v_3 \in V$, Eqs. (2.4)-(2.7) and the following identities hold:*

$$0 = -\rho(\alpha(x_1), \alpha(x_2))(\omega(x_3, x_4, x_5)) - \omega(\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]) \quad (4.1)$$
$$+ \rho(\alpha(x_4), \alpha(x_5))(\omega(x_1, x_2, x_3)) + \omega([x_1, x_2, x_3], \alpha(x_4), \alpha(x_5))$$
$$+ \rho(\alpha(x_5), \alpha(x_3))(\omega(x_1, x_2, x_4)) + \omega(\alpha(x_3), [x_1, x_2, x_4], \alpha(x_5))$$
$$+ \rho(\alpha(x_3), \alpha(x_4))(\omega(x_1, x_2, x_5)) + \omega(\alpha(x_3), \alpha(x_4), [x_1, x_2, x_5]),$$
$$0 = \nu(\alpha(x_1))(A(v), \omega(x_2, x_3, x_4)) + \rho([x_2, x_3, x_4], \alpha(x_1))(A(v)) \quad (4.2)$$
$$+ \rho(\alpha(x_3), \alpha(x_4))\rho(x_1, x_2)(v) - \rho(\alpha(x_2), \alpha(x_4))\rho(x_1, x_3)(v)$$
$$+ \rho(\alpha(x_2), \alpha(x_3))\rho(x_1, x_4)(v),$$
$$0 = \rho(\alpha(x_1), \alpha(x_2))\rho(x_3, x_4)(v) - \rho(\alpha(x_3), \alpha(x_4))\rho(x_1, x_2)(v) \quad (4.3)$$
$$- \rho([x_1, x_2, x_3], \alpha(x_4))(A(v)) - \nu(\alpha(x_4))(A(v), \omega(x_1, x_2, x_3))$$
$$- \rho(\alpha(x_3), [x_1, x_2, x_4])(A(v)) + \nu(\alpha(x_3))(A(v), \omega(x_1, x_2, x_4)).$$

The Fundamental Identity gives the character of ρ, ν and ω.

Proof. The triple $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu, \omega)}, \alpha_{\mathfrak{g} \oplus V})$ defines a 3-Hom-Lie algebra if and only if the Hom-Filippov-Jacobi identity holds on all elements of $\mathfrak{g} \oplus V$. Condition (4.1) is obtained using the Hom-Filippov-Jacobi identity on $\{x_1, x_2, x_3, x_4, x_5\}$ elements of $\mathfrak{g}$.

Similarly, elements $\{x_1, v, x_2, x_3, x_4\}$ gives Eq. (4.2), $\{x_1, x_2, v, x_3, x_4\}$ gives Eq. (4.3), $\{x_1, x_2, v_1, v_2, x_3\}$ gives Eq. (2.4), $\{v_1, x_1, v_2, x_2, x_3\}$ gives Eq. (2.5), $\{v_1, x_1, v_2, v_3, x_2\}$ gives Eq. (2.6) and $\{v_1, v_2, v_3, x_1, x_2\}$ gives Eq. (2.7).

Conversely, if Eqs. (2.4)-(2.7) and Eqs. (4.1)-(4.3) hold, it is straightforward to see that for all $e_1, \ldots, e_5 \in \mathfrak{g} \oplus V$, the Hom-Filippov-Jacobi identity holds. Thus, $(\mathfrak{g} \oplus V, [\cdot, \cdot, \cdot]_{(\rho, \nu, \omega)}, \alpha_{\mathfrak{g} \oplus V})$ is a 3-Hom-Lie algebra. ∎

Remark 4.5. The triple (ρ, ν, ω) is a cochain with respect to the new cohomology defined in Section 3 but it is not necessarily a 2-cocycle.

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