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On a class of power associative LCC-loops

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Abstract: Let LWPC denote the identity $(xy \cdot x) \cdot xz = x((yx \cdot x)z)$, and RWPC the mirror identity. Phillips proved that a loop satisfies LWPC and RWPC if and only if it is a WIP PACC loop. Here, it is proved that a loop Q fulfils LWPC if and only if it is a left conjugacy closed (LCC) loop that fulfils the identity $(xy \cdot x)x = x(yx \cdot x)$. Similarly, RWPC is equivalent to RCC and $x(x \cdot yx) = (x \cdot xy)x$. If a loop satisfies LWPC or RWPC, then it is power associative (PA). The smallest nonassociative LWPC-loop was found to be unique and of order 6 while there are exactly 6 nonassociative LWPC-loops of order 8 up to isomorphism. Methods of construction of nonassociative LWPC-loops were developed.

Key words: left (right) conjugacy closed loop, power associativity, LWPC-loop, RWPC-loop.
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1. Introduction

A quasigroup $(Q, \cdot)$ consists of a non-empty set Q with a binary operation $(\cdot)$ on Q such that given $x, y \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ respectively. We shall sometimes refer to $(Q, \cdot)$ as simply Q. It is usual to set $x = a\backslash b$ and $y = b/a$.

A loop is a quasigroup with a two-sided neutral element 1. We write xy for $x \cdot y$ and stipulate that $\cdot$ have lower priority than juxtaposition among factors to be multiplied-for instance, $x \cdot yz$ stands for $x(yz)$. For an overview on loop theory, see [1, 7, 11].

If a is an element of a loop Q, then $L_a : x \mapsto ax$ permutes Q and is called the left translation of a. Similarly, $R_a : x \mapsto xa$ is called the right translation of a. The loop Q is said to be a left conjugacy closed (LCC) if the

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left translations are closed under conjugation (i.e., for all $x, y \in Q$, there exists $z \in Q$ such that $L_x L_y L_x^{-1} = L_z$). Similarly, right conjugacy closed (RCC) loops are those in which the right translations are closed under conjugation. A loop is said to be conjugacy closed (CC) if it is both LCC and RCC.

Kinyon and Kunen [8] thoroughly analyzed power associative CC-loops (PACC-loops). A loop is power associative (PA) if each of its elements generates a (cyclic) subgroup. By [8], the structure of PACC-loops heavily depends upon the structure of WIP elements. That motivated Phillips [12] to find a short equational basis for WIP PACC loops.

Now, WIP stands for Weak Inverse Property. A loop Q is said to be a WIP-loop if it satisfies the equivalent identities

$$x(yx)^\rho = y^\rho \qquad \text{and} \qquad (xy)^\lambda x = y^\lambda,$$

where $x \cdot x^\rho = 1 = x^\lambda \cdot x$ for all $x \in Q$.

By [12], a loop Q is a WIP PACC-loop if and only if it fulfils the laws

$$(xy \cdot x) \cdot xz = x((yx \cdot x)z)\,, \tag{LWPC}$$

$$zx \cdot (x \cdot yx) = (z(x \cdot xy))x\,. \tag{RWPC}$$

The purpose of this paper is to initiate the study of loops that fulfil only one of the latter identities. Our main result is Theorem 2.2 that proves that an LWPC loop is power associative, and that LWPC loops are exactly the LCC loops in which $(xy \cdot x)x = x(yx \cdot x)$. A mirror result holds for RWPC loops.

During our preliminary search for LWPC-loops, we found two LCC-loops (that are LWPC-loops) of orders 6 and 8 with the property that their right nuclei are abelian groups which are of index two. A general construction of such loops can be found in Drápal [4]. These loops were constructed by means of an arbitrary abelian group and two permutations that satisfy some constraints (cf. Proposition 5.1). We shall be adopting this construction to show that an infinite series of LWPC-loop is feasible.

If Q is a loop and α, β and γ permute Q, then (α, β, γ) is said to be an autotopism of Q if $\alpha(y)\beta(z) = \gamma(yz)$ for all $y, z \in Q$. Autotopisms of Q can be composed through componentwise multiplication, and thus they form a group called the autotopism group denoted by $\mathrm{Atp}(Q)$.

The fact that left translations are closed under conjugation can be expressed equationally by the law $x \cdot y(x\backslash z) = xy/x \cdot z$ (cf. [6]). This law may also be written as $x \cdot yz = (xy/x) \cdot xz$. Hence, Q is LCC if and only if

$$\left(R_x^{-1}L_x,\ L_x, L_x\right) \in \mathrm{Atp}(Q) \qquad \text{for all } x \in Q\,. \tag{1}$$

Writing LWPC as $(x((y/x)/x)\cdot x)\cdot xz = x\cdot yz$ implies that this law holds if and only if

$$\bigl(R_xL_xR_x^{-2}, L_x, L_x\bigr)\in \mathrm{Atp}(Q) \qquad \text{for all } x\in Q\,. \tag{2}$$

Characterizations of LCC and LWPC by autotopisms are of crucial importance for the proof of the main result.

2. LWPC-loop, RWPC-loop and their properties

The following criterion for power associativity will be useful to establish the main result in Theorem 2.2.

Lemma 2.1. *Let x be an element of a loop Q. Suppose that $x^\lambda = x^\rho$, and denote the latter element by x^{-1}. Suppose that for each $i\ge 1$ any bracketing of i occurrences of x yields the same element, and denote this element by x^i. Similarly, let any bracketing of i occurrences of x^{-1} yield an element x^{-i}. Set $x^0 = 1 = (x^{-1})^0$ and $(x^{-i})^{-1} = x^i$. Finally, suppose that $y^jy^{-i} = y^{j-i} = y^{-i}y^j$ whenever $j\ge i\ge 1$ and $y\in\{x,\ x^{-1}\}$. Then x generates a subgroup of Q, and the element x^i attains the usual meaning of the ith power, for every integer i.*

Proof. First note that $x^jx^{-i} = x^{j-i} = x^{-i}x^j$ holds for any positive integers i and j since if $j<i$, then $y^{-j}y^i = y^{i-j} = y^iy^{-j}$ may be used, with $y = x^{-1}$. We have to show that $x^i\cdot x^jx^k = x^ix^j\cdot x^k$ for any i, j and k. If any of them is zero, the corresponding power is equal to 1 and the equality holds. If all of i, j and k are positive, then the equality follows from the assumption on bracketing. If two or more exponents are negative, replace x with $y = x^{-1}$. Thus only the case with exactly one of exponents negative needs to be solved. This means to verify that each of the ensuing triples associates, under the assumption that all of i, j and k are positive integers,

$$\bigl(x^i, x^j, x^{-k}\bigr),\quad \bigl(x^i, x^{-j}, x^k\bigr)\ \text{ and }\ \bigl(x^{-i}, x^j, x^k\bigr).$$

The leftmost and the rightmost triples are mirror symmetric. Hence, only the first two triples will be considered. Now,

$$x^i\cdot x^jx^{-k} = x^ix^{j-k} = x^{i+j-k} = x^{i+j}x^{-k} = x^ix^j\cdot x^{-k}$$

and

$$x^i\cdot x^{-j}x^k = x^ix^{-j+k} = x^{i-j+k} = x^{i-j}x^k = x^ix^{-j}\cdot x^k.$$

∎

Theorem 2.2. *Let Q be a loop. Then*

(i) *Q fulfills LWPC $\Leftrightarrow$ Q is LCC and $\underbrace{(xy \cdot x)x = x(yx \cdot x)}_{P_\lambda(x,y)}$ for all $x, y \in Q$.*

(ii) *Q fulfills RWPC $\Leftrightarrow$ Q is RCC and $\underbrace{x(x \cdot yx) = (x \cdot xy)x}_{P_\rho(x,y)}$ for all $x, y \in Q$.*

If Q fulfills LWPC or RWPC, then Q is power associative.

Proof. Setting $z = 1$ in the LWPC law yields $(xy \cdot x)x = x(yx \cdot x)$. To prove (i) thus means to show LWPC $\Leftrightarrow$ LCC under the assumption of $R_x^2 L_x = L_x R_x^2$, for all $x \in Q$. The equivalence of LCC and LWPC follows immediately from the expression of the identities via autotopisms as in (1) and (2), since $R_x^2 L_x = L_x R_x^2$ implies that $R_x L_x R_x^{-2} = R_x R_x^{-2} L_x = R_x^{-1} L_x$. Nothing else is needed to get (i).

Point (ii) follows by a mirror argument. Let us assume that Q fulfills LWPC. To prove that Q is power associative, let us start by showing that for $k \geq 1$ any bracketing of i occurrences of x yields the same element. Proceed by induction. Cases $k = 1$ and $k = 2$ are clear. The case $k = 3$ follows from LWPC by setting $y = z = 1$. Assume $k \geq 4$. We need to verify that $x^i x^j = xx^{k-1}$ whenever $i \geq 2$ and $i + j = k$. If $i \geq 3$, express x^i as $xx^{i-2} \cdot x$ and set $z = x^{j-1}$. LWPC yields

$$x^i x^j = (xx^{i-2} \cdot x) \cdot xx^{j-1} = x((x^{i-2}x \cdot x) \cdot x^{j-1}) = xx^{k-1}.$$

Assume $i = 2$ and set $y = 1$ in LWPC. Then, $x^2 \cdot xx^{j-1} = x(x^2 x^{j-1}) = xx^{k-1}$. LWPC with $z = 1$ implies $xx = (xx^\rho \cdot x)x = x(x^\rho x \cdot x)$. Hence $x = x^\rho x \cdot x$ and $1 = x^\rho x$. Therefore, $x^\rho = x^\lambda = x^{-1}$.

By Lemma 2.1 it remains to verify that $x^j x^{-i} = x^{j-i} = x^{-i} x^j$ whenever $j \geq i \geq 1$. Proceed by outer induction on $j \geq 2$ and inner induction on $i \geq 1$. To get the case $i = 1$, start from

$$xx^{j-1} = x^j = (xx^{j-2} \cdot x) \cdot xx^{-1} = x \cdot (x^{j-2}x \cdot x)x^{-1} = x \cdot x^j x^{-1}.$$

By cancelling, $x^j x^{-1} = x^{j-1}$. Therefore

$$(x^{-1}x^j \cdot x^{-1})x^{-1} = x^{-1}(x^j x^{-1} \cdot x^{-1}) = x^{j-3} = x^{j-2}x^{-1},$$

and that yields $x^{-1}x^j \cdot x^{-1} = x^{j-2} = x^{j-1}x^{-1}$. Thus, $x^{-1}x^j = x^{j-1}$. Assume $i \geq 2$. Note that $xx^{-i} = x^{-(i-1)}$ follows from the induction assumption since this is the same as $y^{-1}y^i = y^{i-1}$, where $y = x^{-1}$. Hence

$$x \cdot x^{j-i} = x^{j-i+1} = x^j x^{-(i-1)}$$
$$= (xx^{j-2}x) \cdot xx^{-i} = x \cdot (x^{j-2}x \cdot x)x^{-i} = x \cdot x^j x^{-i}.$$

By cancelling, $x^{j-i} = x^j x^{-i}$. To finish up , first observe that $x^{-i} \cdot x^{j-i}x^{-i} = x^{j-3i} = x^{j-i}x^{-i} \cdot x^{-i}$. Indeed, if $j - i \leq 0$, this follows from the earlier part of the proof. If $j - i > 0$, then $x^{j-i}x^{-i} = x^{j-2i} = x^{-i}x^{j-i}$ by the induction assumption (where a switch to $y = x^{-1}$ is needed if $j-i < i$). Since $j-2i$ may be treated similarly as $j-i$, the expression of x^{j-3i} follows. Now, $x^{-i}x^j = x^{j-i}$ may be obtained by cancellation from $(x^{-i}x^j \cdot x^{-i})x^{-i} = x^{-i}(x^j x^{-i} \cdot x^{-i}) = x^{-i} \cdot x^{j-i}x^{-i} = x^{j-i}x^{-i} \cdot x^{-i}$. ∎

Corollary 2.3. *A CC-loop is a power associative WIP-loop if and only if it fulfills the laws $(xy \cdot x)x = x(yx \cdot x)$ and $x(x \cdot yx) = (x \cdot xy)x$.*

Proof. This follows from Theorem 2.2. ∎

Corollary 2.4. *A CC-loop is a power associative WIP-loop if and only if it is a WIP LWPC-loop (alternatively, a WIP RWPC-loop).*

Proof. This follows from Theorem 2.2 and the fact that LCC and RCC are equivalent in a WIP-loop. ∎

Corollary 2.5. (i) *A loop is a LWPC-loop if and only if the conjugates of its left translations are left translations and any left translation commutes with the square of its corresponding right translation.*

(ii) *A loop is a RWPC-loop if and only if the conjugates of its right translations are right translations and any right translation commutes with the square of its corresponding left translation.*

(iii) *A loop is a WIP PACC-loop if and only if the conjugates of both its left translations and right translations are left translations and right translations respectively and left and right translations commute with the squares of their corresponding right and left translations respectively.*

Proof. This follows from Theorem 2.2. ∎

3. Construction of LWPC-loops

Suppose that G and R are abelian groups and that $f : G \times G \to R$ is a mapping. Call it zero preserving if $f(x,0) = 0 = f(0,x)$ for all $x \in G$. Say that f is additive on the right if $f(x, y+z) = f(x,y) + f(x,z)$ for all $x, y, z \in G$. Say that f is additive if it is both right and left additive. Say that f is quadratically triadditive on the left if

$$\begin{array}{rcl} g : G \times G \times G & \longrightarrow & R \\ (x,y,z) & \longmapsto & f(x+y,z) - f(x,z) - f(y,z) \end{array}$$

is a triadditive symmetric mapping (symmetric means that permuting x, y and z has no influence upon the value $g(x,y,z)$).

Define the radical $\mathrm{Rad}(f)$ as the set of all $x \in G$ such that $f(x,y) = 0 = f(y,x)$ for all $y \in G$.

Theorem 3.1. *Let R be a subgroup of an abelian group G, and let $f : G \times G \to R$ be such that $Rad(f) \le R$, f is zero preserving and right additive. Then*

$$x \cdot y = x + y + f(x,y)$$

defines upon G an LCC loop. This loop is associative if and only if f is biadditive, and conjugacy closed if and only if f is quadratically triadditive on the left. The LWPC law is fulfilled if and only if

$$f(2x+y,x) = 2f(x,x) + f(y,x) \qquad \text{for all } x, y \in G\,. \tag{3}$$

If G is an elementary abelian 2-group, then (3) always holds, while for G of odd order, (3) is equivalent to

$$f(x+y,x) = f(x,x) + f(y,x) \qquad \text{for all } x, y \in G\,.$$

Proof. By [3, Theorem 5.3] and [5, Corollary 2.2], only the part relating to (3) needs to be considered. Since f is additive on the right,

$$\begin{aligned} (xy \cdot x)x &= 3x + y + f(x,y) + f(x+y,x) + f(2x+y,x)\,, \\ x(yx \cdot x) &= 3x + y + f(y,x) + f(x+y,x) + f(x,2x+y)\,. \end{aligned}$$

Hence $(xy \cdot x)x = x(yx \cdot x)$ if and only if (3) holds. By Theorem 2.2, this means that (3) characterizes the LWPC loops. Let it be satisfied. If $2z = 0$ for all $z \in G$, then (3) is trivially true. Furthermore, setting $y = 0$ implies that $2f(x,x) = f(2x,x)$.

By adding with itself both the left and the right hand sides and using the additivity on the right, we obtain

$$f(2x+y,2x) = f(2x,2x) + f(y,2x) \qquad \text{for all } x,y \in G\,.$$

If G is of odd order, then $2x$ may be replaced by x. ∎

Theorem 3.1 provides a tool how to construct LWPC loops that are not conjugacy closed. By [5, Theorem 4.6], the construction of Theorem 3.1 covers all LCC loops Q for which there exists a prime p and a central subloop Z such that $|Z| = p$ and Q/Z is an elementary abelian p- group. Classification of such loops up to isomorphism was considered in [5, Section 5], while Section 8 of the same paper proves that all (left) Bol loops of order 8 are LCC, and that all of them may be obtained by the construction of Theorem 3.1. By [8], a nonassociative WIP PACC loop is of order divisible by 16.

By the library of LOOPS package [9] of GAP [10] there are, up to isomorphism, 19 left conjugacy closed loops of order 8 that are not right conjugacy closed. Six of them are Bol loops discovered by Burn [2] and described in [5, Section 8]. We have verified that of the remaining 13 loops none fulfills the LWPC law, and exactly one fulfills the law $x(x \cdot yx) = (x \cdot xy)x$.

Note that if Q is a loop from Theorem 3.1, then the law $x(x{\cdot}yx) = (x{\cdot}xy)x$. holds if and only if

$$3f(x,x) + f(x,y) + f(y,x) = f(x,x) + f(x,y) + f(2x+y,x).$$

In characteristic 2, this is always true. The following thus holds:

Proposition 3.2. *A loop of order* 8 *is LWPC if and only if it is left Bol. Each such loop fulfills the law $x(x \cdot yx) = (x \cdot xy)x$. There are exactly six isomorphism classes of nonassociative LWPC loops of order* 8*. None of them is conjugacy closed or Moufang, and in each of them the left nucleus N_λ is of order* 4*.*

Consider now a loop that possesses a right nucleus of index two, and suppose that the nucleus is isomorphic to an abelian group $(G,+)$. It is easy to see (cf. [4, Proposition 4.2]) that such a loop is isomorphic to a loop $G[f,g]$ that is defined upon $\{0,1\} \times G$ by

$$\begin{aligned} (0,x)(0,y) &= (0,x+y)\,, & (0,x)(1,y) &= (1,g(x)+y)\,,\\ (1,x)(0,y) &= (1,x+y)\,, & (1,x)(1,y) &= (0,f(x)+y), \end{aligned}$$

for all $x,y \in G$, where f and g are permutations of G such that $g(0) = 0$. By

[4, Proposition 5.1], such a loop $G[f,g]$ is LCC if and only if $g^2 = id_G$ and both

$$\begin{aligned} y + f(y+z) &= f\big(z + g(y)\big) + g(y)\,, \\ x + f(y+z) &= f^{-1}\big(z + f(y)\big) + f(y) \end{aligned} \tag{4}$$

are true for all $x, y, z \in G$.

PROPOSITION 3.3. *Let f and g be permutations of G, where G is an abelian group. Suppose that $g(0) = 0$, $g^2 = id_G$ and that (4) holds. The loop $G[f,g]$ fulfills the LWPC law if and only if*

$$\begin{aligned} g\big(f(x+y) + x\big) &= x + f\big(g(y) + x\big)\,, \\ f\big(g\big(f(x) + y\big) + x\big) &= f(x) + g\big(f(y) + x\big) \end{aligned} \tag{5}$$

for all $x, y \in G$.

Proof. By Theorem 2.2, the only step to do is to verify that the two equalities hold if and only if $(ab \cdot a)a = a(ba \cdot a)$ whenever $a = (\varepsilon, x)$ and $b = (\eta, y)$ where $x, y \in G$ and $\varepsilon, \eta \in \{0, 1\}$. The case $\varepsilon = \eta = 0$ is clear.

Assume $\varepsilon = 0$ and $\eta = 1$. Then $ab \cdot a = (1, x + g(x) + y)$ and $(ab \cdot a)a = (1, 2x + g(x) + y)$, while $ba \cdot a = (1, 2x + y)$ and $a(ba \cdot a) = (1, 2x + y + g(x)) = (ab \cdot a)a$.

Thus, $a = (1, x)$ may be assumed. Suppose first that $\eta = 0$. Then $ab \cdot a = (0, f(x+y) + x)$ and $(ab \cdot a)a = (1, g(f(x+y) + x) + x)$, while $ba \cdot a = (0, f(x + g(y)) + x)$ and $a(ba \cdot a) = (1, f(x + g(y)) + 2x)$. Thus, the equality holds if and only if $g(f(x+y) + x) = f(x + g(y)) + x$.

Suppose now that $\eta = 1$. Then, $ab \cdot a = (1, g(f(x) + y) + x)$ and $(ab \cdot a)a = (0, f(g(f(x) + y) + x) + x)$, while

$$\begin{aligned} ba \cdot a &= \big(1, g\big(f(y) + x\big) + x\big)\,, \\ a(ba \cdot a) &= \big(0, f(x) + g\big(f(y) + x\big) + x\big)\,, \end{aligned}$$

yielding thus the second equality of (5). ∎

Assume now that $f(x) = -x$. Then (5) holds if and only if $g(-x) = -g(x)$ for every $x \in G$. Proposition 3.3 together with [4, Proposition 5.7] immediately yield the following statement:

Theorem 3.4. *Let g be a permutation of an abelian group G, $g(0) = 0$, such that $g^2(x) = x$ and $g(-x) = -g(x)$ for every $x \in G$. Suppose also that $g(x) \neq -x$ for at least one $x \in G$. Define an operation $\cdot$ upon $\{0,1\} \times G$ by*

$$(0,x)(\eta,y) = \big(\eta, g^{\eta}(x) + y\big) \quad \text{and} \quad (1,x)(\eta,y) = \big(\eta, (-1)^{\eta}x + y\big)$$

for all $x, y \in G$ and $\eta \in \{0,1\}$.

The operation $\cdot$ describes a nonassociative LWPC loop in which the right nucleus is equal to $\{(0,x); x \in G\}$ and the left nucleus is equal to $\big\{(0,x) : g(x+y) = g(x) - y$ for every $y \in G\big\}$.

Surprisingly, the case $g = id_G$ fulfills the assumptions of Theorem 3.4. For each $n \geq 3$ the operation

$$(\varepsilon,x)(\eta,y) = \big(\varepsilon + \eta, (-1)^{\varepsilon\eta}x + y\big)$$

thus yields an LWPC loop upon $\mathbb{Z}_2 \times \mathbb{Z}_n$. Such a loop is never conjugacy closed since the left nucleus is trivial, while the right nucleus coincides with $\{0\} \times \mathbb{Z}_n$.

The least nonassociative LWPC loop is of order 6. Up to isomorphism this is the only nonassociative LWPC loop of order 6. The latter fact has been verified by using the LOOPS package [9] of GAP [10].

Questions 3.5. (a) Can WIP LWPC loops be described (as a loop variety) by equations that do not use division and/or inverses?

(b) What is the least odd order for which there exists a nonassociative LWPC loop?

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