CERTAIN SIMILARITIES BETWEEN INERTIAL SYSTEMS IN PHYSICS AND STEADILY GROWING SYSTEMS IN ECONOMICS

*Branko HORVAT**

Comparing relativity physics with relativity economics may seem extravagant and bizarre. The most one would expect from such an exercise would be some sort of an intellectual tour de force. Yet, since I first realized certain striking (formal?) similarities some fifteen years ago (1), I have not ceased to think that one day it may be worth while exploring the matter more fully. The text which follows is the result of this adventure. Before we proceed, it will be helpful to compile a list of symbols which will be used.

K = gross fixed capital
R = replacement
G = gross investment
t = time
r = rate of growth
n = life span of fixed assets
k = cost per unit of capital
v = velocity
c = velocity of light
l = length
β = transformation factor for inertial and steady growth systems.
$^\circ$ = superscript denoting a variable in a stationary state (i. e. for r = 0)

1. Lorentz Transformations

Einstein's special theory of relativity is by now so well known that it is unnecessary to quote references in order to derive some of the results we need. Any university textbook in physics will do for the purpose. But for non-physicists short explanations may prove helpful.

In 1881 Michelson and in 1887 Michelson and Morley performed experiments whose task it was to determine the speed of earth in rela-

*) Senior Research Associate, Institute of Economic Studies, Beograd.

tion to an absolute coordinate system fixed in an all-pervading medium called the ether, in which the speed of light was equal in all directions. The experiments produced an unexpected, now famous result: the observed speed of light turned out to be the same regardless of whether the »observer« moved relative to the source of light or not. In 1895 Lorentz explained the phenomenon by assuming that by travelling through a motionless ether a body changes its dimensions. In the direction of the movement a contraction of the body will occur, while its dimensions orthogonal to this direction remain unchanged. The factor of contraction turned out to be*.

$$\beta = \sqrt{1 - \frac{v^2}{c^2}} \tag{1}$$

In 1904 Lorentz fully developed his well known transformation formulae for space coordinates and time. The formulae apply to two systems travelling, relative to each other, at a constant velocity v. Such systems, in which there is no acceleration, are called *intertial systems.*

Lorentz transformations can be easily derived in the following way. Suppose we have two observers, A located at the origin and B travelling at a constant velocity v along a straight line starting from the origin. The respective equations of the wave front of the light travelling from the origin will be

for observer A $\quad c^2t^2 = x^2 + y^2 + z^2$ (2-a)

for observer B $\quad c^2t'^2 = x'^2 + y'^2 + z'^2$ (2-b)

According to the Michelson- Morley experiment both observers observe the same speed of light c.

If the second observer moved in the Newtonian world along the axis x, with two remaining axes being parallel, the coordinates of his position would be determined by

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{3}$$

However, since he moves in an Einsteinian world, in which c = const, the two Newtonian co-ordinates, distance x' and the absolute and uniformly proceeding time t', should be corrected by means of a transformation factor which will preserve the invariance of equations (2). Call this factor $\frac{1}{\beta}$. Express the two coordinates to be corrected by:

$$x' = \frac{1}{\beta}(x - vt)$$

$$t' = \frac{1}{\beta}(t - ax)$$

*) For convenience I use the reciprocal value of originally defined β.

(4)

use (4) in (2-b)

$$c^2 \frac{1}{\beta^2} (t^2 - 2\,a\,t\,x + a^2\,x^2) = \frac{1}{\beta^2} (x^2 - 2\,x\,v\,t + v^2\,t^2) + y'^2 + z'^2$$

$$\frac{1}{\beta^2} (c^2 - v^2)\, t^2 = \frac{1}{\beta^2} (1 - a^2\,c^2)\, x^2 + \frac{2t}{\beta^2} (a\,c^2 - v)\, x + y'^2 + z'^2 \qquad (5)$$

Comparing the coefficients of the same variables in (2-a) and (5) we get

$$\frac{1}{\beta^2} (c^2 - v^2) = c^2$$

$$\frac{1}{\beta^2} (1 - a^2\,c^2) = 1$$

$$2\,t\,(a\,c^2 - v) = 0$$

$$\therefore\ a = \frac{v}{c^2}$$

$$\beta^2 = 1 - \frac{v^2}{c^2}$$

It follows from (4) that the second observer will find his distance from the origin and his time changed into

$$x' = \frac{1}{\beta} (x - vt) = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\qquad (6)$$

$$' = \frac{1}{\beta} \left(t - \frac{vx}{c^2} \right) = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The signs of the square roots are positive and the values are real, since v<c and x′ and t′ can be only positive. Thus we have obtained the transformation factor β as expressed by (1). The meaning of (6) may be summarised in the following way. If observer *A*, having any motion whatever, observes that light for him travels uniformly in all directions with constant velocity *c*, then the observer *B*, moving relative to *A* with a constant velocity *v* along the axis x, will observe — as he should — that light for him also travels uniformly in all directions with the same velocity *c*, provided that in his observations he uses coordinates which are connected with coordinates of *A*′ s system by equations (6), y and z remaining unchanged.

The transformation factor ß has three important properties

$$\beta(v = 0) = 1$$

$$\beta(v = c) = 0 \qquad (7)$$

$$\frac{d\beta}{dv} < 0$$

It declines monotonically between unity — which reflects the applicability of the addition rule in the Newtonian world — and zero — which reflects the fact that in the empirical world there is an absolute limit which no velocity can surpass. This limit is given by the velocity of light c.

2. Contraction of a Moving Body

The transformation of space and time in the systems in motion will have some important consequences concerning the properties of the physical world. They are not quite so easily reconcilable with our intuitive notions about space and time derived from an experience of living in a world of relatively slow motions. For our purposes it suffices to mention two of them: the contraction of a body and the dilatation of time.

Suppose a rod parallel to the x axis travels along this axis at a constant velocity v in relation to the origin 0. In its own coordinate system, S′, rod is, of course, motionless. Its length in the system S′ — the length of being at rest or its proper-length l° — is given by

$$l^{\circ} = x'_2 - x'_1$$

The same length in the system S, in relation to which both S′ and the rod move at the velocity v, is given by

$$l = x_2 - x_1$$

By using our transformation formula (6), the two end points of the rod may be represented by

$$x'_1 = \frac{1}{\beta}(x_1 - vt)$$

$$x'_2 = \frac{1}{\beta}(x_2 - vt)$$

And the length of the rod, as observed from S, shortens into

$$x_2 - x_1 = \beta(x'_2 - x'_1)$$

or $\quad l = \beta l^{\circ}$

Lorentz thought that the contraction of the body was due to the ether through which the body travelled. Einstein insisted that this was so because of the relativity of space and time which were different for each body depending on the motion performed. Many thought — and some still do — that the contraction was just a mathematical result and so arbitrary. Einstein argued that contraction was real since each body has its own space if motions proceed at different velocities.

Clearly, the volume of a body, moving along one of the axes, contracts in the same way, since the other two (orthogonal) dimensions remain unchanged.

$$V = \beta V^{\circ} \tag{8}$$

Again V° is volume of the body in the system in which it is at rest, while V is volume of the same body observed from another system in relation to which the first system and the body travel at a constant velocity v.

3. Dilatation of Time

Suppose it takes time $t'_2 - t'_1$ for a process in the system S' to work itself out. In relation to the system S, the body connected with the process S' has travelled the distance

$$x_2 - x_1 = v(t_2 - t_1)$$

Use the transformation formula (6) for time

$$t'_1 = \frac{1}{\beta}\left(t_1 - \frac{vx_1}{c_2}\right), \qquad t'_2 = \frac{1}{\beta}\left(t_2 - \frac{vx_2}{c_2}\right),$$

to get

$$t'_2 - t'_1 = \frac{1}{\beta}\left[t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)\right] = \frac{1}{\beta}(t_2 - t_1)\left(1 - \frac{v^2}{c^2}\right)$$

$$\therefore t_2 - t_1 = \frac{1}{\beta}(t'_2 - t'_2) \tag{9}$$

The time lengthens. It flows slower for a system in motion. And that holds true for both systems: for an observer in S the time in S' flows at a slower pace but so does time in S as observed from S'.

Again, it was thought — and many still think this way — that it was intuitively obvious that time can flow only in a uniform way. In other words, time was considered absolute, and the result just obtained of purely mathematical significance. Einstein insisted that each body had its own time — as it had its own space — and since time units became variable, it made no sense talking about two events occurring simultaneously or two processes lasting the same time. Each system in motion has its own clock and it is in the nature of things that these clocks cannot be synchronized.

4. Steady Growth Systems in the World of Economics

It is well known that the economic world is somewhat more complex than the physical world. Thus in steady growth systems one cannot expect to find simple regularity of inertial systems. The complexities of the real world in economics can be reduced to manageable proportions if we introduce a set of simplifying assumptions. The full treatment of the problem was undertaken elsewhere (3) (4). Here I shall repeat the analysis of the standard case only.

Growth implies capital. As far as capital is conceived as a means of production, the term primarily refers to fixed assets. As the standard case I take the one when fixed assets preserve their output capacity unimpaired until the end of their service life. This case appears to be also fairly realistic apart from being the most simple one (3).

The following simplifying assumptions will be used:

1. There is no technological progress.

2. Output is proportional to capital, $Y_t = pK_t$. Thus K represents not only gross capital but also output capacity. In the latter case it is implied that the units of measurement are changed by a constant factor p.

3. One type of good is produced and it serves both for capital formation and consumption (this good may be thought of as exchanged for other consumer goods from abroad).

4. Changes in labour and other non-capital costs are neglected.

5. The scrap value of a machine is zero.

6. The investment gestation period is zero.

7. The working with continuous growth rates implies, strictly speaking, perfect divisibility. Thus we assume that our capital good is perfectly malleable.

Assumptions 1 and 2 have been relaxed elsewhere (2). Assumption 3 is simplifying and inessential. The remaining assumptions simplify arithmetic considerably at no cost for the substance of the problem discussed.

I shall also use the following two definitions:

1. The steady growth system is the one that expands at a constant finite rate of growth (r = const.).

2. Real capital cost represents the investment necessary to maintain output capacity intact within the unit period of time. From the national economic planning point of view this seems to be the most meaningful definition of capital cost.

Suppose unit capital investment is made at time t = 0, when, say, the first machine is installed. Since then gross investment expands continuously at the rate r. By the time t gross capital stock — the number of machines in operation — will be equal to all investments made in the last n years; at t no machine installed before t-n is in operation any more.

$$K_t = \int_{t-n}^{t} e^{r\tau}\, d\tau = \frac{1}{r} e^{r(t-n)} \;(e^{rn} - 1) \qquad (10)$$

Replacement at t is, of course, equal to gross investment made n years earlier

$$R_t = G_{t-n} = e^{r\,(t-n)} \qquad (11)$$

Since, according to our definition, replacement represents real capital cost, the cost per unit of capital will be

$$k = \frac{R_t}{K_t} = \frac{r}{e^{rn}-1} \qquad (12)$$

It should be observed that the relation (12) no longer contains t. It follows that unit cost k is not dated and depends only on the rate of growth r and the life span n.

Relation (12) defines unit cost k in a growing system. We might wish to know whether k in a stationary system would be a different one.

$$k^\circ = \lim_{r \to 0} k = \frac{1}{n} \qquad (13)$$

The stationary capital cost is clearly different, which makes it necessary to find out the transformation factor by which capital cost of a stationary system will be transformed into the capital cost of a growing system

$$\frac{R}{R^\circ} = \frac{k}{k^\circ} = \frac{rn}{e^{nr}-1} = \beta \qquad (14)$$

The transformation factor β has the following three important properties

$$\lim_{r \to 0} \beta = 1, \qquad \lim_{r \to \infty} \beta = 0, \qquad \frac{d\beta}{dr} < 0 \qquad (15)$$

We at once recognise our old acquaintance, the relativity factor β. But now it is doing a very different job.

5. Similarities and Differences between Physical and Economic Transformation Factors

It appears that the transformation factor β has the same general characteristics in physics and economics. The content, is, of course, different. Bodies will contract in the moving physical systems; unit costs will shrink in the growing economic systems.

However, there are at least four other important differences.

1. In a relativity physics it makes no sense talking about negative velocities. It makes sense to talk about negative rates of growth in economics. Instead of growing the system is then decaying. It may be easilv found out that the formula (14) for β still applies, only r's assume negative values. Thus the economic ß, unlike the physical ß, is defined for both positive and negative values of the argument. The shape of the function is given in the diagram below.

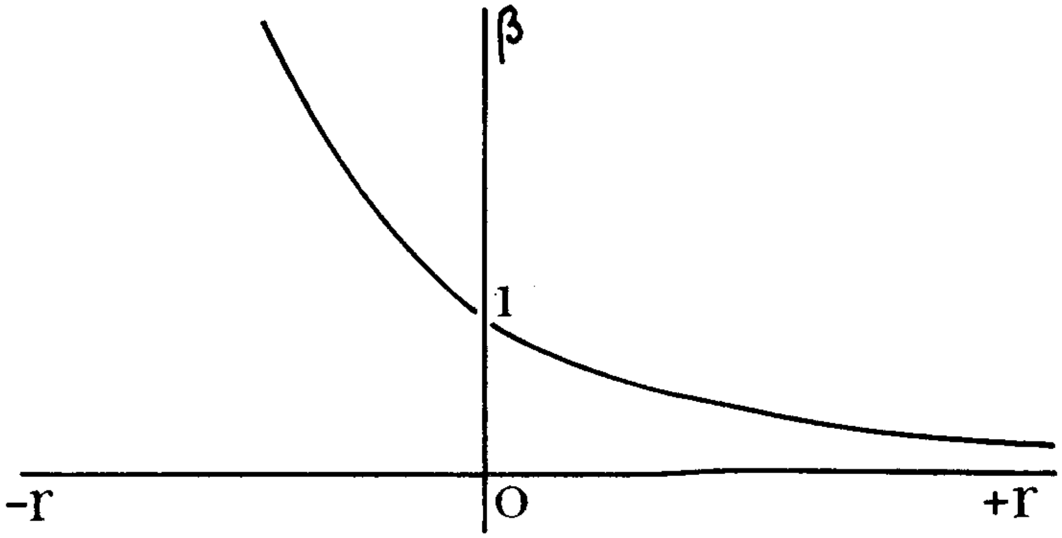

Fig. 1. Transformation function β

At r = 0 the function passes through the point ß = 1. The two branches of the function are not symmetrical. The right hand branch approaches the axis asymptotically towards infinity. The left hand branch stretches towards infinity with a slope $\frac{d\beta}{dr} = -n$. In conjunction with (5) the economic interpretation of ß is straightforward. In decaying systems capital costs are greater than in stationary situations and they increase with the rate of decay. In the growing systems capital costs are smaller than stationary costs and they decrease as the rate of growth increases. Since in the normal case the rate of growth is postitive, in the ensuing analysis we shall focus our atention to the right-hand branch of ß only.

2. Physical β — denote it as ß* — is concave towards the origin, while economic β is convex. The respective slopes when ß meets the axes are:

$$\left.\frac{d\beta^*}{dv}\right|_{v=0} = 0, \qquad \left.\frac{d\beta^*}{dv}\right|_{v=c} = -\infty$$

$$\left.\frac{d\beta}{dr}\right|_{r\to 0} = -\frac{n}{2}, \qquad \left.\frac{d\beta}{dr}\right|_{r\to\infty} = 0$$

Thus in physics the initial increases in v are less felt than in economics, while the final additions to v generate stronger effects than in economics. The tentative conclusion may perhaps be the following one: as far as the precision of analysis is concerned, in physics the Newtonian world in most situations makes full sense, while in economics the stationary world makes little sense, if any (assuming that one is interested in the real world).

3. It has been proved by experiments that v has an absolute limit. Apparently the same cannot be said of r. However, as it is shown elsewhere (2), in this respect there is in fact no difference between the two worlds. The rate of growth of an economic system has its absolute limit as well. But in order not to overburden the analysis from the very beginning, we shall neglect this complication for the time being.

4. If one takes into account that c is a constant, then physical β^* is a function of just one variable, v. Economic β is a function of two variables, r and n.

6. »Contraction« of Fixed Capital Cost and »Dilatation« of Economic Time

Growing systems imply changes which cannot be explained in the usual way, as functions of technology or otherwise. They are functions of the speed of change exclusively. Since speed or velocity are defined as processes per unit of time, changes will be observed only in connections with the quantities possessing a time dimension. Capital is such a quantity. Capital expended on the prodution of a unit of output, i. e. capital cost, will be subject to transformations depending on how fast a system grows.

Since β is defined as a ratio of dynamic (i. e. pertaining to growing systems) and static (i. e. pertaining to stationary systems) capital costs, it follows directly:

$$k = \beta k^{\circ} \tag{16}$$

capital cost diminishes as the rate of growth increases. The contraction of the cost quantity is thus analogous to the contraction of the volume of a body in physical systems in motion.

Again, starting from the definition of ß, and using the standard model as a point of departure, we get the following relation

$$\beta = \frac{k}{k^{\circ}} = \frac{R}{K} : \frac{R^{\circ}}{K^{\circ}} = \frac{1}{n} : \frac{1}{n^{*}}$$

Unit capital cost in a stationary economy that keeps the output capacity of fixed assets intact until they are scrapped is $\frac{1}{n^{*}}$ In an analogous way we assume that in a growing economy unit capital cost will be $\frac{1}{n}$ without specifying the dimensions of n. Now, n^{*} is a number indicating how many times annual R° is contained in K°. Similar interpretation applies to n in relation to growing R and K. However, n° is not just a dimensionless pure number, as it might appear by interpreting it as a ratio of two capital quantities, R° and K°. R° represents investment *per unit of time*. Thus n° has the time dimension. Obviously, n has to imply a time dimension as well. In fact n° represents the number of time units or years an asset is expected to last in a stationary system. Analogously n can be interpreted as the number of years an asset would last in a system growing relatively to the system S° which is taken as a standard of reference. It follows:

$$n = \frac{1}{\beta} n^\circ$$

the effects of growth on capital costs are such that it appears as if n° stationary years were lengthened $\frac{1}{\beta}$ times into n dynamic years.

The general case slightly complicates matters. Instead of a straightforward interpretation of n°, we shall now use the time dimension of k° to provide the necessary link. Denoting the length of time in the stationary world as $t^\circ_2 - t^\circ_1$, and in the dynamic world as $t_2 - t_1$, and defining β, which is a pure number, in terms of dimensions of constituent factors, we get

$$\beta = \frac{k}{k_\circ} = \frac{K(t_2 - t_1)^{-1}}{K} : \frac{K^\circ(t^\circ_2 - t^\circ_1)^{-1}}{K^\circ} = \frac{t^\circ_2 - t^\circ_1}{t_2 - t_1}$$

$$\therefore \; t_2 - t_1 = \frac{1}{\beta}(t^\circ_2 - t^\circ_1) \qquad (17)$$

However, unlike before, the time interval $(t_2^\circ - t_1^\circ)$ does not represent the calendar service life of the asset from the time of installation to the time of scrapping (n°). What sort of time is it then? Artificial? No! The decay of the capacity made simple calendar time meaningless. The economic time is the weighted average of the time duration of all particles of an asset. Denote this weighted average as m. Under proportional decay proceeding at the rate ρ an asset of the unit value will on the average last

$$m = \int_0^n K_t\, dt = \int_0^n e^{-\rho n}\, dt = \frac{1}{\rho}(1 - e^{-\rho n}) \qquad (18)$$

In a stationary situation replacement and maintenance will represent a constant proportion of the capital stock

$$M + R^\circ = \frac{1}{m} K^\circ \qquad (19)$$

Using (33) and (35) of (3) we have

$$k^{*\circ} = \rho + k^\circ = \rho + \frac{\rho}{e^{\rho n} - 1} = \frac{\rho}{1 - e^{-\rho n}} = \frac{1}{m} \qquad (20)$$

It appears that m represents the economic time of assets and is equal in (20) and (18) as it should be. From (18) it is also obvious that weighted m is shorter than calendar n, $m < n$.

Expression (17) is a generalization and corresponds to the dilatation of time in the physical world. An increase in the rate of growth is equivalent to the creation of time. Each economic system has its own, specific, economic time.

7. Concluding remarks

The mathematics of deriving physical and economic β's seems to suggest certain tentative conclusions about the nature of relativity phenomena. Such phenomena are characteristic only for *systems in motion*. Motion implies that the time dimension must be present. The time dimension is not independent of other dimensions in the system; it is an *essential component of the changing system itself*. For this to happen time must be positive and finite in relation to the change observed. In other words, there must be an *inherent limitation to the speed of change*. For light to travel a certain distance, a certain finite time is necessary. In fact, there is an absolute minimum of time that the structure of the universe allows for travelling a certain distance. The corresponding limitation in an economic system is to be found in the fact that fixed assets have a finite lifespan (For $n = 0$ or $n = \infty$ relativity phenomena disappear). Because of such limitations, which seem to be inherent in the universe, simple additions cannot be applied to changes nor can time be considered absolute and uniform. For the sake of completeness it may be mentioned that once we pass from inertial systems and steady growth to accelerations, we encounter new dynamic phenomena. In the physical world acceleration generates a gravitational field which is absent in an inertial system. I cannot see a close analogy to gravitational fields in economics within the framework of the present analysis. But there is a definite analogy between accelerations in both systems. Acceleration generates diminishing returns, and they are, in turn, responsible for some novel features of the growing systems. More specifically, the rate of growth can no more be infinitely great; it has an absolute limit. The generalization of the analysis enables us to relax those two crucial assumptions about technological progress and proportionality of output to capital (2). However, all this is a subject for a separate study.

REFERENCES

1. B. Horvat, »Depreciation Multiplier and a Generalized Theory of Fixed Capital Cost«, *Manchester School*, 1958, 136—59; revised version in *Towards a Theory of Planned Economy*, Institute of Economic Sciences, Belgrade, 1964.

2. ————, »A Model of Maximal Economic Growth,« *Kyklos*, 1972, 215—28.

3. ————, »Real Fixed Capital Costs Under Steady Growth,« *European Economic Review*, 1973, forthcoming.

4. ————, »Fixed Capital Cost, Depreciation Multiplier and the Rate of Interest,« *European Economic Review*, 1973, forthcoming.

IZVJESNE SLIČNOSTI IZMEĐU INERCIJALNIH SISTEMA U FIZICI I UNIFORMNO RASTUĆIH SISTEMA U EKONOMIJI

Branko HORVAT

Rezime

Kretanje, bilo fizikalno bilo ekonomsko (rast), dovodi do relativitetnih fenomena. Razlog je u tome što fizikalni i ekonomski sistemi uključuju vrijeme kao fundamentalnu dimenziju. Ne postoji apsolutno, uniformno, univerzalno vrijeme. Svaki sistem ima svoje vrijeme. Uniformno rastući sistemi u ekonomiji odgovaraju inercijalnim sistemima u fizici. Što je viša stopa rasta, niži su troškovi osnovnih sredstava i sporiji je tok vremena. Transformacioni faktor β *pretvara više, stacionarne, troškove fiksnih fondova u niže, dinamičke. Numerička vrijednost tog faktora ovisi o vremenskom profilu osnovnih sredstava, ali osnovne karakteristike su uvijek iste:* $\lim_{r \to 0} \beta = 1$, $\lim_{r \to \infty} \beta = 0$, $\frac{d\beta}{dr} < 0$. *Ekonomski* β *odgovara Lorentzovom transformacionom faktoru u fizici. Ispituju se razlike i sličnosti tih dvaju transformacionih faktora.*