SOME CONDITIONS OF MACROECONOMIC STABILITY IN MULTIREGIONAL MODELS

Ranko BON)*

This investigation was a part of a comparative study of the three multiregional input-output (MRIO) models: column coefficient, row coefficient, and gravity coefficient.[1]) The objectives of the research were twofold: (1) to examine the causes underlying negative values in the inverse generated by the row coefficient model, as well as negative projection generated by the model; and (2) to explain why the column coefficient model did not present any of these problems.

The first section provides a brief introduction to the two MRIO models. In the first part of the second section, several theorems concerning the required properties of the technical coefficient matrix that ensure the generation of non-negative inverses and non-negative projections of Leontief's input-output model are employed and extended to MRIO models. Two new theorems concerning the properties of the regional trade coefficient matrix that ensure the generation of non-negative inverses and non-negative projection in MRIO models in general are provided. Next, the results concerning MRIO models in general, which were derived in the first part, are applied in the analysis of the two MRIO models. The objective of these two parts is to determine whether the two models satisfy the conditions that ensure non-negative inverses and projections. An economic interpretation of the relationship between the column coefficient and row coefficient models is then presented.

The results of this research provide: (1) construction rules for the regional trade coefficient matrix which ensure that the projections generated by MRIO models will be non-negative; (2) on the basis of these rules, a test of regional technology and regional trade data that ensures non-negative projections for well-constructed MRIO models; and (3)

*) This paper is a somewhat modified version of the author's doctoral dissertation, defended in June 1975 at the Department of Urban Studies and Planning, Massachusetts Institute of Technology, Cambridge, Massachusetts. The author worked for two years as a research assistant in the Multiregional Input-Output (MRIO) Project at M.I.T., lead by Karen Polenske, Associate Professor of Regional Economics and Planning. The author wishes to thank Prof. Karen Polenske (thesis supervisor), Prof. Aaron Fleisher (thesis advisor), Dr. Malte Möhr, Mark Schuster and Nathaniel Ng for numerous comments and suggestions that helped shape this work.

[1]) For a comparative analysis of the column coefficient and gravity coefficient models, see Fenci and Ng [5]. For a comparison of the results of the application of the column coefficient and row coefficient models using 1963 MRIO data for the United States, aggregated into three regions and three industrial sectors, contact: Ranko Bon, Linhartova 9, 61000 Ljubljana, tel. (061) 326-855.

an explanation of the malfunction of the row coefficient model, which concentrates on the violation of the rules.

Finally, the policy implications of this investigation extend the conclusions of Hawkins and Simon [7] from the single-region economy, to the multiregional economy: if the production system is internally consistent, it will be consistent with *any* schedule of consumption goods, the latter representing a set of policy variables.

MULTIREGIONAL INPUT-OUTPUT MODELS

Multiregional input-output models are essentially conventional input-output models modified to incorporate interregional trade.[2]) These models are founded on one basic economic principle: the total output of an industry is equal to the sum of intermediate demands by various industries (including the industry itself) and demands by final users of the industry's products.

Mathematically, this relationship can be expressed as a set of linear equations:

$$x_i = \sum_{j=1}^{m} a_{ij} x_j + y_i \quad \text{(all i)}, \tag{1}$$

where

a_{ij} = technical coefficient representing the amount of input of commodity i required by industry j to produce one unit output of commodity j.

x_i = total supply of commodity i.

x_j = total production of commodity j.

y_i = final demand of commodity i.

i,j = 1, . . . , m.

Assuming no trade between regions, an input-output model for m industries and n regions can be represented by the following set of linear equations:

$$x_i^{og} = \sum_{j=1}^{m} a_{ij}^{g}\, x_j^{go} + y_i^{g} \quad \text{(all i)}, \tag{2}$$

where

a_{ij}^{g} = technical coefficient representing the amount of input of commodity i required by industry j located in region g to produce one unit of output of commodity j.

[2]) The reader who is not familiar with multiregional input-output models is advised to refer to Yan [17] for a detailed analysis of national input-output models and to Miernyk [9] for an introduction to regional input-output models. More advanced material on multiregional models can be found in Polenske [12; 13].

x_i^{og} = total supply of commodity i in region g.

x_j^{go} = total produciton of commodity j in region g.

y_i^g = final demand of commodity i in region g.

$i,j = 1, \ldots, m.$

$g = 1, \ldots, n.$

If equation (2) is to be used to describe a multiregional model, it must be further modified to account for the commodities traded between regions. The following two sections will describe the column coefficient and row coefficient models, respectively, since each of the two models utilizes a different accounting scheme for interregional trade.[3])

Column Coefficient Model[4])

Interregional trade is described in the column coefficient model by means of the following relationship:

$$x_i^{gh} = c_i^{gh} \, x_i^{oh} \text{ (all i)},$$

where

x_i^{gh} = amount of commodity i produced in region g that is shipped to region h.

x_i^{oh} = total amount of commodity i consumed in region h.

c_i^{gh} = trade parameter, indicating the fraction of total consumption of commodity i in region h that is produced in and shipped from region g.

i = 1, . . , m.

g, h = 1, . . . , n.

Equations (2) and (3) are combined to obtain the following set of linear equations (in matrix notation):

$$X = C(AX + Y), \tag{4}$$

where

X = nm · 1 vector of regional outputs, x_i^{go}, arranged as a column vector with m outputs for each of the n regions.

C = nm · nm diagonal block matrix of regional trade coefficients,

---

3) For a more detailed description of the accounting frameworks, see Polenske [13].
4) This is the version as first described by Chenery and Clark [2] and Moses [10].

$$c_i^{gh} = x_i^{gh} / x_i^{oh}$$

where

$$\sum_{g=1}^{m} c_i^{gh} = 1,$$

with each of the diagonals of the n·n submatrices C_i containing the coefficients for m traded commodities and all off-diagonal elements equal to zero.

A = nm · nm block diagonal matrix of regional technical coefficients,

$$a_{ij}^h = x_{ij}^h / x_{oj}^h$$

where

$$\sum_{i=1}^{m} a_{ij}^h < 1,$$

with each of the n submatrices A^h along the principal diagonal containing the m·m coefficient matrix derived from each of the n regional input-output tables, and the elements in all blocks off the principal diagonal being equal to zero.

Y = nm · 1 vector of regional final demands, y_i^h, arranged as a column vector with m elements representing the amount of commodity i purchased by final users in each of the n regions.

In the implementation of the column coefficient model, specified by equation (4), Y is the unknown and is eliminated from the right-hand side of the equation as follows:[5])

$$X = CAX + CY$$

$$X - CAX = CY$$

$$(I - CA)X = CY$$

$$X = (I - CA)^{-1} CY, \qquad (5)$$

or

$$X = (C^{-1} - A)^{-1} Y. \qquad (6)$$

To calculate the regional outputs, X, from equations (5) or (6), matrices A and C and the vector Y must first be obtained.

[5]) It should be noted that in the formulation of equation (6), it is implied that $|C| \neq 0$ since equations (5) and (6) are equivalent only under this condition. This means, among other things, that C cannot have zero columns or zero rows. In economic terms, it is implied that if there is an industry i in the economy, then commodity i must be both produced and consumed in region g. Consequently, this formulation may be of restricted applicability in regional analysis. More precisely, it is contingent upon the level of aggregation of the data employed. It should be added, however, that this problem has not appeared so far in the work with the model, even though this formulation is typical used in empirical work. Therefore, this implicit assumption is likely to be reasonable for highly aggregated data.

Row Coefficient Model

Since there are many similarities between the column coefficient and the row coefficient models, the latter having been conceived as the »mirror image« of the former, the row coefficient model will be described in less detail.

Interregional trade is described in the row coefficient model by means of the following relationship:

$$x_i^{gh} = r_i^{gh} \ x_i^{go} \quad \text{(all i)}, \tag{7}$$

where

x_i^{gh} = amount of commodity i produced in region g that is shipped to region h.

x_i^{go} = total amount of commodity i produced in region g.

r_i^{gh} = trade parameter, indicating the fraction of total production of commodity i in region g that is shipped to region h.

$i = 1, \ldots, m.$
$g, h = 1, \ldots, n.$

Equations (2) and (7) are combined to obtain the following set of linear equations (in matrix notation):

$$R'X = AX + Y, \tag{8}$$

where

X = nm · 1 vector of regional outputs.

R′ = transpose of R, where R is an nm · nm diagonal block matrix of regional trade coefficients,

$$r_i^{gh} = x_i^{gh} / x_i^{go}$$

where

$$\sum_{h=1}^{m} r_i^{gh} = 1,$$

with each of the diagonals of the n · n submatrices R_i containing the coefficients for m traded commodities and all off-diagonal elements being equal to zero.

A = nm · nm block diagonal matrix of regional technical coefficients.

Y = nm · 1 vector of regional final demands.

In the implementation of the row coefficient model, specified by equation (8), X is the unknown and is eliminated from the right-hand side of the equations as follows:[6])

$$(R' - A)X = Y$$

$$X = (R' - A)^{-1}Y, \tag{9}$$

or

$$X = [I - (R')^{-1}A]\,(R')^{-1}\,Y. \tag{10}$$

To calculate the regional outputs, X, from equations (9) or (10), matrices A and R' and the vector Y must first be obtained.

MACROECONOMIC STABILITY OF MULTIREGIONAL INPUT-OUTPUT MODELS

A real n-square matrix $A = \| a_{ij} \|$ is called *positive (non-negative)* if $a_{ij} > 0$ $(a_{ij} \geq 0)$ for $i, j = 1, \ldots, n$. If A is positive (non-negative), it is denoted by $A > 0$ $(A \geq 0)$.

The properties of positive matrices were first investigated by Perron, and then amplified and generalized for non-negative matrices by Frobenius. Wielandt provided considerably simplified proofs for Frobenius's results. Positive and non-negative square matrices have played an important role in the probabilistic theory of Markov chains, as well as in the more recent study of linear models in economics, and particularly in connection with Leontief's input-output model. The matrices of interest in this study were first noted by Minkowski.[7])

In this section, first, the problem under investigation is rigorously stated. Second, several well-known theorems concerning the properties of the technical coefficient matrix that ensure the generation of non-negative projections of Leontief's input-output model are summarized and stated without proof. Third, these theorems are applied to the multiregional input-output models. And fourth, two new theorems concerning the properties of the regional trade coefficient matrix that ensure the generation of non-negative projections in MRIO models in general, are proved.

Then, the results derived in the first part of this section are applied in an analysis of the column coefficient and row coefficient models, respectively, to determine whether the two models satisfy the conditions that ensure non-negative projections. A formal argument is presented demonstrating that the mathematical properties of the column coefficient model are compatible with these conditions, while the opposite is true in the case of the row coefficient model.

---

[6]) It should be noted that in the formulation of equation (10), it is implied that $|R'| \neq 0$ since equations (9) and (10) are equivalent under this condition (see Footnote 5).

[7]) For a historical outline of the underlying concepts, the basic theorems on positive and non-negative matrices, and an extensive bibliography, see Bellman [1, pp. 286—315].

Finally, an economic interpretation is given of the formal argument concerning the structure of the two models developed in the preceding two parts. It is argued that the present formulation of the row coefficient model is not a consistent »mirror image« of the column coefficient model, as it was intended to be.

Multiregional Input-Output Models

Consider the general formulation of an MRIO model that corresponds to equations (5) and (10) for the column coefficient model and the row coefficient model, respectively:

$$X = (I - \Theta A)^{-1}\,\Theta Y, \tag{11}$$

where

X = vector of regional outputs.

Θ = diagonal block regional trade coefficient matrix.

A = block diagonal regional technical coefficient matrix.

Y = vector of regional final demands, $y_i^h \geq 0$ for i = 1, . . . , m and h = 1, ... , n.

It is assumed that Θ, A, and Y are independent, and that $[I - \Theta A] \neq 0$.[8])

To be economically meaningful, all the elements of X must be *positive* for indecomposable ΘA, and *non-negative* for decemposable ΘA.[9])

This will be ensured if

$$(I - \Theta A)^{-1} \tag{12}$$

is positive for indecomposable ΘA and *non-negative* for decomposable ΘA. If so much as one negative element appears in matrix (12), then

[8]) Matrices Θ, A, and ΘA for an n-region, m-industry economy can be found in Appendix A.

[9]) An n-square matrix A (n > 1) is said to be *indecomposable* if for no permutation matrix T does

$$A_T = TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square. Otherwise A is *decomposable*. If $A_{12} = 0$, A is *completely decomposable*. (The terms »irreducible« and »reducible« are often used instead of indecomposable and decomposable). A permutation matrix is obtained by permuting the columns of an identity matrix. TAT^{-1} is obtained by performing the same permutation on the rows and on the columns of A. These concepts can be economically interpreted as follows: If n industries are connected by two-way links directly or indirectly, the system is indecomposable. »In an indecomposable matrix a dollar spent anywhere will eventually, after at most n—1 'rounds,' leak at least in part to every other sector« (Solow [16, p. 34]). If k (k ≤ n) industries are connected by one-way links, the system is decomposable. The system is completely decomposable if there are no links between two or more groups of industries. These separable groups can then be analyzed separately. For discussion of the economic significance of these concepts see Dorfman, Samuelson, and Solow [4, pp. 254—255], and Solow [16, pp. 33—35].

there is at least one ΘY that will lead to economically meaningless negative outputs. Suppose the (g, h) element of the matrix is negative; then if ΘY is a vector with very small elements except for a large

$$\sum_{h=1}^{n} \theta_i^{gh} y_i^h$$

the element x_i^g of X will be negative. Indeed, since matrix (12) represents both direct and indirect interindustry requirements of a productive system, negative values in the matrix cannot be meaningfully interpreted in economic terms. As Solow put it,

> In static input-output terms, the main formal question is the existence of a solution with no negative outputs; essentially, this is a way of asking if the system of industries is capable of supporting itself. [16, p. 30]

The problem under investigation can therefore be stated as follows:

(1) What are the necessary and sufficient conditions on Θ that ensure that matrix (12) is *positive* for indecomposable ΘA and *non-negative* for decomposable ΘA, given that A has the following properties:

$$0 \leqslant a_{ij} < 1 \text{ (all i,j)}^{10}, \tag{13}$$

and

$$\sum_{i=1}^{u} a_{ij} < 1 \text{ (all j)}. \tag{14}$$

(2) Which MRIO models statisfy the conditions on Θ to be derived under (1).

Several theorems concerning a particular class of positive and non-negative matrices arise in connection with the solution of the system of linear equations of the form:

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + y_i \text{ (all i)} \tag{1}$$

which is associated with Leontief's input-output model. These will now be summarized and stated without proof.[11])

It should be borne in mind that the conditions which ensure that the elements of (12) will be positive or non-negative, presented below, are sufficient, but not necessary. These results will be strengthened

---

10) It should be noted that throughout this section, whenever a symbol has no superscripts, the subscripts denote only the position of an element within a matrix.

11) For proofs, see Bellman [1, pp. 286—315, especially p. 298], Debreu and Herstein [3], Hadley [6, pp. 118—119], Hawkins and Simon [7], Marcus and Minc [8, pp. 121—133], Rogers [15, pp. 405—438, and especially pp. 418—420], and Solow [16].

in the second part of the argument that follows to yield a set of more general sufficient conditions. The first set of conditions will be presented because it yields results directly applicable in the treatment of the column coefficient model, while the second set of conditions will be of use in the discussion of the row coefficient model.

THEOREM 1: $(I-A)^{-1}$ will be *positive* if $A > 0$ and condition (14) is satisfied, implying that A is indecomposable, or if conditions (13) and (14) are satisfied and A is indecomposable,[12]) and *non-negative* if conditions (13) and (14) are satisfied and A is decomposable.

This condition, *mutatis mutandis,* applies to multiregional models as well.

COROLLARY: $(I-\Theta A)^{-1}$ will be *positive* if conditions (15a) and (16) below are satisfied, implying that ΘA is indecomposable, or if conditions (15b) and (16) are satisfied and ΘA is indecomposable, and *non-negative* if conditions (15b) and (16) are satisfied and ΘA is decomposable:

$$0 < d_{ij} < 1 \quad \text{(all i,j)}, \tag{15a}$$

or

$$0 \leqslant d_{ij} < 1 \quad \text{(all i,j)}, \tag{15b}$$

and

$$\sum_{i=1}^{n} d_{ij} < 1 \quad \text{(all j)}, \tag{16}$$

where

$$d_{ij} = \sum_{k=1}^{n} \theta_{ik}\, a_{kj}.^{13)} \tag{17}$$

The properties of Θ that satisfy conditions on ΘA still remain to be established.

THEOREM 2: When conditions (13) and (14) are satisfied, conditions (15a) or (15b) and (16) for indecomposable ΘA and conditions (15b) and (16) for decomposable ΘA will be satisfied if the following sufficient conditions on Θ are satisfied:

---

[12]) There is an equivalent theorem by Hawkins and Simon [7] that, although contained in Theorem 1, provides an important economic interpretation of the phenomenon under investigation (see Apendix B). It should be noted that the conditions of Theorem 1 are often referred to as Hawkins-Simon conditions, even though their original result has subsequently been considerably improved and sharpened. For discussions of Hawkins-Simon conditions, the reader is advised to refer to Dorfman, Samuelson, and Solow [9, especially pages 215, 254—257, and 500], and Solow [16, especially pages 31 and 32].

[13]) It should be noted that due to the constuction of Θ and A all the sums d_{ij} will have only one term, all other terms being equal to zero (see Appendix A). It should also be noted that all the elements of ΘA will be positive if all the elements along the diagonals of all the blocks of Θ are positive, and if all the element in the blocks on the principal diagonal of A are positive. In other words, ΘA may be indecomposable regardless of the fact that both Θ and A are completely decomposable (note that diagonal block matrix Θ can be transformed into a block diagonal matrix by regrouping rows and corresponding columns with the same pattern of elements in blocks along the principal diagonal). Whether ΘA will be indecomposable or decomposable will depend upon the particular economic system under investigation.

$$0 \leqslant \theta_{ik} \leqslant 1 \quad \text{(all i,k)}, \tag{18}$$

and

$$\sum_{i=1}^{n} \theta_{ik} \leqslant 1 \quad \text{(all k)}. \tag{19}$$

The proof will be provided in two parts: (a) *sufficiency:* it will be assumed that conditions (18 and (19) are satisfied, and (b) *necessity:* it will be assumed that conditions (18) or (19) or both are not satisfied.

Sufficiency. Condition (16) will be examined first. From condition (16) and equation (17) it follows that

$$\sum_{i=1}^{n} d_{ij} = \sum_{i=1}^{n} \sum_{k=1}^{n} \theta_{ik} a_{kj}, \tag{20}$$

$$= \sum_{k=1}^{n} \sum_{k=1}^{n} \theta_{ik} a_{kj},$$

$$= \sum_{k=1}^{n} \left[a_{kj} \sum_{i=1}^{n} \theta_{ik} \right].$$

Let

$$\sum_{i=1}^{n} \theta_{ik} = 1 \quad \text{(all k)};$$

since

$$\sum_{k=1}^{n} a_{kj} < 1 \quad \text{(all j)},$$

it follows that

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \theta_{ik} a_{kj} < 1.$$

Condition (16) is therefore satisfied, given conditions (18) and (19). Now conditions (15a) and (15b) will be examined. First, it follows by implication of the above result that $d_{ij} < 1$ since (16) is satisfied. Second, $d_{ij} \geqq 0$ follows from conditions (13) and (18), ensuring that all the elements of A and Θ, respectively, are non-negative. The possibility of $d_{ij} > 0$ has already been shown in Footnote 13 (page 59); all the elements of ΘA will be positive if all the elements along the diagonals of all the blocks of Θ are positive and if all the elements in the blocks on the principal diagonal of A are positive. Conditions (15a) and (15b) are therefore satisfied as well.

Necessity. Now suppose that either condition (18) or (19) or both are not satisfied.

(i) Suppose $\theta_{ik} < 0$ (all i, k); in this case neither condition (15a) nor (15b) will be satisfied since

$$\sum^{n} \theta_{ik} a_{kj} < 0.$$

If so much as one element of Θ is smaller than zero, then there is at least one A that will lead to the violation of conditions (15a) and (15b). Suppose the (g, h) element of Θ is negative; then if column vector a_{m} of A has very small elements except for a large a_{hm}, the element d_{gm} of ΘA will be negative.

(ii) Suppose $\theta_{ik} > 1$ (all i, k); using the same method of proof as was used in the treatment of sufficient conditions above, it can be shown that in this case conditions (15a), (15b), and (16) will not be statisfied since

$$\sum_{i=1}^{n} \theta_{ik} > 1 \quad \text{(all k)},$$

and consequently

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \theta_{ik}\, a_{kj} > 1$$

for some values of a_{kj}, $0 \leq a_{kj} < 1$. If so much as one element of Θ is greater than one, there is at least one A that will lead to the violation of conditions (15a), (15b), and (16). Suppose the (g, h) element of Θ is greater than one; then if column vector a_{m} of A has very small elements except for a large a_{hm}, the element d_{gm} of ΘA will be greater than one.

(iii) Suppose

$$\sum_{i=1}^{n} \theta_{ik} > 1 \quad \text{(all k)};$$

in this case conditions (15a), (15b), and (16) will not be satisfied for the reasons discussed in (ii) above.

Therefore, when conditions (13), (14), (18), and (19) are satisfied, the elements of matrix (12) will be *positive* when ΘA is indecomposable and *non-negative* when ΘA is decomposable. Furthermore, given that all the elements of ΘY are non-negative, all the elements of X will be *positive* when ΘA is indecomposable and *non-negative* when ΘA is decomposable.

The above theorems can be strengthened; that is, the above conditions can be relaxed to provide a set of more general sufficient conditions. These conditions will also be of value in providing one more step toward an economic interpretation of the phenomena discussed.

As Bellman [1, p. 299] pointed out, if

$$\sum_{i=1}^{n} a_{ij} = 1 \quad \text{(all j)},$$

then 1 is a characteristic root of A, thus $|I—A| = 0$. In other words, the system of equations describing an economy composed of unstable sectors, with unitary marginal propensities to spend, cannot yield an equilibrium solution; the system is singular. It is reasonable, however, to suppose that conditions (14) and (16) can be relaxed. It should be borne in mind, nevertheless, that the unitary marginal propensity to spend represents the upper limit beyond which this condition cannot be relaxed. The argument will again proceed from the theorem concerning the system of equations (1), which will be stated without proof,[14]) to its application to equation (11).

THEOREM 3: $(I-A)^{-1}$ will be *positive* if

$$0 < a_{ij} < 1 \quad \text{(all i,j)}, \tag{21}$$

$$\sum_{i=1}^{n} a_{ij} < 1 \quad \text{(at least one j)}, \tag{22a}$$

and

$$\sum_{i=1}^{n} a_{ij} \leqslant 1 \quad \text{(all j)}, \tag{23}$$

implying that A is indecomposable, or if conditions (13), (22a), and (23) are satisfied and A is indecomposable; $(I-A)^{-1}$ will be *non-negative* if conditions (13) and (23) are satisfied and

$$\sum_{i=1}^{h} a^{*}_{ij} < 1 \quad \text{(at least one j; } h < n) \tag{22b}$$

for all h-square submatrices A* along the principal diagonal of A, where each submatrix A* represents a partition of A, implying that A is decomposable.[15])

Again, this result, *mutatis mutandis*, applies to multiregional models also.

COROLLARY: $(I-\Theta A)^{-1}$ will be *positive* if

$$0 < d_{ij} < 1 \quad \text{(all i,j)}, \tag{15a}$$

$$\sum_{i=1}^{n} d_{ij} < 1 \quad \text{(at least one j)}, \tag{24a}$$

and

$$\sum_{i=1}^{n} d_{ij} \leqslant 1 \quad \text{(all j)}, \tag{25}$$

implying that ΘA is indecomposable, or if conditions (15b), (24a), and (25) are satisfied and ΘA is indecomposable; $(I-\Theta A)^{-1}$ will be *non-negative* if conditions (15b) and (25) are satisfied and

---

[14]) For the proof, see Solow [16, pp. 36—38].
[15]) See footnote 10.

$$\sum_{i=1}^{h} d^*_{ij} < 1 \quad \text{(at jeast one j; } h < n) \tag{24b}$$

for all h-square submatrices ΘA^* along the principal diagonal of ΘA, where each submatrix ΘA^* represents a partition of ΘA, implying that ΘA is decomposable.

The properties of Θ that satisfy conditions on ΘA will be established next.

THEOREM 4: When conditions (13) and (14) are satisfied, conditions (15a) or (15b), (24a), and (25) for indecomposable ΘA will be satisfied if the following conditions on Θ are satisfied:

$$0 \leqslant \theta_{ik} < 1/a_{kj} \quad \text{(all i,k),)} \tag{26}$$

and

$$\sum_{i=1}^{n} \theta_{ik} \leqslant 1 \quad \text{(at least one k);} \tag{27a}$$

when conditions (13) and (14) are satisfied, conditions (15b), (24b), and (25) for decomposable ΘA will be satisfied if, in addition to condition (26), the following condition on Θ is satisfied:

$$\sum_{i=1}^{h} \theta^*_{ik} \leq 1 \quad \text{(at least one k; } h < n) \tag{27b}$$

for all h-square submatrices ΘA^* along the principal diagonal of ΘA, where each submatrix ΘA^* represents a partition of ΘA.

Proof: Condition (26) follows directly from conditions (15a) and (15b) on ΘA, and condition (13) on A. Conditions (27a) and (27b) follow from the proof of sufficiency of the less general conditions (18) and (19) above; what was true for all k previosly, obviously holds for at least one k for indecomposable matrices, and for at least as many k as there are partitions of decomposable matrices.

Consequently, when conditions (13), (14), (26), and (27a) are satisfied, and when ΘA is indecomposable, the elements of matrix (12) will be *positive;* when conditions (13), (14), (26), and (27b) are satisfied, and when ΘA is decomposable, the elements of the matrix will be *non-negative.*

According to Solow, it follows that if an economic system is indecomposable

> It is sufficient for an industry to be able to lift itself by its own bootstraps ... and for all other industries to be just able to keep themselves going; if the interindustrial relationships are such that each of the latter industries shares at some stage the leverage of the first industry, then the system is capable of equilibrium at positive consumption. [16, p. 39].

If it is decomposable, the above applies to each decomposable sector. Furthermore, according to Solow,

> If the unstable sectors are uncoupled or only unilaterally coupled to the stable part of the economy, the instability may become general. If they are multilaterally coupled, then the stable sectors will always tame the unstable ones. [16, p. 35]

On the most general level, the policy implications of these conclusions are obvious: by proper coupling of the unstable sectors with stable sectors, the stability of the system as a whole can be achieved. Since indecomposability is essentially a property of connectedness, Solow suggests the need for further research into the topological properties of linear systems. Such research could lead to concrete policy proposals concerning the coupling of industrial sectors.

Column Coefficient Model

Consider the formulation of the column coefficient model that corresponds to the general formulation of an MRIO model specified by equation (11):

$$X = (I - CA)^{-1} CY, \qquad (5)$$

where $C = \|c_{ij}\|$ has the following properties:

$$0 \leq c_{ij} \leq 1 \quad \text{(all i,j)}, \qquad (28)$$

and

$$\sum_{i=1}^{n} c_{ij} = 1 \quad \text{(all j).}^{16)} \qquad (29)$$

Since (5) and (11) are equivalent, and since (28) and (29) satisfy (18) and (19), it follows that the column coefficient MRIO model satisfies the conditions on C that ensure that

$$(I - CA)^{-1} \qquad (30)$$

is *positive* for indecomposable CA and *non-negative* for decomposable CA. Consequently, given that all the elements of CY are non-negative, all the elements of X are *positive* for indecomposable CA and *non-negative* for decomposable CA. In other words, the column coefficient MRIO model is structurally correct.[17)]

[16)] It is interesting to note that these properties are shared by Markov matrices, associated with finite Markov chains, and also that the research concerning the properties of positive and non-negative matrices started in connection with these matrices and was only later extended in connection with linear economic models. For a historical outline of the underlying concepts and an extensive bibliography, see Bellman [1, pp. 263—280].

[17)] Moses [10, p. 830] briefly argued that the column coefficient model is consistent, although he did not provide a rigorous proof of his argument. Also, his argument is incomplete since it does not take into consideration the distinctions between positive and non-negative matrices and between indecomposable and decomposable matrices.

Row Coefficient Model

Consider the formulation of the row coefficient model that corresponds to the general formulation of an MRIO model specified by equation (11):

$$X = \Big[I - (R')^{-1}A\Big]^{-1}(R')^{-1}Y, \qquad (10)$$

where $R' = \|\bar{r}_{ij}\|$ has the following properties:

$$0 \leq \bar{r}_{ij} \leq 1 \quad \text{(all i,j)} \qquad (31)$$

and

$$\sum_{i=1}^{n} \bar{r}_{ij} = 1 \quad \text{(all j)}^{19} \qquad (32)$$

The properties of $(R')^{-1} = \|\bar{r}^*_{ij}\|$ will be examined next. First, the conditions (18) and (19) will be assumed to hold, and second, more general conditions (26) and (27a) od (27b) will be assumed to be satisfied.

Assuming that conditions (13) and (14) are satisfied,

$$[I - (R')^{-1}A]^{-1}$$

will be *positive* if conditions (34) and (35) below are satisfied and $(R')^{-1}$ A is indecomposable and *non-negative* if conditions (34) and (35) are satisfied and $(R')^{-1}$ A is decomposable:

$$0 \leq \bar{r}^*_{ij} \leq 1 \quad \text{(all i,j)}, \qquad (34)$$

and

$$\sum_{i=1}^{n} \bar{r}^*_{ij} \leq 1 \quad \text{(all j)}. \qquad (35)$$

It will now be shown that elements of $(R')^{-1}$ do not satisfy conditions (34) and (35).

Given properties (31) and (32) of R', the absolute value of the dominant characteristic root of R' is equal to one (Bellman [1, p. 270]). Now the characteristic root of a matrix and its inverse are inverses of each other (Rogers [15, pp. 410—411]):

$$\beta_i = 1/\lambda_i,$$

where λ_i is a characteristic root of R', and β_i is a characteristic root of $(R')^{-1}$. Consequently, since there is a characteristic root of R' the absolute value of which is smaller than one, the absolute value of the corresponding characteristic root of $(R')^{-1}$ will be greater than one. Indeed, the absolute value of the dominant characteristic root of R' will corre-

[19]) See footnote 16.

spond to the absolute value of the smallest characteristic root of $(R')^{-1}$ and will be equal to one. Therefore, the elements of $(R')^{-1}$ will take both negative values and values greater than one. It follows that $(R')^{-1}$ does not satisfy conditions (34) and (35), which correspond to conditions (18) and (19 for MRIO models in general, that is, that $(R')^{-1}A$ does not satisfy conditions (15a) or (15b) and (16).

The more general conditions (26) and (27a) or (27b) will be considered next. Regardless of whether $(R')^{-1}A$ is indecomposable or decomposable, condition (26) must be satisfed by $(R')^{-1}$. From the preceding argument it is obvious that this condition is not satisfied. It is sufficient to point out that $(R')^{-1}$ is not non-negative as the lower bound of this condition requires. The row coefficient model is, therefore, structurally incorrect. It can be shown that this conclusion holds even when the assumption that $|R'|$ is non-singular, made explicity in Footnote 6, is dropped. Suppose $|R'| = 0$. The least restrictive formulation of the row coefficient model, that is,

$$X = (R' - A)^{-1}Y, \tag{9}$$

will be considered in this case. Now a matrix will have a *positive* inverse if it is indecomposable and all its elements on the principal diagonal are positive while all the off-diagonal elements are *negative;* a matrix will have a *non-negative* inverse if it is decomposable and all its elements on the principal diagonal are positive while the off-diagonal elements are *non-positive* (Debreu and Herstein [3, pp. 602—603]). Given the properties of R′ and A, it is obvious that there is nothing in the structure of the row coefficient model that prevents the elements on the principal diagonal of R′ from being smaller than the corresponding elements of A. In other words, the elements on the principal diagonal of (R′ — A) may be non-positive. Furthermore, the off-diagonal elements of (R′ — A) can be positive, negative, or equal to zero. Consequently, the assumption that R′ is singular does not modify the above conclusions. In the next part of this section, it will be demonstrated that a certain number of the off-diagonal elements of (R′ — A) will be positive by structural necessity. It follows that the row coefficient model as presently formulated will generate negative inverses even when the elements on the principal diagonal of (R′ — A) are positive.

The Relationship Between the Column Coefficient and Row Coefficient Models

The objective of this part is to provide an economic interpretation of the formal argument presented earlier and to re-examine the structure of the two models in more detail in light of this interpretation. The column coefficient and row coefficient models will be developed following Chenery and Clark [2] in order to trace the economic reasoning underlying the two models. For simplicity, Chenery and Clark consider a 2-region, n-industry model that can be easily extended to any number of regions. For each industry i, there is a set of accounting relations describing the flows between the two regions, as shown in Table 1.

Table 1

Interregional Accounts for Industry

Producing Region	Consuming Region *g*	Consuming Region *h*	Total Shipments (Production)
g	x_i^{gg}	x_i^{gh}	x_i^g
h	x_i^{hg}	x_i^{hg}	x_i^h
Total Consumption (Supply)	z_i^g	z_i^h	

From this Table, it follows that the production of industry i in region g can be defined as:

$$x_i^g = x_i^{gg} + x_i^{gh}, \tag{36}$$

while the supply of industry i in region g can be defined as:

$$z_i^g = x_i^{gg} + x_i^{hg}. \tag{37}$$

The set of input-output balance equations,

$$z_i^g = \sum_{j=1}^{n} a_{ij}^g x_j^g + y_i^g \quad \text{(all i)}, \tag{1a}$$

cannot be solved since there are 2n equations and 6n variables: 2n autonomous demands, 2n production levels, and 2n import levels. In order to solve this set of equations for given final demands, therefore, an assumption about either supply or production must be made. An assumption concerning supply sources will first be made, leading to the column coefficient model: imports are a fixed fraction of the total supply of each commodity. (Chenery and Clark [2] call these proportions »supply coefficients«.) These coefficients are defined as:

$$x_i^{gh} = c_i^{gh} z_i^h. \tag{38}$$

As Chenery and Clark [2, p. 67] point out, »the supply coefficent therefore extends the idea of a given marginal propensity to import each commodity to any number of regions.«

This fixed-supply assumption makes it possible to express the total production of industry i in region g, x_i^g, as a function of the total demands in all regions:

$$x_i^g = c_i^{gg} z_i^g + c_i^{gh} z_i^h \quad \text{(all i)}. \tag{39}$$

It is now possible to solve for the production levels corresponding to given final demands in all regions by substituting from the set of equations (1a) into (39) and collecting terms:

$$x_i^g = \left[\sum_{j=1}^{n} c_i^{gg} a_{ij}^g x_j^g + \sum_{j=1}^{n} c_i^{gh} a_{ij}^h x_j^h \right] +$$

$$\left[c_i^{gg} y_i^g + c_i^{gh} y_i^h \right] \quad \text{(all i).} \tag{40}$$

In other words, the total production of industry i in region g, x_i^g is equal to the amounts of commodity i used for further production in both regions plus the shipments to both regions for final demand.

Now an assumption concerning production will be made, leading to the row coefficient model: the exports are a fixed fraction of the total production of each commodity. These proportions, which may be called »production coefficients,« are defined as:

$$x_i^{gh} = r_i^{gh} x_i^g. \tag{41}$$

This fixed-production assumption makes it possible to express the total supply of commodity i in region g, z_i^g as a function of the total production in all regions:

$$z_i^g = r_i^{gg} x_i^g + r_i^{hg} x_i^h \quad \text{(all i).} \tag{42}$$

By substituting from the set of equations (1a) into (42) and collecting terms, the following set of equations is obtained:

$$x_i^g = 1/r_i^{gg} \left[\sum_{i=1}^{n} a_{ij}^g x_j^g + y_i^g \right] - 1/r_i^{gg} \left[r_i^{hg} x_i^h \right] \quad \text{(all i).} \tag{43}$$

That is, the total production of industry i in region g is equal to the amount used for further production in region g plus the shipments to the final demand in region g *minus* the amount exported to region h.

The clue to the understanding of the problems encountered in the testing of the row coefficient model lies in the interpretation of these negative terms. It is important to emphasize that the economic interpretation of equations (41) and (42) is straightforward, as demonstrated above, although the occurrence of the implied pattern of trade in actual economies is implausible (except, perhaps, for a certain class of commodities).[20]) Difficulties arise, when equation (1a) and (42) are combined. As Richardson pointed out,

[19]) It should be noted that if there are k regions, there may be at most (k — 1) negative terms in each equation (43).

[20]) In her work on fruit and vegetable shipments, Polenske [11] found that the output estimates of the row coefficient model were relatively more accurate than those of either the column coefficient or gravity coefficient models. As Richardson pointed out,

> The critical question is wheter changes in output have more or less impact on the regional distribution of shipments than fluctuations in total demand. Two possible explanations were suggested for the fruit and vegetable results: the strong interest of producers in retaining market links with each major consumer market, leading to rationing of available suplies in periods of low output; the indifference of consumers toward the regional origin of the goods consumed, leading to more regional substitution and hence variation in the import coefficienst. [14, p. 67]

Similar results can be expected for other commodities to the extent to which the commodities share these properties.

> The main feature of the row coefficient model, that the proportion of the output industry i in region r [g in the text above] sold to region s [h in the text above] remains constant irrespective of changes in the level of demand in any of the regions, is theoretically implausible, and infringes the Walrasian assumptions of input-output models that output changes are generated only by shifts in demand and price changes by shifts in supply. [14, pp. 66—67]

In other words, in the case of the column coefficient model, the output of an industry is determined solely by the demand for its products, while in the case of the row coefficient model, the output of an industry is determined by the demand for its products and also by some characteristics of the technology employed in the process of production (whence the notion of »production coefficients«). More precisely, in the latter case it is implied that demand changes are determined by changes in output. Equation (43) is therefore self-contradictory. The conflict between these two economic principles (demand determines output and output determines demand) is expressed by the fact that the technical and trade coefficients in the column coefficient model represent *inputs* and *imports*, respectively, while they represent *inputs* and *exports* in the row coefficient model.

Until now the argument was made in terms of the sets of equations (40) and (43). In order to shed some additional light on the relationship between the economic and mathematical reasons for the failure of the row coefficient model, the structure of the matrices $(C^{-1}—A)$ and $(I—CA)$ for the column coeficient model and the matrices $(R'—A)$ and $[I—(R')—A]$ for the row coefficient model will now be re-examined in more detail. (Special attention will be given to the sign of matrix elements because of the important role negative elements in equation (43) play in the explanation of the reasons for the problems with the row coefficient model.) These matrices must have positive elements on the principal diagonal and negative (non-positive) off-diagonal elements if their inverses are to be positive (non-negative). For simplicity, a 2-region, 2-industry economy will be considered.[21]) If it is assumed that intraregional trade in each commodity is greater than interregional trade in that commodity, it follows that all the elements on the principal diagonal of C^{-1} and $(R')^{-1}$ are positive, while all the off-diagonal elements are negative or non-positive. Finally, if the above assumption holds, it also follows that the elements on the principal diagonal of C^{-1} and $(R')^{-1}$ are greater than or equal to one.

Consider matrix $(C^{-1} — A)$. Elements on the principal diagonal are positive because the lements on the principal diagonal of C^{-1} are positive and greater than one, while the elements on the principal diagonal of A are positive and smaller than one. Off-diagonal elements of $(C^{-1}—A)$ are non-positive because the off-diagonal elements of C^{-1} are non-positive,

[21]) Matrices C, C^{-1}, CA, R' $(R')^{-1}$ and $(R')^{-1}$ A in terms of interregional trade flows for a 2-region, 2-industry economy can be found in Appendix *C*.

while the off-diagonal elements of A are positive or non-negative. Therefore, matrix (C^{-1} — A) satisfies the conditions that ensure positive or non-negative inverses.

Consider matrix (I — CA). Elements on the principal diagonal are positive because the elements on the principal diagonal of CA are positive and smaller than one. Off-diagonal elements of (I — CA) are negative or non-positive because the off-diagonal elements of CA are positive or non-negative. It follows that matrix (I — CA) also satisfies the conditions that will ensure positive or non-negative inverses.

Now consider matrix (R′ — A). Elements on the principal diagonal are not always positive because the elements on the principal diagonal of R′ (representing intraregional trade) are positive and smaller than or equal to one, while the elements on the principal diagonal of A (representing intraindustry transactions) are positive and smaller than one. Indeed, there is no reason why intraregional trade of commodity i in region g, r_i^{gg} should generally be greater than intraindustry transactions of commodity i in region g, a The first represents a shipment of the commodity within the region, while the second represents a purchase of the commodity by the industry producing it. Off-diagonal elements of (R′ — A) can be positive, negative, or equal to zero because the off-diagonal elements of both R′ and A are positive or non-negative. It should also be noted that a certain number of the off-diagonal elements of (R′ — A) will always be positive because of the interregional trade elements in matrix R′; these elements represent exports, as was determined in the discussion of equation (43). In other words, matrix (R′ — A) does not generally satisfy the conditions that ensure positive or non-negative inverses.

Finally, consider matrix $[I - (R')^{-1}A]$. Again, the elements on the principal diagonal are not always positive because the elements on the principal diagonal of $(R')^{-1}$ are greater than one, which means that the elements on the principal diagonal of $(R')^{-1}A$ may be greater than one. Off-diagonal elements of $[I - (R')^{-1}A]$ can be positive, negative, and equal to zero because the off-diagonal elements of $(R')^{-1}A$ can be positive, negative, and equal to zero. Again, it should be noted that a certain number of the off-diagonal elements of $[I - (R')^{-1}A]$ will always be positive; these elements represent exports, as was the case with matrix $(R' - A)$. Consequently, matrix $[I - (R')^{-1}A]$ also fails to satisfy the above conditions.

It can be concluded that the structures of the column coefficient and row coefficient models are not fully symmetrical, as they were intended to be. The mathematical properties of the row coefficient model demand that the technical coefficient matrix be redefined to represent *outputs*, and not *inputs*, if the row coefficient model is indeed to be the »mirror image« of the column coefficient model is also be internally consistent. One of the objectives of future research will therefore be to examine the economic implications of this requirement.

CONCLUSIONS

The evaluation of the theoretical column coefficient and row coefficient models made in this study has confirmed results of earlier empirical testing of the models. The column coefficient model always generates an inverse with all the elements larger than zero (positive), as well as positive projections. The row coefficient model always generates an inverse with a large proportion of elements smaller than zero (negative). Also, the row coefficient model frequently generates negative projections.

Three conclusions can be drawn from the theoretical and empirical evidence:

1. All multiregional input-output models of the general formulation given by equation (11) must be constructed in accordance with construction rules (13), (14), (26) and (27a), that ensure that matrix (12) will be *positive* of ΘA is indecomposable, or with construction rules (13), (14), and (27b), that ensure that matrix (12) will be *non-negative* if ΘA is decomposable. Furthemore, given that all the elements of the final demand vector, Y, are non-negative, all the elements of the regional output vector, X, will be *positive* if matrix (12) is positive and *non-negative* if the matrix is non-negative. The policy implications of this conclusion were mentioned in the introduction: if a productive system is internally consistent, any schedule of regional final demands (policy variables) can be produced.

2. Regional trade and technology data (matrices Θ and A) for well-constructed multiregional input-output models can be tested using the conditions discussed in point (1) above. The consistency of the data with conditions (13), (14), (26), and (27a) ensures that matrix (12) will be *positive* if ΘA is indecomposable, and the consistency of the data with conditions (13), (14), (26), and (27b) ensures that matrix (12) will be *non-negative* if ΘA is decomposable. Furthermore, given that all the elements of the final demand vector, Y, are non-negative, all the elements of the regional output vector, X, will be *positive* if matrix (12) is positive and *non-negative* if matrix (12) is non-negative.

3. Unlike the structure of the column coefficient model, the very structure of the row coefficient model violates the conditions that ensure that matrix (12) will be positive or non-negative. The objective of future research in this area will be to construct a multiregional input-output model that represents a consistent »mirror image« of the column coefficient model, which the present formulation of the row coefficient model is not. Now that the structure of the MRIO models is better understood, the research will proceed toward restructuring of the row coefficient model in accordance with the construction rules discussed in this work.

(Rad primljen februara 1976)

APPENDIX A
MATRICES Θ, *A*, AND Θ*A* FOR AN *n*-REGION, m-INDUSTRY ECONOMY

$$
H = \begin{bmatrix}
\begin{bmatrix} \theta^{11}_1 & & \bigcirc \\ & \theta^{11}_2 & \\ & & \ddots \\ \bigcirc & & \theta^{11}_m \end{bmatrix} &
\begin{bmatrix} \theta^{12}_1 & & \bigcirc \\ & \theta^{11}_2 & \\ & & \ddots \\ \bigcirc & & \theta^{12}_m \end{bmatrix} & \cdots &
\begin{bmatrix} \theta^{1n}_1 & & \bigcirc \\ & \theta^{1n}_2 & \\ & & \ddots \\ \bigcirc & & \theta^{1n}_m \end{bmatrix} \\
\begin{bmatrix} \theta^{21}_1 & & \bigcirc \\ & \theta^{21}_1 & \\ & & \ddots \\ \bigcirc & & \theta^{21}_m \end{bmatrix} &
\begin{bmatrix} \theta^{22}_1 & & \bigcirc \\ & \theta^{22}_2 & \\ & & \ddots \\ \bigcirc & & \theta^{22}_m \end{bmatrix} & \cdots &
\begin{bmatrix} \theta^{2n}_1 & & \bigcirc \\ & \theta^{2n}_2 & \\ & & \ddots \\ \bigcirc & & \theta^{2n}_m \end{bmatrix} \\
\vdots & \vdots & \ddots & \vdots \\
\begin{bmatrix} \theta^{n1}_1 & & \bigcirc \\ & \theta^{n1}_2 & \\ & & \ddots \\ \bigcirc & & \theta^{n1}_m \end{bmatrix} &
\begin{bmatrix} \theta^{n2}_1 & & \bigcirc \\ & \theta^{n2}_2 & \\ & & \ddots \\ \bigcirc & & \theta^{n2}_m \end{bmatrix} & \cdots &
\begin{bmatrix} \theta^{nn}_1 & & \bigcirc \\ & \theta^{nn}_1 & \\ & & \ddots \\ \bigcirc & & \theta^{nn}_m \end{bmatrix}
\end{bmatrix}
$$

$$A = \begin{bmatrix} \begin{bmatrix} a^1_{11} & a^1_{12} & \cdots & a^1_{1m} \\ a^1_{21} & a^1_{22} & \cdots & a^1_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^1_{m1} & a^1_{m2} & \cdots & a^1_{mm} \end{bmatrix} & & & \bigcirc \\ & \begin{bmatrix} a^2_{11} & a^2_{12} & \cdots & a^2_{1m} \\ a^2_{21} & a^2_{22} & \cdots & a^2_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^2_{m1} & a^2_{m2} & \cdots & a^2_{mm} \end{bmatrix} & & \\ & & \ddots & \\ \bigcirc & & & \begin{bmatrix} a^n_{11} & a^n_{12} & \cdots & a^n_{1m} \\ a^n_{21} & a^n_{22} & \cdots & a^n_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^n_{m1} & a^n_{m2} & \cdots & a^n_{mm} \end{bmatrix} \end{bmatrix}$$

$$\Theta A = \begin{bmatrix}
\theta_1^{11} a_{11}^1 & \theta_1^{11} a_{12}^1 & \cdots & \theta_1^{11} a_{1m}^1 & \theta_1^{12} a_{11}^2 & \theta_1^{12} a_{12}^2 & \cdots & \theta_1^{12} a_{1m}^2 & \cdots & \theta_1^{1n} a_{11}^n & \theta_1^{1n} a_{12}^n & \cdots & \theta_1^{1n} a_{1m}^n \\
\theta_2^{11} a_{21}^1 & \theta_2^{11} a_{22}^1 & \cdots & \theta_2^{11} a_{2m}^1 & \theta_2^{12} a_{21}^2 & \theta_2^{12} a_{22}^2 & \cdots & \theta_2^{12} a_{2m}^2 & \cdots & \theta_2^{1n} a_{21}^n & \theta_2^{1n} a_{22}^n & \cdots & \theta_1^{1n} a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
\theta_m^{11} a_{m1}^1 & \theta_m^{11} a_{m2}^2 & \cdots & \theta_m^{11} a_{mm}^1 & \theta_m^{12} a_{m1}^2 & \theta_m^{12} a_{m2}^2 & \cdots & \theta_m^{12} a_{mm}^2 & \cdots & \theta_m^{1n} a_{m1}^n & \theta_m^{1n} a_{m2}^n & \cdots & \theta_m^{1n} a_{mm}^n \\
\theta_1^{21} a_{11}^1 & \theta_1^{21} a_{12}^1 & \cdots & \theta_1^{21} a_{1m}^1 & \theta_1^{22} a_{11}^2 & \theta_1^{22} a_{12}^2 & \cdots & \theta_1^{22} a_{1m}^2 & \cdots & \theta_1^{2n} a_{11}^n & \theta_1^{2n} a_{12}^n & \cdots & \theta_1^{2n} a_{1m}^n \\
\theta_2^{21} a_{21}^1 & \theta_2^{21} a_{22}^1 & \cdots & \theta_2^{21} a_{2m}^1 & \theta_2^{22} a_{21}^2 & \theta_2^{22} a_{22}^2 & \cdots & \theta_2^{22} a_{2m}^2 & \cdots & \theta_2^{2n} a_{21}^n & \theta_2^{2n} a_{22}^n & \cdots & \theta_2^{2n} a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
\theta_m^{21} a_{m1}^1 & \theta_m^{21} a_{m2}^1 & \cdots & \theta_m^{21} a_{mm}^1 & \theta_m^{22} a_{m1}^2 & \theta_m^{22} a_{m2}^2 & \cdots & \theta_m^{22} a_{mm}^2 & \cdots & \theta_m^{2n} a_{m1}^n & \theta_m^{2n} a_{m2}^n & \cdots & \theta_m^{2n} a_{mm}^n \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots & & \vdots \\
\theta_1^{n1} a_{11}^1 & \theta_1^{n1} a_{12}^1 & \cdots & \theta_1^{n1} a_{1m}^1 & \theta_1^{n2} a_{11}^2 & \theta_1^{n2} a_{12}^2 & \cdots & \theta_1^{n2} a_{1m}^2 & \cdots & \theta_1^{nn} a_{11}^n & \theta_1^{nn} a_{12}^n & \cdots & \theta_1^{nn} a_{1m}^n \\
\theta_2^{n1} a_{21}^1 & \theta_2^{n1} a_{22}^1 & \cdots & \theta_2^{n1} a_{2m}^1 & \theta_2^{n2} a_{21}^2 & \theta_2^{n2} a_{22}^2 & \cdots & \theta_1^{n2} a_{2m}^2 & \cdots & \theta_2^{nn} a_{21}^n & \theta_2^{nn} a_{22}^n & \cdots & \theta_2^{nn} a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
\theta_m^{n1} a_{m1}^1 & \theta_m^{n1} a_{m2}^1 & \cdots & \theta_m^{n1} a_{mm}^1 & \theta_m^{n2} a_{m1}^2 & \theta_m^{n2} a_{m2}^2 & \cdots & \theta_m^{n2} a_{mm}^2 & \cdots & \theta_m^{nn} a_{m1}^n & \theta_m^{nn} a_{m2}^n & \cdots & \theta_m^{nn} a_{mm}^n
\end{bmatrix}$$

APPENDIX B

A NOTE ON HAWKINS-SIMON CONDITIONS OF MACROECONOMIC STABILITY

Hawkins and Simon [7] show that a sufficint condition on A which ensures that all the elements of $(I - A)^{-1}$ are positive is that all the principal minors of $(I - A)$ are positive, given that A is indecomposable. Furthermore, it is a corollary of this theorem that a sufficient conditions that all the elements of X satisfying $(I - A)^{-1}$ be positive for any Y is that all the principal minors of $(I - A)$ are positive. Hawkins an Simon provide the following economic interpretation of their theorem and corollary:

> From the corollary, we see that if the production equations are internally consistent in permitting the production of some fixed schedule of consumption goods, then these consumption goods can be obtained in any desired proportion from this production system. Hence the system will be consistent with *any* schedule of consumption goods.
>
> The condition that all principal minors must be positive means, in economic terms, that the group of industries corresponding to each minor must be capable of supplying more than its own needs for the group of products produced by this group of industries ... For example, if the principal minor involving the ith and jth commodities is negative, this means that the quantity of the ith commodity required to produce one unit of the jth commodity is greater than the quantity of the ith commodity that can be produced with an input of one unit of the jth commodity. Under these circumstances, the production of these two commodities could not be continued, for they would exhaust each other in their joint production. [7, p. 248]

For discussions of Hawkins-Simon conditions, see Dorfman, Samuelson, and Solow [4], and Solow [16].

APPENDIX C

MATRICES C, C^{-1}, CA, R', $(R')^{-1}$, AND $(R')^{-1}A$ CONSTRUCTED FROM INTERREGIONAL TRADE FLOWS FOR A 2-REGION, 2-INDUSTRY ECONOMY

$$C = \begin{bmatrix} \dfrac{x_1^{11}}{x_1^{11} + x_1^{21}} & 0 & \dfrac{x_1^{12}}{x_1^{12} + x_1^{22}} & 0 \\ 0 & \dfrac{x_2^{11}}{x_2^{11} + x_2^{21}} & 0 & \dfrac{x_2^{12}}{x_2^{12} + x_2^{22}} \\ \dfrac{x_1^{21}}{x_1^{11} + x_1^{21}} & 0 & \dfrac{x_1^{22}}{x_1^{12} + x_1^{22}} & 0 \\ 0 & \dfrac{x_2^{21}}{x_2^{11} + x_2^{21}} & 0 & \dfrac{x_2^{22}}{x_2^{12} + x_2^{22}} \end{bmatrix}$$

Note: Each element x_i^{gh} represents a flow of commodity i from region g to region h.

$$C^{-1} = \begin{bmatrix} \dfrac{x_1^{12} x_1^{11} + x_1^{22} x_1^{21}}{x_1^{22} x_1^{11} - x_1^{12} x_1^{21}} & 0 & \dfrac{x_1^{12} x_1^{11} + x_1^{12} x_1^{21}}{x_1^{12} x_1^{21} - x_1^{22} x_1^{11}} & 0 \\ 0 & \dfrac{x_2^{22} x_2^{11} + x_2^{22} x_2^{21}}{x_2^{22} x_2^{11} - x_2^{12} x_2^{21}} & 0 & \dfrac{x_2^{12} x_2^{11} + x_2^{12} x_2^{21}}{x_2^{12} x_2^{21} - x_2^{22} x_2^{11}} \\ \dfrac{x_1^{12} x_1^{21} + x_1^{22} x_1^{21}}{x_1^{12} x_1^{21} - x_1^{22} x_1^{11}} & 0 & \dfrac{x_1^{22} x_1^{11} + x_1^{12} x_1^{11}}{x_1^{22} x_1^{11} - x_1^{12} x_1^{21}} & 0 \\ 0 & \dfrac{x_2^{12} x_2^{21} + x_2^{22} x_2^{21}}{x_2^{12} x_2^{21} - x_2^{22} x_2^{11}} & & \dfrac{x_2^{22} x_2^{11} + x_2^{12} x_2^{11}}{x_2^{22} x_2^{11} - x_2^{12} x_2^{21}} \end{bmatrix}$$

$$CA = \begin{bmatrix} \frac{x_1^{11}}{x_1^{11} + x_1^{21}} \cdot a_{11}^1 & \frac{x_1^{11}}{x_1^{11} + x_1^{21}} \cdot a_{12}^1 & \frac{x_1^{12}}{x_1^{12} + x_1^{22}} \cdot a_{11}^2 & \frac{x_1^{12}}{x_1^{12} + x_1^{22}} \cdot a_{12} \\ \frac{x_2^{11}}{x_2^{11} + x_2^{21}} \cdot a_{21}^1 & \frac{x_2^{11}}{x_2^{11} + x_2^{21}} \cdot a_{22}^1 & \frac{x_2^{12}}{x_2^{12} + x_2^{22}} \cdot a_{21}^2 & \frac{x_2^{12}}{x_2^{12} + x_2^{22}} \cdot a_{22}^2 \\ \frac{x_1^{21}}{x_1^{11} + x_1^{21}} \cdot a_{11}^1 & \frac{x_1^{21}}{x_1^{11} + x_1^{21}} \cdot a_{12}^1 & \frac{x_1^{22}}{x_1^{12} + x_1^{22}} \cdot a_{11}^2 & \frac{x_1^{22}}{x_1^{12} + x_1^{22}} \cdot a_{12}^2 \\ \frac{x_2^{21}}{x_2^{11} + x_2^{21}} \cdot a_{21}^1 & \frac{x_1^{21}}{x_2^{11} + x_2^{21}} \cdot a_{22}^1 & \frac{x_2^{22}}{x_2^{12} + x_2^{22}} \cdot a_{21}^2 & \frac{x_2^{22}}{x_2^{12} + x_2^{22}} \cdot a_{22}^2 \end{bmatrix}$$

$$R' = \begin{bmatrix} \dfrac{x_1^{11}}{x_1^{11} + x_1^{12}} & 0 & \dfrac{x_1^{21}}{x_1^{21} + x_1^{22}} & 0 \\ 0 & \dfrac{x_2^{11}}{x_2^{11} + x_2^{12}} & 0 & \dfrac{x_2^{21}}{x_2^{21} = x_2^{22}} \\ \dfrac{x_1^{12}}{x_1^{11} + x_1^{12}} & 0 & \dfrac{x_1^{22}}{x_1^{21} + x_1^{22}} & 0 \\ 0 & \dfrac{x_2^{12}}{x_2^{11} + x_2^{12}} & 0 & \dfrac{x_2^{22}}{x_2^{21} + x_2^{22}} \end{bmatrix}$$

Note: Each element x_i^{gh} represents a flow of commodity i from region g to region h.

$$
(R')^{-1} = \begin{bmatrix}
\dfrac{x_1^{22}x_1^{11} + x_1^{22}x_1^{12}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} & 0 & \dfrac{x_1^{21}x_1^{11} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} & 0 \\[2em]
0 & \dfrac{x_2^{22}x_2^{11} + x_2^{22}x_2^{12}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} & 0 & \dfrac{x_2^{21}x_2^{11} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \\[2em]
\dfrac{x_1^{12}x_1^{22} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} & 0 & \dfrac{x_1^{22}x_1^{11} + x_1^{21}x_1^{11}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} & 0 \\[2em]
0 & \dfrac{x_2^{12}x_2^{22} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} & 0 & \dfrac{x_2^{22}x_2^{11} + x_2^{21}x_2^{11}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}}
\end{bmatrix}
$$

$$(R')^{-1}A = \begin{bmatrix} \frac{x_1^{22}x_1^{11} + x_1^{22}x_1^{12}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} \cdot a_{11}^1 & \frac{x_1^{22}x_1^{11} + x_1^{22}x_1^{12}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} \cdot a_{12}^1 & \frac{x_1^{21}x_1^{11} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} \cdot a_{11}^2 & \frac{x_1^{21}x_1^{11} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} \cdot a_{22}^2 \\ \frac{x_2^{22}x_2^{11} + x_2^{22}x_2^{12}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \cdot a_{21}^1 & \frac{x_2^{22}x_2^{11} + x_2^{22}x_2^{12}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \cdot a_{22}^1 & \frac{x_2^{21}x_2^{11} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \cdot a_{21}^2 & \frac{x_2^{21}x_2^{11} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \cdot a_{22}^2 \\ \frac{x_1^{12}x_1^{22} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} \cdot a_{11}^1 & \frac{x_1^{12}x_1^{22} + x_1^{12}x_1^{21}}{x_1^{12}x_1^{21} - x_1^{22}x_1^{11}} \cdot a_{12}^1 & \frac{x_1^{22}x_1^{11} + x_1^{21}x_1^{11}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} \cdot a_{11}^2 & \frac{x_1^{22}x_1^{11} + x_1^{21}x_1^{11}}{x_1^{22}x_1^{11} - x_1^{12}x_1^{21}} \cdot a_{12}^2 \\ \frac{x_2^{12}x_2^{22} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \cdot a_{21}^1 & \frac{x_2^{12}x_2^{22} + x_2^{12}x_2^{21}}{x_2^{12}x_2^{21} - x_2^{22}x_2^{11}} \cdot a_{22}^1 & \frac{x_2^{21}x_2^{11} + x_2^{21}x_2^{11}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \cdot a_{21}^2 & \frac{x_2^{22}x_2^{11} + x_2^{21}x_2^{11}}{x_2^{22}x_2^{11} - x_2^{12}x_2^{21}} \cdot a_{22}^2 \end{bmatrix}$$

BIBLIOGRAPHY

1. Bellman, R. *Introduction to Matrix Analysis.* New York: McGraw-Hill, Inc., 1970.
2. Chenery, H. B. and Clark, P. G. *Interindustry Economics.* New York: John Wiley & Sons, 1967.
3. Debreu, G. and Herstein. I. M. »Nonnegative Square Matrices.« *Econometrica.* Vol. 21 (1953), 597—607.
4. Dorfman. R., Samuelson, P. A. and Solow, R. M. *Linear Programming and Economic Analysis.* New York: McGraw-Hill, Inc., 1958.
5. Fencl, Z. and Ng, N. K. »Comparison Tests of the Column Coefficient and the Gravity Coefficient Models.« DOT Report No. 6. Prepared for the University Research Program, U. S. Department of Transportation, April 1974.
6. Hadley, G. *Linear Algebra.* Reading, Massachusetts: Addison-Wesley, Inc., 1964.
(1955), 803—832.
7. Hawkins, D and Simon, H. A. »Note: Some Conditions of Macro-Economic Stability.« *Econometrica,* Vol. 17 (1949), 245—248.
8. Marcus, M. and Minc, H. *A Survey of Matrix Theory and Matrix Inequalities.* Boston: Allyn and Bacon, Inc., 1964.
9. Miernyk, W. H. *The Elements of Input-Output Analysis.* New York: Random House, 1965.
10. Moses, L. N. »The Stability of Interregional Trading Patterns and Input-Output Analysis.« *The American Economic Review,* Vol. 45, No. 5
11. Polenske, Karen R. »A Case Study of Transportation Models Used in Multiregional Analysis.« Unpublished Ph. D. dissertation, Harvard University, May 1966.
12. ———. »An Empirical Test of Interregional Input-Output Models: Estimation of 1965 Japanese Production.« *The American Economic Review,* Vol. 60, No. 2 (1970), 76—82.
13. ———. »The Implementation of a Multiregional Input-Output Model of the United States.« *Input-Output Techniques.* Edited by A. Brody and A. P. Carter, Amsterdam: North-Holland Publishing Co., 1972, 171—189.
14. Richardson, H. W. *Input-Output and Regional Economics.* New York: John Wiley & Sons, 1974.
15. Rogers, A. *Matrix Methods in Urban and Regional Analysis.* San Francisco: Holden-Day, 1971.
16. Solow, R. M. »On the Structure of Linear Models.« *Econometrica,* Vol. 20 (1952), 29—46.
17. Yan, Chiou-shung. *Introduction to Input-Output Economics.* New York: Holt, Rinehart, and Winston, Inc., 1969.

NEKI USLOVI MAKROEKONOMSKE STABILNOSTI U MULTIREGIONALNIM MODELIMA

Ranko BON

R e z i m e

Ovo istraživanje predstavlja deo komparativne analize tri multiregionalna input-output (MRIO) modela: model kolona-koeficijenata, model red-koeficijenata i model gravitacijskih koeficijenata. Istraživanje je imalo dva cilja: (1) ispitati uzroke pojavljivanja negativnih vrednosti u inverznoj matrici modela red-koeficijenata, kao i negativnih

projekcija koje ovaj model generira i (2) objasniti zašto se u modelu kolona-koeficijenata ovakvi problemi ne pojavljuju.

Rezultati ovog istraživanja daju: (1) pravila konstrukcije regionalnih matrica koeficijenata snabdevanja (ili trgovine) koja garantuje da će projekcije koje generiraju MRIO modeli biti nenegativne, (2) test podataka o regionalnoj tehnologiji i trgovini na bazi ovih pravila i (3) objašnjenje ponašanja modela red-koeficijenata, u čijoj konstrukciji ova pravila nisu bila poštovana, kao i ponašanja modela kolona-koeficijenata, čija konstrukcija zadovoljava gornja pravila.

Konačno, implikacije ovog istraživanja za ekonomsku politiku proširuju zaključke Hawkins-a i Simon-a sa jednoregionalne na više-regionalnu ekonomiju: ako je proizvodni sistem konzistentan, onda će biti konzistentan i za b i l o k o j i *vektor konačne potrošnje, koji predstavlja skup promenljivih ekonomske politike.*