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A class of nonlocal impulsive differential equations with conformable fractional derivative

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ABSTRACT

In this paper, we deal with the Duhamel formula, existence, uniqueness, and stability of mild solutions of a class of nonlocal impulsive differential equations with conformable fractional derivative. The main results are based on the semigroup theory combined with some fixed point theorems. We also give an example to illustrate the applicability of our abstract results

RESUMEN

En este artículo, tratamos la fórmula de Duhamel, la existencia, unicidad y estabilidad de soluciones mild de una clase de ecuaciones diferenciales no locales impulsivas con derivadas fraccionarias conformables. Los resultados principales se basan en teoría de semigrupos, combinada con algunos teoremas de punto fijo. También entregamos un ejemplo para ilustrar la aplicabilidad de nuestros resultados abstractos.

Keywords and Phrases: Functional-differential equations with fractional derivatives; Groups and semigroups of linear operators; Nonlocal conditions; Impulsive conditions; Conformable fractional derivatives.

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1 Introduction

Fractional calculus has attracted the attention of many researchers, due to its wide range of applications in modeling of various natural phenomena in different fields of sciences and engineering including: physics, engineering, biology, finance, chemistry [3, 26, 31, 35, 38, 41, 43, 44, 45, 46, 47, 48]. For better understanding these phenomena, several definitions of fractional derivatives have been introduced such as Riemann-Liouville and Caputo definitions, for more details we refer to the books [31, 41]. Unfortunately, these definitions are very complicated to handle in real models. However, in [30] a new definition of fractional derivative named conformable fractional derivative was initiated. This novel fractional derivative is very easy and satisfies all the properties of the classical derivative. The advantage of the conformable fractional derivative is very remarkable compared to other fractional derivatives in many comparisons. Indeed, for example, in the work [15] the authors gave the solution of conformable-fractional telegraph equations in terms of the classical exponential function, however for the Caputo-fractional telegraph equations considered in the very good papers [19, 20, 36], the fundamental solution cannot be given in terms of the exponential function as in the conformable-fractional case, and therefore the authors have been introduced the so-called Mittag-Leffler function. Another comparison, we notice that the constants of increases of the norms of the control bounded operators W and W^{-1} in the application of the work [27] are given directly in a simple way in terms of the exponential function, contrary, for the Caputo fractional derivative in the application of the nice work [51] these constants are given in terms of the so-called Mittag-Leffler function. For more details and conclusions concerning the uses and applications of conformable fractional calculus, we refer to the works [2, 4, 5, 7, 8, 10, 11, 12, 13, 14, 16, 17, 22, 23, 24, 25, 28, 29, 42, 49].

On the other hand, impulsive differential equations are crucial in description of dynamical processes with short-time perturbations [6, 32, 50]. Actually, the Cauchy problem of impulsive differential equations attracts the attention of many authors [1, 9, 33, 34, 37]. For example, Liang *et al.* [33] have proved the existence and uniqueness of mild solutions for the Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), \quad t \in [0, \tau], \quad t \neq t_1, t_2, \ldots, t_n, \\ x(0) = x_0 + g(x), \\ x(t_i^+) = x(t_i^-) + h_i(x(t_i)), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

by using the following classical Duhamel formula:

$$x(t) = T(t)[x_0 + g(x)] + \sum_{0<t_i<t} T(t-t_i)h_i(x(t_i)) + \int_0^t T(t-s)f(s, x(s))ds, \tag{1.2}$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by the linear part A on a Banach space $(X, \| \, . \, \|)$ [40] and $x_0 \in X$. The expression $x(t_i^+) = x(t_i^-) + h_i(t_i)$ means the impulsive condition, with $x(t_i^+)$, $x(t_i^-)$ are the right and left limits of $x(.)$ at $t = t_i$, respectively. The condition $x(0) = x_0 + g(x)$

represents the nonlocal condition, which can be applied in physics with better effects than the classical initial condition [18, 21, 39]. The functions $f : [0, \tau] \times X \longrightarrow X$, $h_i : X \longrightarrow X$ and $g : \mathcal{C} \longrightarrow X$ satisfied some assumptions, with $\mathcal{C}$ is the space of functions $x(.)$ defined from $[0, \tau]$ into X such that $x(.)$ is continuous on each interval $]t_i, t_{i+1}]$ and $x(t_i^+)$, $x(t_i^-)$ exist.
The analogous of equation (1.1) in the frame of the Caputo fractional derivative have been considered by Mophou [37], when the author proved the existence and uniqueness of mild solutions for the following fractional Cauchy problem

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in [0, \tau], \quad t \neq t_1, t_2, \ldots, t_n, \quad 0 < \alpha < 1, \\ x(0) = x_0 + g(x), \\ x(t_i^+) = x(t_i^-) + h_i(x(t_i)), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.3}$$

by using the following fractional Duhamel formula:

$$\begin{aligned} x(t) &= T(t)[x_0 + g(x)] + \sum_{0<t_i<t} T(t - t_i)h_i(x(t_i)) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t - s) f(s, x(s)) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} T(t - s) f(s, x(s)) ds, \end{aligned} \tag{1.4}$$

with Γ is the Gamma function and ${}^cD^\alpha x(t)$ presents the Caputo fractional derivative.

In the present work, we are interested in studying of equation (1.1) in the frame of the conformable fractional derivative. Precisely, we will be concerned with the study of the existence, uniqueness, and stability of mild solutions for the following conformable fractional Cauchy problem

$$\begin{cases} \dfrac{d^\alpha x(t)}{dt^\alpha} &= Ax(t) + f(t, x(t)), \quad t \in [0, \tau], \quad t \neq t_1, t_2, \ldots, t_n, \quad 0 < \alpha < 1, \\ x(0) &= x_0 + g(x), \\ x(t_i^+) &= x(t_i^-) + h_i(x(t_i)), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.5}$$

where $\dfrac{d^\alpha x(t)}{dt^\alpha}$ is the conformable fractional derivative.
The main novelty of this paper is to prove the analogous of Duhamel formulas (1.2) and (1.4) for the Cauchy problem (1.5) as follows:

$$\begin{aligned} x(t) &= T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) \\ &+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds. \end{aligned} \tag{1.6}$$

Then, based on this conformable fractional Duhamel formula, we discuss some results concerning the existence, uniqueness, and stability of the mild solution of the conformable fractional Cauchy problem (1.5).

This paper is organized as follows. In section 2, we briefly recall some tools related to the conformable fractional calculus. In section 3, we prove the main results. Section 4 is devoted to a concrete application of the main abstract results.

2 Preliminaries

Recalling some preliminary facts on the conformable fractional calculus.

Definition 2.1 ([30]). *Let $\alpha \in]0,1]$. The conformable fractional derivative of order α of a function $x(.): \ [0,+\infty[\longrightarrow \mathbb{R}$ is defined by*

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{\varepsilon \longrightarrow 0} \frac{x(t+\varepsilon t^{1-\alpha}) - x(t)}{\varepsilon}, \ \ \textit{for} \ \ t>0 \ \ \textit{and} \ \ \frac{d^\alpha x(0)}{dt^\alpha} = \lim_{t \longrightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha},$$

provided that the limits exist.

The fractional integral $I^\alpha(.)$ associated with the conformable fractional derivative is defined by

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s)ds.$$

Theorem 2.2 ([30]). *If $x(.)$ is a continuous function in the domain of $I^\alpha(.)$, then we have*

$$\frac{d^\alpha (I^\alpha(x)(t))}{dt^\alpha} = x(t).$$

Definition 2.3 ([41]). *The Laplace transform of a function $x(.)$ is defined by*

$$\mathcal{L}(x(t))(\lambda) := \int_0^{+\infty} e^{-\lambda t} x(t)dt, \quad \lambda > 0.$$

It is remarkable that the above transform is not adequate to solve conformable fractional differential equations. For this reason, we consider the following definition, which appeared in [2].

Definition 2.4 ([2]). *The fractional Laplace transform of order α of a function $x(.)$ is defined by*

$$\mathcal{L}_\alpha(x(t))(\lambda) := \int_0^{+\infty} t^{\alpha-1} e^{-\lambda \frac{t^\alpha}{\alpha}} x(t)dt, \quad \lambda > 0.$$

The following proposition gives us the actions of the fractional integral and the fractional Laplace transform on the conformable fractional derivative, respectively.

Proposition 2.5 ([2]). *If $x(.)$ is a differentiable function, then we have the following results*

$$I^\alpha\left(\frac{d^\alpha x(.)}{dt^\alpha}\right)(t) = x(t) - x(0),$$
$$\mathcal{L}_\alpha\left(\frac{d^\alpha x(t)}{dt^\alpha}\right)(\lambda) = \lambda \mathcal{L}_\alpha(x(t))(\lambda) - x(0).$$

We end this preliminaries by the following remark.

Remark 2.6 ([14]). *For two arbitrary functions $x(.)$ and $y(.)$, we have*

$$\mathcal{L}_\alpha\left(x\left(\frac{t^\alpha}{\alpha}\right)\right)(\lambda)=\mathcal{L}(x(t))(\lambda),$$

$$\mathcal{L}_\alpha\left(\int_0^t s^{\alpha-1}x\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)y(s)ds\right)(\lambda)=\mathcal{L}(x(t))(\lambda)\mathcal{L}_\alpha(y(t))(\lambda).$$

3 Main results

We first prove the conformable fractional Duhamel formula (1.6). To do so, for $t\in[0,t_1]$, we apply the fractional Laplace transform in equation (1.5), we obtain

$$\mathcal{L}_\alpha(x(t))(\lambda)=(\lambda-A)^{-1}[x_0+g(x)]+(\lambda-A)^{-1}\mathcal{L}_\alpha(f(t,x(t)))(\lambda).$$

According to the inverse fractional Laplace transform and Remark (2.6), we get

$$x(t)=T\left(\frac{t^\alpha}{\alpha}\right)[x_0+g(x)]+\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds,$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by the linear part A on the Banach space X, that is, $(T(t))_{t\geq 0}$ is one parameter family of bounded linear operators on X satisfying the following properties

(1) $T(0)=I$,

(2) $T(s+t)=T(s)T(t)$ for all $t,s\in\mathbb{R}^+$,

(3) $\lim\limits_{t\downarrow 0}\parallel T(t)x-x\parallel=0$ for each fixed $x\in X$,

(4) $\lim\limits_{t\downarrow 0}\dfrac{T(t)x-x}{t}=Ax$, for $x\in X$, provided that the limit exists.

As in [37], we assume that the solution of equation (1.5) is such that at the point of discontinuity t_k, we have $x(t_k^-)=x(t_k)$. Hence, one has

$$x(t_1^-)=T\left(\frac{t_1^\alpha}{\alpha}\right)[x_0+g(x)]+\int_0^{t_1} s^{\alpha-1}T\left(\frac{t_1^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds.$$

For $t\in(t_1,t_2]$, using the fractional Laplace transform in equation (1.5), we obtain

$$\begin{aligned}x(t)&=T\left(\frac{t^\alpha-t_1^\alpha}{\alpha}\right)x(t_1^+)+\int_{t_1}^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds\\&=T\left(\frac{t^\alpha-t_1^\alpha}{\alpha}\right)[x(t_1^-)+h_1(x(t_1))]+\int_{t_1}^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds.\end{aligned}$$

Replacing $x(t_1^-)$ by its expression in the above equation, we get

$$\begin{aligned}x(t)=&\,T\left(\frac{t^\alpha-t_1^\alpha}{\alpha}\right)\left[T\left(\frac{t_1^\alpha}{\alpha}\right)(x_0+g(x))+\int_0^{t_1} s^{\alpha-1}T\left(\frac{t_1^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds+h_1(x(t_1))\right]\\&+\int_{t_1}^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds.\end{aligned}$$

By using a computation, the above equation becomes

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)[h_1(x(t_1))] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

In particular, for $t = t_2^-$, one has

$$x(t_2^-) = T\left(\frac{t_2^\alpha}{\alpha}\right)[x_0 + g(x)] + T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right)[h_1(x(t_1))] + \int_0^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

As the same, for $t \in (t_2, t_3]$, we obtain

$$\begin{aligned} x(t) &= T\left(\frac{t^\alpha - t_2^\alpha}{\alpha}\right) x(t_2^+) + \int_{t_2}^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds \\ &= T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)[x(t_2^-) + h_2(x(t_2))] + \int_{t_2}^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds. \end{aligned}$$

Hence, replacing $x(t_2^-)$ by its expression, we have

$$\begin{aligned} x(t) = T\left(\frac{t^\alpha - t_2^\alpha}{\alpha}\right) \Bigg[& T\left(\frac{t_2^\alpha}{\alpha}\right)[x_0 + g(x)] + T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right)[h_1(x(t_1))] \\ & + \int_0^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds + h_2(x(t_2)) \Bigg] \\ & + \int_{t_2}^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds. \end{aligned}$$

Using a computation, we get

$$\begin{aligned} x(t) = {} & T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + T\left(\frac{t^\alpha - t_1^\alpha}{\alpha}\right)[h_1(x(t_1))] + T\left(\frac{t^\alpha - t_2^\alpha}{\alpha}\right)[h_2(x(t_2))] \\ & + \int_{t_2}^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds. \end{aligned}$$

Repeating the same process, we obtain the following conformable fractional Duhamel formula

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

Definition 3.1. *A function $x \in \mathcal{C}$ is called a mild solution of conformable fractional Cauchy problem* (1.5) *if*

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

In the rest of this paper, we endow the space $\mathcal{C}$ with the norm $\mid x \mid_c := \sup_{t \in [0,\tau]} \parallel x(t) \parallel$. It is well known that the space $(\mathcal{C}, \mid . \mid_c)$ becomes a Banach space. We also denote by $|.|$ the norm in the space $\mathcal{L}(X)$ of bounded operators defined form X into itself.

To prove the main results, we need to use the following assumptions:

(H_1) The function $f(t, .) : X \longrightarrow X$ is continuous and for all $r > 0$ there exists a function $\mu_r \in L^\infty([0, \tau], \mathbb{R}^+)$ such that $\sup_{\|x\| \leq r} \parallel f(t, x) \parallel \leq \mu_r(t)$, for all $t \in [0, \tau]$.

(H_2) The function $f(.,x) : [0,\tau] \longrightarrow X$ is continuous, for all $x \in X$.

(H_3) There exists a constant $L_1 > 0$ such that $\| g(y) - g(x) \| \leq L_1 \mid y - x \mid_c$, for all $x, y \in \mathcal{C}$.

(H_4) There exist constants $C_i > 0$ such that $\| h_i(y(t_i)) - h_i(x(t_i)) \| \leq C_i \mid y - x \mid_c$, for all $x, y \in \mathcal{C}$.

Theorem 3.2. *If $(T(t))_{t>0}$ is compact and $(H_1) - (H_4)$ are satisfied, then the conformable fractional Cauchy problem (1.5) has at least one mild solution, provided that*

$$\left(L_1 + \sum_{i=1}^{n} C_i\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| < 1.$$

Proof. Let $B_r = \{x \in \mathcal{C}, \mid x \mid_c \leq r\}$, where

$$r \geq \frac{\sup\limits_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left[\| x_0 \| + \| g(0) \| + \sum\limits_{i=1}^{n} \| h_i(0) \| + \frac{\tau^\alpha}{\alpha} \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)}\right]}{1 - \left(L_1 + \sum\limits_{i=1}^{n} C_i\right) \sup\limits_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right|}.$$

In order to use the Krasnoselskii fixed-point theorem, we consider the following operators Γ_1 and Γ_2 defined by

$$\Gamma_1(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)), \quad x \in B_r,$$
$$\Gamma_2(x)(t) = \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds, \quad x \in B_r.$$

It is very easy to justify that the operator $\Gamma := \Gamma_1 + \Gamma_2$ is well defined, that is, $\Gamma(x) \in \mathcal{C}$ for all $x \in \mathcal{C}$. The rest of the proof will be given in four steps:

Step 1: Prove that $\Gamma_1(x) + \Gamma_2(y) \in B_r$, whenever $x,\ y \in B_r$.

Let $x,\ y \in B_r$, we have

$$\begin{aligned}\Gamma_1(x)(t) + \Gamma_2(y)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i))\\ + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, y(s))ds.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\| \Gamma_1(x)(t) + \Gamma_2(y)(t) \| \leq & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| [\, \| x_0 \| + \| g(0) \| + \| g(x) - g(0) \| \,]\\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} [\, \| h_i(0) \| + \| h_i(x(t_i)) - h_i(0) \| \,]\\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \int_0^t s^{\alpha-1} \| f(s, y(s)) \| ds.\end{aligned}$$

By using assumptions (H_1), (H_3) and (H_4), we get

$$\begin{aligned}\| \Gamma_1(x)(t) + \Gamma_2(y)(t) \| \leq & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| [\, \| x_0 \| + \| g(0) \| + L_1 \mid x \mid_c] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} [\, \| h_i(0) \| + C_i \mid x \mid_c] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \int_0^t s^{\alpha-1} ds.\end{aligned}$$

According to the fact that $x,\ y \in B_r$, we conclude that

$$\begin{aligned}\| \Gamma_1(x)(t) + \Gamma_2(y)(t) \| \leq & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| [\, \| x_0 \| + \| g(0) \| + L_1 r] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} [\, \| h_i(0) \| + C_i r] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \int_0^t s^{\alpha-1} ds.\end{aligned}$$

Taking the supremum, we get

$$\begin{aligned}\mid \Gamma_1(x) + \Gamma_2(y) \mid_c \leq & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| [\| x_0 \| + \| g(0) \| + L_1 r] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{i=1}^{n} [\, \| h_i(0) \| + C_i r] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \int_0^\tau s^{\alpha-1} ds \\ = & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| [\, \| x_0 \| + \| g(0) \| + L_1 r] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{i=1}^{n} [\, \| h_i(0) \| + C_i r] \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \frac{\tau^\alpha}{\alpha} \\ \leq & \; r.\end{aligned}$$

Hence, the above inequality combined with the continuity of the function $\Gamma_1(x)(.) + \Gamma_2(y)(.)$ on $[0,\tau]$ show that $\Gamma_1(x) + \Gamma_2(y) \in B_r$, for all $x,\ y \in B_r$.

Step 2: Prove that Γ_1 is a contraction operator on B_r.

For $x, y \in \mathcal{C}$, we have

$$\Gamma(y)(t) - \Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right)[h_i(y(t_i)) - h_i(x(t_i))].$$

Consequently, one has

$$\begin{aligned}\| \Gamma(y)(t) - \Gamma(x)(t) \| \leq & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \| g(y) - g(x) \| \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} \| h_i(y(t_i)) - h_i(x(t_i)) \| .\end{aligned}$$

Using assumptions (H_3) and (H_4), we get

$$\| \Gamma(y)(t) - \Gamma(x)(t) \| \leq \left(L_1 + \sum_{i=1}^{n} C_i\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid y - x \mid_c .$$

Taking the supremum in above equation, we obtain

$$\mid \Gamma(y) - \Gamma(x) \mid_c \leq \left(L_1 + \sum_{i=1}^{n} C_i\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid y - x \mid_c .$$

This implies that Γ_1 is a contraction operator on B_r.

Step 3: Prove that Γ_2 is continuous.

Let $(x_n) \subset B_r$ such that $x_n \longrightarrow x$ in B_r. We have

$$\Gamma_2(x_n)(t) - \Gamma_2(x)(t) = \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) [f(s, x_n(s)) - f(s, x(s))] ds.$$

Then, by using a computation, we obtain

$$\mid \Gamma_2(x_n) - \Gamma_2(x) \mid_c \leq \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \int_0^\tau s^{\alpha-1} \| f(s, x_n(s)) - f(s, x(s)) \| ds.$$

Using assumption (H_1), we get $\| s^{\alpha-1}[f(s, x_n(s)) - f(s, x(s))] \| \leq 2\mu_r(s)s^{\alpha-1}$ and $f(s, x_n(s)) \longrightarrow f(s, x(s))$ as $n \longrightarrow +\infty$.

According to the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\longrightarrow+\infty} \mid \Gamma_2(x_n) - \Gamma_2(x) \mid_c = 0.$$

Thus, the operator Γ_2 is continuous.

Step 4: Prove that Γ_2 is compact by using the Arzelà-Ascoli theorem.

`Claim 1:` We prove that $\Gamma_2(B_r)$ is equicontinuous.

Let $t_1, t_2 \in [0, \tau]$ such that $t_1 < t_2$. Then, we have

$$\begin{aligned}\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) = & \int_0^{t_1} s^{\alpha-1} \left[T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)\right] f(s, x(s)) ds \\ & + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds \\ = & \left[T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I)\right] \int_0^{t_1} s^{\alpha-1} T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds \\ & + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds.\end{aligned}$$

By using a computation and assumption (H_1), we obtain

$$\| \Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) \| \leq \frac{\sup\limits_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)}}{\alpha} \left[(t_2^\alpha - t_1^\alpha) + \tau^\alpha \left| T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I \right| \right].$$

According to [40], the compactness of $(T(t))_{t>0}$ assures that $\lim\limits_{t_2 \longrightarrow t_1} \left| T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I \right| = 0$. Hence, combining this fact with the above inequality, we conclude that $\Gamma_2(x)$, $x \in B_r$ are equicontinuous on $[0,\tau]$.

`Claim 2:` We prove that the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X.

For some fixed $t \in]0,\tau]$ let $\varepsilon \in]0,t[$, $x \in B_r$ and define the operator Γ_2^ε as follows

$$\Gamma_2^\varepsilon(x)(t) = T\left(\frac{\varepsilon^\alpha}{\alpha}\right) \int_0^{(t^\alpha - \varepsilon^\alpha)^{\frac{1}{\alpha}}} s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha - \varepsilon^\alpha}{\alpha}\right) f(s, x(s)) ds.$$

Since $(T(t))_{t>0}$ is compact, then the set $\{\Gamma_2^\varepsilon(x)(t),\ x \in B_r\}$ is relatively compact in X. By using a computation combined with assumption (H_1), we get

$$\| \Gamma_2^\varepsilon(x)(t) - \Gamma_2(x)(t) \| \leq \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \frac{\varepsilon^\alpha}{\alpha}.$$

Therefore, we deduce that the $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X. For $t = 0$ the set $\{\Gamma_2(x)(0),\ x \in B_r\}$ is compact. Thus, the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X for all $t \in [0,\tau]$. By using the Arzelà-Ascoli theorem, we conclude that the operator Γ_2 is compact.

In conclusion, by the above steps combined with the Krasnoselskii fixed-point theorem, we conclude that $\Gamma_1 + \Gamma_2$ has at least one fixed point in $\mathcal{C}$, which is a mild solution of conformable fractional Cauchy problem (1.5). □

To obtain the uniqueness of the mild solution, we need the following assumption:

(H_5) There exists a constant $L_2 > 0$ such that $\| f(t,y) - f(t,x) \| \leq L_2 \| y - x \|$, for all $x, y \in X$ and $t \in [0,\tau]$.

Theorem 3.3. *Assume that $(H_2) - (H_5)$ hold, then the conformable fractional Cauchy problem* (1.5) *has an unique mild solution, provided that*

$$\left(L_1 + \sum_{i=1}^n C_i + \frac{\tau^\alpha}{\alpha} L_2\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| < 1.$$

Proof. Define the operator $\Gamma : \mathcal{C} \longrightarrow \mathcal{C}$ by:

$$\Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right) [x_0 + g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right) h_i(x(t_i)) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds.$$

For $x, y \in \mathcal{C}$, we have

$$\Gamma(y)(t) - \Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right)[h_i(y(t_i)) - h_i(x(t_i))]$$
$$+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s, y(s)) - f(s, x(s))]ds.$$

Then, we obtain

$$\begin{aligned}\| \Gamma(y)(t) - \Gamma(x)(t) \| \leq & \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \| g(y) - g(x) \| \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \sum_{0<t_i<t} \| h_i(y(t_i)) - h_i(x(t_i)) \| \\ & + \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \int_0^t s^{\alpha-1} \| f(s, y(s)) - f(s, x(s)) \| ds.\end{aligned}$$

According to assumptions (H_3), (H_4) and (H_5), we conclude that

$$\| \Gamma(y)(t) - \Gamma(x)(t) \| \leq \left(L_1 + \sum_{i=1}^n C_i + \frac{\tau^\alpha}{\alpha} L_2\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid y - x \mid_c .$$

Taking the supremum, we obtain

$$\mid \Gamma(y) - \Gamma(x) \mid_c \leq \left(L_1 + \sum_{i=1}^n C_i + \frac{\tau^\alpha}{\alpha} L_2\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \mid y - x \mid_c .$$

Since $\left(L_1 + \sum_{i=1}^n C_i + \frac{\tau^\alpha}{\alpha} L_2\right) \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| < 1$ then Γ is a contraction operator on the Banach space $(\mathcal{C}, \mid . \mid_c)$. Hence, by using the Banach contraction principle, we conclude that the operator Γ has an unique fixed point in $\mathcal{C}$, which is the mild solution of the conformable fractional Cauchy problem (1.5). □

Now, we are in position to prove the continuous dependence of the mild solution to the initial condition. Precisely, we have the following result.

Theorem 3.4. *Assume that the conditions of Theorem* (3.3) *are satisfied. Let x_0, $y_0 \in X$ and denote by x and y the solutions associated with x_0 and y_0, respectively. Then, we have the following estimate*

$$\mid y - x \mid_c \leq \frac{\alpha \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right|}{\alpha - \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left(\alpha L_1 + \sum_{i=1}^n \alpha C_i + L_2 \tau^\alpha\right)} \| y_0 - x_0 \| .$$

Proof. For $t \in [0, \tau]$, we have

$$y(t) - x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[y_0 - x_0 + g(y) - g(x)] + \sum_{0<t_i<t} T\left(\frac{t^\alpha - t_i^\alpha}{\alpha}\right)[h_i(y(t_i)) - h_i(x(t_i))]$$
$$+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s, y(s)) - f(s, x(s))]ds.$$

Then, by using a computation combined with assumptions (H_1), (H_3) and (H_4), we obtain

$$\| y(t) - x(t) \| \leq \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left[\| y_0 - x_0 \| + \left(L_1 + \frac{L_2\tau^\alpha}{\alpha} + \sum_{i=1}^{n} C_i\right) | y - x |_c \right].$$

Taking the supremum, we get

$$| y - x |_c \leq \sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left[\| y_0 - x_0 \| + \left(L_1 + \frac{L_2\tau^\alpha}{\alpha} + \sum_{i=1}^{n} C_i\right) | y - x |_c \right].$$

Thus, we deduce the desired estimate

$$| y - x |_c \leq \frac{\alpha \sup\limits_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right|}{\alpha - \sup\limits_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right)\right| \left(\alpha L_1 + \sum\limits_{i=1}^{n} \alpha C_i + L_2\tau^\alpha\right)} \| y_0 - x_0 \| .$$

□

4 Application

We consider the nonlocal impulsive partial differential equation with conformable fractional derivative of the form

$$\begin{cases} \dfrac{\partial^{\frac{1}{2}} u(t,\xi)}{\partial t^{\frac{1}{2}}} = -\dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + \displaystyle\int_0^t \frac{|\cos(u(t-s,\xi))|}{1+|\sin(u(t-s,\xi))|} ds, \quad (t,\xi) \in [0,1]\times]0,\pi[, \quad t \neq \dfrac{1}{2}, \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\ u(0,\xi) = \dfrac{1}{n^2}[u(t_1,\xi) + 2u(t_2,\xi) + 3u(t_3,\xi) + \cdots + nu(t_n,\xi)], \quad \xi \in [0,\pi], \\ \lim\limits_{\varepsilon \longrightarrow 0^+} u\left(\dfrac{1}{2}+\varepsilon,\xi\right) = \lim\limits_{\varepsilon \longrightarrow 0^+} u\left(\dfrac{1}{2}-\varepsilon,\xi\right) + \dfrac{|u(\frac{1}{2},\xi)|}{n + |u(\frac{1}{2},\xi)|}, \quad \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where $n \in \mathbb{N}$ such that $3 < n$ and $0 < t_1 < t_2 < t_3 < \cdots < t_n < 1$ are given real constants.

Let $X = L^2([0,\pi],\mathbb{R})$ and define the operator A as follows

$$A = -\frac{\partial^2(.)}{\partial \xi^2}, \quad D(A) = \{\varphi \in X : \ \varphi, \ \dot{\varphi} \text{ are absolutely continuous, } \ddot{\varphi} \in X \text{ and } \varphi(0) = \varphi(\pi) = 0\}.$$

It is well known that the operator A generates a compact semigroup $(T(t))_{t\geq 0}$ on X such that $\sup\limits_{t\geq 0}|T(t)| \leq 1$.

Next, we consider the change $x(t)(\xi) = u(t,\xi)$ and the following notations

$$f(t,x(t)) = \int_0^t \frac{|\cos(x(t-s))|}{1+|\sin(x(t-s))|} ds,$$

$$g(x) = \frac{1}{n^2}\sum_{i=1}^{n} ix(t_i),$$

$$h_1\left(x\left(\frac{1}{2}\right)\right) = \frac{|x(\frac{1}{2})|}{n + |x(\frac{1}{2})|}.$$

Then, equation (4.1) becomes as follows:

$$\begin{cases} \dfrac{d^{\frac{1}{2}}x(t)}{dt^{\frac{1}{2}}} & = Ax(t) + f(t, x(t)), t \in [0,1], \quad t \neq \frac{1}{2}, \\ x(0) & = g(x), \\ x(\frac{1}{2}^{+}) & = x(\frac{1}{2}^{-}) + h_1(x(\frac{1}{2})). \end{cases} \tag{4.2}$$

In this concrete application, we have $L_1 = \dfrac{1+2+3+\cdots+n}{n^2} = \dfrac{n+1}{2n}$, $C_1 = \dfrac{1}{n}$ and $\sup\limits_{t\in[0,1]} \left| T\left(\dfrac{t^\alpha}{\alpha}\right) \right| = \sup\limits_{t\in[0,1]} \left| T(2\sqrt{t}) \right|$.

Now, returning back to Theorem (3.2), we obtain that

$$(L_1 + C_1) \sup_{t\in[0,1]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| = \left(\frac{n+1}{2n} + \frac{1}{n}\right) \sup_{t\in[0,1]} |T(2\sqrt{t})| \leq \frac{n+1}{2n} + \frac{1}{n} = \frac{n+3}{2n} < 1.$$

Hence, we conclude that the above equation has at least one mild solution.

Conclusion

In this work, we have proved the Duhamel formula, existence, uniqueness, and stability of mild solutions of a class of nonlocal impulsive differential equations in the frame of the conformable fractional derivative. The main results are obtained by using the semigroup theory combined with some fixed point theorems. The ideas of this paper can be extended to other models in physics, biology, chemistry, economics and so forth.

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