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Variational methods to second-order Dirichlet boundary value problems with impulses on the half-line

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ABSTRACT

In this paper, the existence of solutions for a second-order impulsive differential equation with a parameter on the half-line is investigated. Applying Lax-Milgram Theorem, we deal with a linear Dirichlet impulsive problem, while the nonlinear case is established by using standard results of critical point theory.

RESUMEN

En este artículo, se investiga la existencia de soluciones de una ecuación diferencial de segundo orden impulsiva con un parámetro en la semi-recta. Aplicando el Teorema de Lax-Milgram, tratamos un problema lineal impulsivo de Dirichlet, mientras que el caso no lineal es establecido usando resultados estándar de teoría de punto crítico.

Keywords and Phrases: Dirichlet boundary value problem, half-line, Lax-Milgram theorem, critical points, impulsive differential equation.

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1 Introduction

In recent years, many researchers have extensively applied variational methods to study boundary value problems (BVPs) for impulsive differential equations on the finite intervals. More precisely, employing critical point theory, Nieto and O'Regan [8] studied a linear Dirichlet boundary value problem with impulses

$$\begin{cases} -u''(t) + \lambda u(t) &= \sigma(t), \quad \text{a.e. } t \in [0,T], \\ \triangle u'(t_j) &= d_j, \quad j \in \{1,2,\dots,l\}, \\ u(0) = u(T) &= 0, \end{cases} \tag{1.1}$$

and a nonlinear impulsive problem

$$\begin{cases} -u''(t) + \lambda u(t) &= f(t,u(t)), \quad \text{a.e. } t \in [0,T], \\ \triangle u'(t_j) &= I_j(u(t_j^-)), \quad j \in \{1,2,\dots,l\}, \\ u(0) = u(T) &= 0, \end{cases} \tag{1.2}$$

where λ is a positive parameter.

Moreover, the study of solutions for impulsive BVPs on the infinite intervals by using variational methods has received considerably more attention, see for example [1, 2, 3, 9, 10], and the references therein.

In the present paper, our aim is to improve some assumptions made in [8] in order to extend problems (1.1) and (1.2) on the half-line via variational approach.

This paper is organized as follows. In Section 2 we state some preliminaries. In Section 3 we consider the linear Dirichlet problem with impulses in the derivative. Due to the Lax-Milgram Theorem, we show the existence of weak solutions that are precisely the critical points of some functionals. The last section is to deal with the nonlinear Dirichlet problem. To investigate the existence of solutions, we use standard results of critical point theory. Also, some examples are given to illustrate our main results.

2 Preliminaries

We cite some basic and celebrated theorems from critical point theory which are crucial tools in the proof of our main results.

Let H be a Hilbert space.

Theorem 2.1 (Lax-Milgram [4, 5]). *Let $a : H \times H \to \mathbb{R}$ be a bounded bilinear form. If a is coercive, i.e., there exists $\alpha > 0$ such that $a(u,u) \geq \alpha \|u\|^2$ for every $u \in H$, then for any $\sigma \in H'$ (the conjugate space of H) there exists a unique $u \in H$ such that*

$$a(u,v) = (\sigma, v), \quad \textit{for every} \quad v \in H.$$

Moreover, if a is also symmetric, then the functional $\varphi : H \to \mathbb{R}$ defined by

$$\varphi(v) = \frac{1}{2}a(v, v) - (\sigma, v)$$

attains its minimum at u.

Theorem 2.2 ([7])**.** *If φ is weakly lower semi-continuous (w.l.s.c.) on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Now, let us recall some necessary concepts that will be needed in our argument. Let us define the following reflexive Banach space

$$H_0^1(0,\infty) = \left\{u : [0,\infty) \to \mathbb{R} \text{ is absolutely continuous, } u, u' \in L^2(0,\infty),\ u(0) = u(\infty) = 0\right\},$$

equipped with the norm

$$\|u\| = \left(\int_0^{+\infty} |u(t)|^2 dt + \int_0^{+\infty} |u'(t)|^2 dt\right)^{\frac{1}{2}}.$$

Set the space

$$C_{l,p}[0,+\infty) = \{u \in C([0,+\infty),\mathbb{R}) : \lim_{t\to\infty} p(t)u(t) \text{ exists}\}$$

with the norm

$$\|u\|_{\infty,p} = \sup_{t\in[0,+\infty)} p(t)|u(t)|,$$

where the function $p : [0; +\infty) \to (0, +\infty)$ is continuously differentiable and bounded, satisfying

$$C = 2\max(\|p\|_{L^2}, \|p'\|_{L^2}) < +\infty.$$

Concerning the above spaces, we get the following vital embeddings.

Lemma 2.3 ([6])**.** *The space $H_0^1(0,\infty)$ embeds continuously in $C_{l,p}[0,\infty)$, more precisely $\|u\|_{\infty,p} \le C\|u\|$ for every $u \in H_0^1(0,\infty)$.*

Lemma 2.4 ([6])**.** *The embedding $H_0^1(0,\infty) \hookrightarrow C_{l,p}[0,\infty)$ is compact.*

3 Impulsive linear problem

We consider the following linear Dirichlet boundary value problem with impulses in the derivative at the prescribed instants t_j, $j \in \mathbb{N}^* = \{1, 2, 3, \dots\}$

$$\begin{cases} -u''(t) + \lambda u(t) &= \sigma(t), \quad \text{a.e. } t \in [0,\infty),\ t \neq t_j, \\ \triangle u'(t_j) &= d(t_j), \quad j \in \mathbb{N}^*, \\ u(0) = u(+\infty) &= 0, \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{R}$, $\sigma \in L^2(0,\infty)$, $0 = t_0 < t_1 < t_2 < \cdots < t_j < \cdots < t_m \to \infty$, as $m \to \infty$, are the impulse points, $d : [0,\infty) \to \mathbb{R}$ satisfies $\sum_{j=1}^{\infty} \frac{d(t_j)}{p(t_j)} < \infty$ and $\triangle u'(t_j) = u'(t_j^+) - u'(t_j^-)$ for $u'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u'(t)$.

Now, multiply the equation in problem (3.1) by $v \in H_0^1(0,\infty)$, and then integrate over $(0,+\infty)$, we obtain

$$-\int_0^{+\infty} u''v + \lambda \int_0^{+\infty} uv = \int_0^{+\infty} \sigma v.$$

We have

$$-\int_0^{+\infty} u''v = -\sum_{j=0}^{\infty} \int_{t_j}^{t_{j+1}} u''v,$$

and

$$\int_{t_j}^{t_{j+1}} u''v = u'(t_{j+1}^-)v(t_{j+1}^-) - u'(t_j^+)v(t_j^+) - \int_{t_j}^{t_{j+1}} u'v'.$$

Consequently,

$$\begin{aligned} -\int_0^{+\infty} u''v &= \sum_{j=1}^{\infty} \triangle u'(t_j)v(t_j) + u'(0)v(0) - u'(\infty)v(\infty) + \int_0^{+\infty} u'v' \\ &= \sum_{j=1}^{\infty} d(t_j)v(t_j) + \int_0^{+\infty} u'v'. \end{aligned}$$

This leads to define the bilinear form $a : H_0^1(0,\infty) \times H_0^1(0,\infty) \to \mathbb{R}$, by

$$a(u,v) = \int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv, \tag{3.2}$$

and the linear operator $l : H_0^1(0,\infty) \to \mathbb{R}$ by

$$l(v) = \int_0^{+\infty} \sigma v - \sum_{j=1}^{\infty} d(t_j)v(t_j). \tag{3.3}$$

Definition 3.1. *We say that a function u is a weak solution of the impulsive problem (3.1) if $u \in H_0^1(0,\infty)$ such that $a(u,v) = l(v)$ is valid for any $v \in H_0^1(0,\infty)$.*

In what follows we refer to problem (3.1) as (LP).

It is easily verified that a and l defined by (3.2), (3.3) respectively are continuous, and a is coercive if $\lambda > 0$.

Consider the functional $\varphi : H_0^1(0,\infty) \to \mathbb{R}$, defined by

$$\varphi(u) = \frac{1}{2} \int_0^{+\infty} u'^2 + \frac{\lambda}{2} \int_0^{+\infty} u^2 - \int_0^{+\infty} \sigma u + \sum_{j=1}^{\infty} d(t_j)u(t_j). \tag{3.4}$$

It is clear that φ is differentiable at any $u \in H_0^1(0,\infty)$ and

$$\varphi'(u)v \quad = \quad \int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv - \int_0^{+\infty} \sigma v + \sum_{j=1}^{\infty} d(t_j)v(t_j) = a(u,v) - l(v).$$

Thus, a critical point of (3.4) gives us a weak solution of the problem (LP).

Definition 3.2. *We mean by a classical solution of the problem (LP) a function $u \in H^2(t_j, t_{j+1})$ for all $j \in \mathbb{N}^*$, where*

$$H^2(t_j, t_{j+1}) = \left\{u : [0,\infty) \to \mathbb{R} \quad \textit{is absolutely continuous}, \ u', u'' \in L^2(t_j, t_{j+1})\right\},$$

and u satisfies the first equation of (3.1) a.e. on $[0,\infty)$ with $u(0) = u(\infty) = 0$, the limits $u'(t_j^+)$, $u'(t_j^-)$, $j \in \mathbb{N}^*$ exist and the impulse conditions hold.

Lemma 3.3. *If $u \in H_0^1(0,\infty)$ is a weak solution of (LP), then u is a classical solution of (LP).*

Proof. Since $u \in H_0^1(0,\infty)$, it is evident that $u(0) = u(\infty) = 0$.

For $j \in \{1, 2, \dots\}$, choose any $v \in H_0^1(0,\infty)$ such that $v(t) = 0$ for $t \in [0, t_j] \cup [t_{j+1}, +\infty)$. Then

$$\int_{t_j}^{t_{j+1}} u'v' + \lambda \int_{t_j}^{t_{j+1}} uv = \int_{t_j}^{t_{j+1}} \sigma v.$$

Hence, $-u'' + \lambda u = \sigma$ a.e. on (t_j, t_{j+1}). So, $u \in H^2(t_j, t_{j+1})$ and satisfies the previous equation a.e. on $[0,\infty)$.

Multiplying $-u'' + \lambda u = \sigma$ by $v \in H_0^1(0,\infty)$ and integrating over $[0,\infty)$, we get

$$\sum_{j=1}^{\infty} \triangle u'(t_j)v(t_j) = \sum_{j=1}^{\infty} d(t_j)v(t_j).$$

Therefore, $\triangle u'(t_j) = d(t_j)$ for every $j \in \mathbb{N}^*$, and the impulsive conditions are satisfied. □

Lemma 3.4. *If $u \in H_0^1(0,\infty)$ is a critical point of φ defined by (3.4), then u is a weak solution of the impulsive Dirichlet problem (LP).*

Proof. Let $u \in H_0^1(0,\infty)$. The assumption that u is a critical point of φ means that $\varphi'(u)v = 0$, for all $v \in H_0^1(0,\infty)$. Thus,

$$\int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv - \int_0^{+\infty} \sigma v + \sum_{j=1}^{\infty} d(t_j)v(t_j) = 0, \quad \forall v \in H_0^1(0,\infty).$$

Hence,

$$\int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv = \int_0^{+\infty} \sigma v - \sum_{j=1}^{\infty} d(t_j)v(t_j), \quad \forall v \in H_0^1(0,\infty).$$

This implies that $a(u,v) = l(v)$ is valid for any $v \in H_0^1(0,\infty)$. As a result, u is a weak solution of the (LP). □

In view of Lax-Milgram theorem, we formulate the following main result.

Theorem 3.5. *If $\lambda > 0$, then the Dirichlet impulsive problem (LP) has a weak solution $u \in H_0^1(0,\infty)$ for any $\sigma \in L^2(0,\infty)$. Moreover, $u \in H^2(0,\infty)$ and u is a classical solution and minimizes the functional (3.4) and hence it is a critical point of (3.4).*

Proof. For $\lambda > 0$, it follows that the bilinear a is coercive. The fact that a is continuous, by applying Theorem 2.1, for any $\sigma \in L^2(0,\infty)$, there exists a unique $u \in H_0^1(0,\infty)$ such that $a(u,v) = l(v)$ for all $v \in H_0^1(0,\infty)$. So, the problem (LP) has a weak solution $u \in H_0^1(0,\infty)$.

Owing to Lemma 3.3, a weak solution of (LP) is a classical solution. In addition, a is symmetric, then the functional φ attains its minimum at u which is exactly a critical point of φ since it is differentiable. $\square$

Example 3.6. *As an example, let $\lambda = 1$ and $p(t) = \frac{1}{1+t^2}$.*

This impulsive boundary value problem

$$\begin{cases} -u''(t) + u(t) &= \frac{1}{1+t}, \quad a.e.\ \ t \in [0,\infty), \\ \triangle u'(j) &= e^{-j}, \quad j \in \mathbb{N}^*, \\ u(0) = u(+\infty) &= 0, \end{cases} \tag{3.5}$$

has a solution.

4 Impulsive nonlinear problem

In the nonlinear situation we consider the following impulsive boundary value problem

$$\begin{cases} -u''(t) + \lambda u(t) &= f(t,u(t)), \quad \text{a.e. } \ t \in [0,\infty),\ t \neq t_j, \\ \triangle u'(t_j) &= g(t_j) I_j(u(t_j^-)), \quad j \in \mathbb{N}^*, \\ u(0) = u(+\infty) &= 0, \end{cases} \tag{4.1}$$

where λ is a positive parameter, the functions $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $I_j : \mathbb{R} \to \mathbb{R}$, $j \in \mathbb{N}^*$, and $g : [0,\infty) \to [0,\infty)$ are continuous with $\sum_{j=1}^{\infty} g(t_j) < \infty$.

We refer to problem (4.1) as (NP).

Definition 4.1. *A weak solution of (NP) is a function $u \in H_0^1(0,\infty)$ such that*

$$\int_0^{+\infty} u'v' + \lambda \int_0^{+\infty} uv + \sum_{j=1}^{\infty} g(t_j) I_j(u(t_j)) v(t_j) - \int_0^{+\infty} f(t,u(t))dt = 0,$$

for every $v \in H_0^1(0,\infty)$.

Setting $F(t,u) = \int\limits_0^u f(t,s)ds$, we define the functional $\varphi : H_0^1(0,\infty) \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{2}\int\limits_0^{+\infty} u'^2(t)dt + \frac{\lambda}{2}\int\limits_0^{+\infty} u^2(t)dt + \sum_{j=1}^{\infty} g(t_j)\int\limits_0^{u(t_j)} I_j(s)ds - \int\limits_0^{+\infty} F(t,u(t))dt. \tag{4.2}$$

Now we present our principal results for this part.

Theorem 4.2. *Suppose that the following conditions hold:*

(H_1) *There exists a positive bounded function $M \in L^1(0,+\infty)$ with $\frac{M}{p} \in L^1(0,+\infty)$ such that*

$$|f(t,u)| \le M(t) \quad \textit{for} \quad (t,u) \in [0,+\infty) \times \mathbb{R}.$$

(I_1) *There exist $M_j > 0$, $j \in \mathbb{N}^*$, satisfying $\sum_{j=1}^{\infty} M_j g(t_j) < \infty$ and $\sum_{j=1}^{\infty} \frac{M_j g(t_j)}{p(t_j)} < \infty$, such that the impulsive functions I_j are bounded i.e.,*

$$|I_j(u)| \le M_j \quad \textit{for every} \quad u \in \mathbb{R},\ j \in \{1,2,\dots\}.$$

Then there is a critical point of φ, and (NP) has at least one solution.

Proof. Claim 1. φ is weakly lower semi-continuous (w.l.s.c).

Let $(u_n) \subset H_0^1(0,\infty)$ be a sequence such that $u_n \rightharpoonup u$ in $H_0^1(0,\infty)$, when $n \to \infty$. Then,

$$\|u\| \le \liminf_{n\to\infty} \|u_n\|,$$

and by Lemma 2.4 we have that (u_n) converges to u in $C_{l,p}[0,\infty)$, hence $u_n(t)$ converges to $u(t)$ for all $t \in [0,\infty)$.

From (H_1) and (I_1), using the continuity of f and I_j, $j \in \mathbb{N}^*$, together with the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}\liminf_{n\to+\infty} \varphi(u_n) &= \liminf_{n\to+\infty} \left[\frac{1}{2}\int\limits_0^{+\infty} u_n'^2 + \frac{\lambda}{2}\int\limits_0^{+\infty} u_n^2 + \sum_{j=1}^{\infty} g(t_j)\int\limits_0^{u_n(t_j)} I_j(s)ds - \int\limits_0^{+\infty} F(t,u_n(t))dt\right] \\ &\ge \frac{1}{2}\int\limits_0^{+\infty} u'^2 + \frac{\lambda}{2}\int\limits_0^{+\infty} u^2 + \sum_{j=1}^{\infty} g(t_j)\int\limits_0^{u(t_j)} I_j(s)ds - \int\limits_0^{+\infty} F(t,u(t))dt = \varphi(u).\end{aligned}$$

Thus, φ is w.l.s.c.

Claim 2. φ is coercive.

For any $u \in H_0^1(0,\infty)$, the fact that $\lambda > 0$, there exists $\alpha > 0$ such that

$$\varphi(u) \ge \alpha\|u\|^2 + \sum_{j=1}^{\infty} g(t_j)\int\limits_0^{u(t_j)} I_j(s)ds - \int\limits_0^{+\infty} F(t,u(t))dt.$$

Using conditions (H_1), (I_1) and Lemma 2.3, we have

$$\begin{aligned}\varphi(u) &\geq \alpha\|u\|^2 - \sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)}p(t_j)|u(t_j)| - \int_0^{+\infty}\frac{M(t)}{p(t)}p(t)|u(t)|dt\\ &\geq \alpha\|u\|^2 - \|u\|_{\infty,p}\sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)} - \|u\|_{\infty,p}\int_0^{+\infty}\frac{M(t)}{p(t)}dt\\ &\geq \alpha\|u\|^2 - C\|u\|\sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)} - C\|u\|\left\|\frac{M}{p}\right\|_{L^1}\\ &\geq \alpha\|u\|^2 - C\left(\sum_{j=1}^{\infty}\frac{M_j g(t_j)}{p(t_j)} + \left\|\frac{M}{p}\right\|_{L^1}\right)\|u\|,\end{aligned}$$

for some $C > 0$. Then, the above inequality implies that $\lim\limits_{\|u\|\to+\infty}\varphi(u) = +\infty$. Hence, φ is coercive.

Applying Theorem 2.2, φ possesses a minimum which is a critical point of φ. Finally, by (H_1) and (I_1), it is easy to check that φ is continuous and differentiable for any $u \in H_0^1(0,\infty)$ and that

$$\varphi'(u)v = \int_0^{+\infty} u'v' + \lambda\int_0^{+\infty} uv + \sum_{j=1}^{\infty} g(t_j)I_j(u(t_j))v(t_j)dt - \int_0^{+\infty} f(t,u(t))v(t)dt. \tag{4.3}$$

Therefore, a critical point of φ is a weak solution of the problem (NP). □

Remark 4.3. *Assume $M \in L^2(0,\infty)$ in (H_1), then it is easy to see that a weak solution u is in $H^2(0,\infty)$.*

Example 4.4. *Take $\lambda = 1$, $p(t) = e^{-t}$, $M(t) = e^{-2t}$, $g(t) = e^{-2t}$, $M_j = \dfrac{1}{j}$ and $I_j(s) = \dfrac{1}{j+s^2}$, $j \in \mathbb{N}^*$.*

The following IBVP:

$$\begin{cases} -u''(t) + u(t) &= e^{-3t}, \quad a.e.\ \ t \in [0,\infty),\\ \triangle u'(j) &= \dfrac{e^{-2j}}{j+u^2(j)}, \quad j \in \mathbb{N}^*,\\ u(0) = u(+\infty) &= 0,\end{cases}$$

has at least one solution. (See Figure 1)

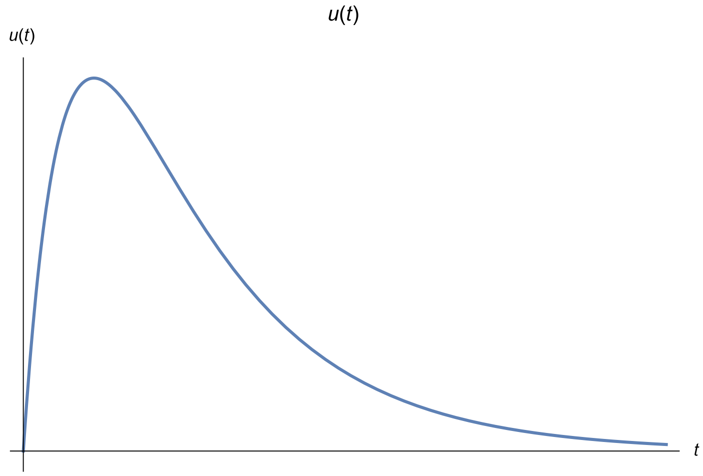

Figure 1

Theorem 4.5. *Assume the following conditions are satisfied:*

(H_2) *The function f is sublinear i.e., there exist a constant $\gamma \in [0,1)$ and positive functions $a, b \in L^1(0,\infty)$ with $\frac{a}{p}, \frac{b}{p^\gamma}, \frac{b}{p^{\gamma+1}} \in L^1[0,\infty)$ such that*

$$|f(t,u)| \leq a(t) + b(t)|u|^\gamma \quad \textit{for} \quad (t,u) \in [0,+\infty) \times \mathbb{R}.$$

(I_2) *There exist constants $\delta \in [0,1)$ and $a_j, b_j > 0$, $j \in \{1,2,\dots\}$ with $\sum\limits_{j=1}^{\infty} a_j g(t_j)$, $\sum\limits_{j=1}^{\infty} \frac{a_j g(t_j)}{p(t_j)}$, $\sum\limits_{j=1}^{\infty} \frac{b_j g(t_j)}{p^\delta(t_j)}$, $\sum\limits_{j=1}^{\infty} \frac{b_j g(t_j)}{p^{\delta+1}(t_j)}$ are convergent series, such that the impulsive functions I_j have sublinear growths i.e.,*

$$|I_j(u)| \leq a_j + b_j|u|^\delta \quad \textit{for every} \quad u \in \mathbb{R},\ j \in \{1,2,\dots\}.$$

Then there is a critical point of φ, and (NP) has at least one solution.

Proof. Claim 1. φ is weakly lower semi-continuous.

Under (H_2) and (I_2), arguing analogously to the proof of Theorem 4.2, we find the weak lower semi-continuity of φ.

Claim 2. φ is coercive.

In view of conditions (H_2), (I_2) and (4.2), for any $u \in H_0^1(0,\infty)$, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^{+\infty} u'^2 + \frac{\lambda}{2}\int_0^{+\infty} u^2 + \sum_{j=1}^{\infty} g(t_j)\int_0^{u(t_j)} I_j(s)ds - \int_0^{+\infty} F(t,u(t))dt \\ &\geq \alpha\|u\|^2 - \sum_{j=1}^{\infty} g(t_j)\int_0^{u(t_j)} (a_j + b_j|s|^\delta)ds - \int_0^{+\infty}\left(a(t)|u(t)| + \frac{b(t)}{\gamma+1}|u(t)|^{\gamma+1}\right)dt\end{aligned}$$

$$
\begin{aligned}
&\geq \alpha\|u\|^2 - \sum_{j=1}^{\infty} g(t_j)\left(\frac{a_j}{p(t_j)}p(t_j)|u(t_j)| + \frac{b_j}{(\delta+1)p^{\delta+1}(t_j)}|p(t_j)u(t_j)|^{\delta+1}\right)\\
&- \int_0^{+\infty} \frac{a(t)}{p(t)}p(t)|u(t)|dt - \frac{1}{(\gamma+1)}\int_0^{+\infty}\frac{b(t)}{p^{\gamma+1}(t)}|p(t)u(t)|^{\gamma+1}dt\\
&\geq \alpha\|u\|^2 - \|u\|_{\infty,p}\sum_{j=1}^{\infty}\frac{a_j g(t_j)}{p(t_j)} - \|u\|_{\infty,p}^{\delta+1}\sum_{j=1}^{\infty}\frac{b_j g(t_j)}{p^{\delta+1}(t_j)} - \|u\|_{\infty,p}\left\|\frac{a}{p}\right\|_{L^1}\\
&- \|u\|_{\infty,p}^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}.
\end{aligned}
$$

Hence, by Lemma 2.3, we get

$$
\begin{aligned}
\varphi(u) &\geq \alpha\|u\|^2 - C\|u\|\sum_{j=1}^{\infty}\frac{a_j g(t_j)}{p(t_j)} - C^{\delta+1}\|u\|^{\delta+1}\sum_{j=1}^{\infty}\frac{b_j g(t_j)}{p^{\delta+1}(t_j)} - C\|u\|\left\|\frac{a}{p}\right\|_{L^1}\\
&- C^{\gamma+1}\|u\|^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}\\
&\geq \alpha\|u\|^2 - C\left(\left\|\frac{a}{p}\right\|_{L^1} + \sum_{j=1}^{\infty}\frac{a_j g(t_j)}{p(t_j)}\right)\|u\| - C^{\delta+1}\left(\sum_{j=1}^{\infty}\frac{b_j g(t_j)}{p^{\delta+1}(t_j)}\right)\|u\|^{\delta+1}\\
&- C^{\gamma+1}\left\|\frac{b}{p^{\gamma+1}}\right\|_{L^1}\|u\|^{\gamma+1}.
\end{aligned}
$$

Since $\delta, \gamma \in [0,1)$, then $\lim\limits_{\|u\|\to+\infty}\varphi(u) = +\infty$. This means, φ is coercive.

Using Theorem 2.2, φ has a minimum, which is a critical point of φ. Finally, from (H_2) and (I_2), we get the differentiability of φ such that its differentiable is defined by (4.3). Consequently, (NP) has at least one solution. □

Remark 4.6. *In (H_2), assume $a, \dfrac{b}{p^\gamma} \in L^2(0,\infty)$, then a weak solution u is in $H^2(0,\infty)$.*

Example 4.7. *Consider the following problem*

$$
\begin{cases}
-u''(t) + u(t) &= e^{-2t}\sqrt{|u(t)|} + e^{-3t}, \quad a.e.\ \ t\in[0,\infty),\\
\triangle u'(j) &= e^{-2j}\left(\dfrac{1}{j^2} + \dfrac{|s|^{\frac14}}{j}\right), \quad j\in\mathbb{N}^*,\\
u(0) = u(+\infty) &= 0,
\end{cases}
$$

where $\lambda = 1$, $p(t) = e^{-t}$, $g(t) = e^{-2t}$, $a_j = \dfrac{1}{j^2}$, $b_j = \dfrac{1}{j}$ and $I_j(s) = \dfrac{1}{j^2} + \dfrac{|s|^{\frac14}}{j}$, $j\in\mathbb{N}^*$.

By simple calculations, all conditions in Theorem 4.5 are satisfied, then (4.1) has at least one solution.

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