Cubo A Mathematical Journal

CUBO, A Mathematical Journal
Vol. 24, no. 01, pp. 63–82, April 2022
DOI: 10.4067/S0719-06462022000100063

The topological degree methods for the fractional $p(\cdot)$-Laplacian problems with discontinuous nonlinearities

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ABSTRACT

In this article, we use the topological degree based on the abstract Hammerstein equation to investigate the existence of weak solutions for a class of elliptic Dirichlet boundary value problems involving the fractional $p(x)$-Laplacian operator with discontinuous nonlinearities. The appropriate functional framework for this problems is the fractional Sobolev space with variable exponent.

RESUMEN

En este artículo, usamos el grado topológico basado en la ecuación abstracta de Hammerstein para investigar la existencia de soluciones débiles para una clase de problemas elípticos de valor en la frontera de Dirichlet que involucran el operador $p(x)$-Laplaciano fraccional con no linealidades discontinuas. El marco funcional apropiado para estos problemas es el espacio de Sobolev fraccional con exponente variable.

Keywords and Phrases: Fractional $p(x)$-Laplacian, weak solution, discontinuous nonlinearity, topological degree theory.

2020 AMS Mathematics Subject Classification: 35R11, 35J60, 47H11, 35A16.

Accepted: 05 November, 2021
Received: 12 May, 2021

1 Introduction and main result

The study of fractional Sobolev spaces and the corresponding nonlocal equations has received a tremendous popularity in the last two decades considering their intriguing structure and great application in many fields, such as social sciences, fractional quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process, game theory, and Levy process, see [34, 35] and references therein for more details.

On the other hand, in recent years, a great deal of attention has been paid to the study of differential equations and variational problems involving $p(x)$-growth conditions since they can be used to model a variety of physical phenomena that occur in the fields of elastic mechanics, electro-rheological fluids ("smart fluids"), and image processing, etc. The readers are guided to [19, 20, 27] and its references.

It is only normal to wonder what results can be obtained when the fractional $p(\cdot)$-Laplacian is used instead of the $p(\cdot)$-Laplacian. The fractional $p(\cdot)$-Laplacian has also recently been investigated in elliptic problems; see [8, 10, 25, 26]. U. Kaufmann *et al.* [26] presented a new class of fractional Sobolev spaces with variable exponents in a recent paper. The authors in [8, 9] showed some additional basic properties on this function space as well as the associated nonlocal operator.

They used the critical point theory in [4] to prove the existence of solutions for fractional $p(\cdot)$-Laplacian equations. K. Ho and Y.-H. Kim [25] managed to obtain fundamental imbeddings for a new fractional Sobolev space with variable exponents, which is a generalization of previously defined fractional Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with Lipschitz boundary and let $p : \overline{\Omega}\times\overline{\Omega} \to (1,\ +\infty)$ be a continuous bounded function. The purpose of this paper is to establish the existence of nontrivial weak solutions for the following fractional $p(x)$-Laplacian problems with discontinuous nonlinearities.

$$\begin{cases} (-\triangle_{p(x)})^s u(x) + |u(x)|^{q(x)-2}u(x) + \lambda H(x,u) \in -\left[\underline{\psi}(x,u), \overline{\psi}(x,u)\right] & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $ps < N$ with $0 < s < 1$ and $(-\triangle_{p(x)})^s$ is the fractional $p(x)$-Laplacian operator defined by

$$(-\Delta)^s_{p(x)} u(x) = p.v. \int_{\mathbb{R}^N\backslash B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}} dy, \quad x \in \mathbb{R}^N \tag{1.2}$$

$\forall x \in \Omega$, where $p.v.$ is a commonly used abbreviation in the principal value sense and let $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,\ y) \leq p(x,\ y) \leq p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,\ y) < +\infty, \tag{1.3}$$

p is symmetric *i. e.*

$$p(x,\ y) = p(y,\ x),\ \forall (x,\ y) \in \overline{\Omega}\times\overline{\Omega}; \tag{1.4}$$

and $B_\varepsilon(x) := \{y \in \mathbb{R}^N \ : \ |x-y| < \varepsilon\}$.
Let as denote by:

$$\widetilde{p}(x) = p(x,\ x),\ \forall x \in \overline{\Omega}.$$

Furthermore, the Carathéodory's functions H satisfy only the growth condition, for all $s \in \mathbb{R}$ and a. e. $x \in \Omega$.

$$(H_0) \qquad |H(x,s)| \leq \varrho(e(x) + |s|^{q(x)-1}),$$

where ϱ is a positive constant, $e(x)$ is a positive function in $L^{p'(x)}(\Omega)$.

In the simplest case $p = 2$, we have the well-known fractional Laplacian, a large amount of papers were written on this direction see [6, 15]. Moreover, if $s = 1$, we get the classic Laplacian. Some related results can be found in [21, 39, 40, 41, 42]. Notice that when $s = 1$, the problems like (1.1) have been studied in many papers, we refer the reader to [1, 5, 24], in which the authors have used various methods to get the existence of solutions for (1.1). In the case when $p = p(x)$ is a continuous function, problem (1.1) has also been studied by many authors. For more information, see [11, 23].

In order to prove the existence of nontrivial weak solutions, the main difficulties are reflected in the following aspect, we cannot directly use the topological degree methods in a natural way because the nonlinear term ψ is discontinuous. In order to overcome the discontinuous difficulty, we will transform this Dirichlet boundary value problem involving the fractional p-Laplacian operator with discontinuous nonlinearities into a new one governed by a Hammerstein equation. Then, we shall employ the topological degree theory developed by Kim in [29, 28] for a class of weakly upper semi-continuous locally bounded set-valued operators of (S_+) type in the framework of real reflexive separable Banach spaces, based on the Berkovits-Tienari degree [12]. The topological degree theory was constructed for the first time by Leray-Schauder [31] in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Furthermore, Browder [14] has developed a topological degree for operators of class (S_+) in reflexive Banach spaces, see also [37, 38]. Among many examples, we refer the reader to the classical works [2, 3, 18, 45] for more details.

To this end, we always assume that $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a possibly discontinuous function, we "fill the discontinuity gaps" of ψ, replacing ψ by an interval $\left[\underline{\psi}(x,u), \overline{\psi}(x,u)\right]$, where

$$\underline{\psi}(x,s) = \liminf_{\eta \to s} \psi(x,\eta) = \lim_{\delta \to 0^+} \inf_{|\eta - s| < \delta} \psi(x,\eta),$$

$$\overline{\psi}(x,s) = \limsup_{\eta \to s} \psi(x,\eta) = \lim_{\delta \to 0^+} \sup_{|\eta - s| < \delta} \psi(x,\eta).$$

Such that

(H_1) $\overline{\psi}$ and $\underline{\psi}$ are super-positionally measurable (*i. e.*, $\overline{\psi}(\cdot, u(\cdot))$ and $\underline{\psi}(\cdot, u(\cdot))$ are measurable on Ω for every measurable function $u : \Omega \to \mathbb{R}$).

(H_2) ψ satisfies the growth condition:

$$|\psi(x,s)| \leq b(x) + c(x)|s|^{\gamma(x)-1},$$

for almost all $x \in \Omega$ and all $s \in \mathbb{R}$, where $b \in L^{\gamma'(x)}(\Omega)$, $c \in L^{\infty}(\Omega)$, where $1 < \gamma(x) < p(x)$ for all $x \in \overline{\Omega}$.

First of all, we define the operator $\mathcal{N}$ acting from $W_0^{s,p(x,y)}(\Omega)$ into $2^{\left(W_0^{s,p(x,y)}(\Omega)\right)^*}$ by

$$\begin{aligned}\mathcal{N}u = \big\{\varphi \in \left(W_0^{s,p(x,y)}(\Omega)\right)^* \ \setminus \ \exists\, h \in L^{p'(x)}(\Omega);\\ \underline{\psi}(x,u(x)) \leq h(x) \leq \overline{\psi}(x,u(x)) \ \text{ a. e. } x \in \Omega\\ \text{and } \langle \varphi, v\rangle = \int_\Omega hv dx \quad \forall v \in W_0^{s,p(x,y)}(\Omega)\big\}.\end{aligned}$$

In this spirit, we consider $F : W_0^{s,p(x,y)}(\Omega) \longrightarrow \left(W_0^{s,p(x,y)}(\Omega)\right)^*$ such that

$$\langle Fu, v\rangle = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}} dxdy, \tag{1.5}$$

for all $v \in W_0^{s,p(x,y)}(\Omega)$ and the operator $A : W_0 \to W_0^*$ setting by

$$\langle Au, v\rangle = \int_\Omega |u(x)|^{q(x)-2}(u(x)v(x) + \lambda H(x,u))v(x)dx, \quad \forall u, v \in W_0,$$

where the spaces $W_0^{s,p(x,y)}(\Omega) := W_0$ will be introduced in Section 2.

Next, we give the definition of weak solutions for problem (1.1).

Definition 1.1. *A function $u \in W_0^{s,p(x,y)}(\Omega)$ is called a weak solution to problem* (1.1)*, if there exists an element $\varphi \in \mathcal{N}u$ verifying*

$$\langle Fu, v\rangle + \langle Au, v\rangle + \langle \varphi, v\rangle = 0, \quad \textit{for all} \quad v \in W_0^{s,p(x,y)}(\Omega).$$

Now we are in a position to present our main result.

Theorem 1.2. *Assume that ψ satisfies $(H_1), (H_2)$ and H satisfies (H_0). Then, the problem* (1.1) *has a weak solution u in $W_0^{s,p(x,y)}(\Omega)$.*

2 Preliminaries

2.1 Lebesgue and fractional Sobolev spaces with variable exponent

In this subsection, we first recall some useful properties of the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$. For more details we refer the reader to [22, 30, 44].

Denote

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) | \inf_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ := \max\{h(x),\ x \in \overline{\Omega}\}, \quad h^- := \min\{h(x),\ x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define the variable exponent Lebesgue spaces

$$L^{p(x)}(\Omega) = \left\{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\}.$$

Endowed with *Luxemburg norm*

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 \,|\, \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

where

$$\rho_{p(\cdot)}\ (u) = \int_\Omega |u(x)|^{p(x)}dx,\ \forall u \in L^{p(x)}$$

$(L^{p(x)}(\Omega), ||.||_{p(x)})$ is a Banach space, separable and reflexive. Its conjugate space is $L^{p'(x)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ for all $x \in \Omega$. We have also the following result

Proposition 2.1. *([22]) For any $u \in L^{p(x)}(\Omega)$ we have*

(i) $\|u\|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$,

(ii) $\|u\|_{p(x)} \geq 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^+}$,

(iii) $\|u\|_{p(x)} \leq 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^-}$.

From this proposition, we can deduce the inequalities

$$\|u\|_{p(x)} \leq \rho_{p(\cdot)}(u) + 1, \tag{2.1}$$

$$\rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}. \tag{2.2}$$

If $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{q(x)}(\Omega) \to L^{p(x)}(\Omega)$.

Next, we present the definition and some results on fractional Sobolev spaces with variable exponent that was introduced in [8, 26]. Let s be a fixed real number such that $0 < s < 1$, and let $q : \overline{\Omega} \to (0, \infty)$ and $p : \overline{\Omega} \times \overline{\Omega} \to (0, \infty)$ be two continuous functions. Furthermore, we suppose that the assumptions (1.3) and (1.4) be satisfied, we define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$W = W^{s,,q(x),p(x,y)}(\Omega) = \left\{u \in L^{q(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}}dxdy < +\infty,\right.$$
$$\left.\text{for some } \lambda > 0\right\}.$$

We equip the space W with the norm

$$\|u\|_W = \|u\|_{q(x)} + [u]_{s,p(x,y)},$$

where $[\cdot]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = \inf\left\{\lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy \leq 1\right\}.$$

The space $(W,\ \|\cdot\|_W)$ is a Banach space (see [17]), separable and reflexive (see [8, Lemma 3.1]).

We also define W_0 as the subspace of W which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||_W$. From [7, Theorem 2.1 and Remark 2.1]

$$\|\cdot\|_{W_0} := [\cdot]_{s,p(x,y)}$$

is a norm on W_0 which is equivalent to the norm $\|\cdot\|_W$, and we have the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}$. So the space $(W_0,\ \|\cdot\|_{W_0})$ is a Banach space separable and reflexive.

We defne the modular $\rho_{p(\cdot,\cdot)} : W_0 \to \mathbb{R}$ by

$$\rho_{p(\cdot,\cdot)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.$$

The modular ρ_p checks the following results, which is similar to Proposition 2.1 (see [43, Lemma 2.1])

Proposition 2.2. *([30]) For any $u \in W_0$ we have*

(i) $\|u\|_{W_0} \geq 1 \Rightarrow \|u\|_{W_0}^{p^-} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{W_0}^{p^+}$,

(ii) $\|u\|_{W_0} \leq 1 \Rightarrow \|u\|_{W_0}^{p^+} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{W_0}^{p^-}$.

2.2 Some classes of operators and an outline of Berkovits degree

Now, we introduce the theory of topological degree which is the major tool for our results. We start by defining some classes of mappings. Let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle\cdot\,,\cdot\rangle$ between X^* and X in this order. The symbol $\rightharpoonup$ stands for weak convergence. Let Y be another real Banach space.

Definition 2.3.

(1) *We say that the set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is bounded, if* F *maps bounded sets into bounded sets;*

(2) *we say that the set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is locally bounded at the point* $u \in \Omega$*, if there is a neighborhood* V *of* u *such that the set* $\mathrm{F(V)} = \bigcup_{u\in V} \mathrm{F}u$ *is bounded.*

Definition 2.4. *The set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is called*

(1) *upper semicontinuous (u.s.c.) at the point* u*, if, for any open neighborhood* V *of the set* $\mathrm{F}u$*, there is a neighbhorhood* U *of the point* u *such that* $\mathrm{F}(U) \subseteq V$*. We say that* F *is upper semicontinuous (u.s.c) if it is u.s.c at every* $u \in X$*;*

(2) *weakly upper semicontinuous (w.u.s.c.), if* $\mathrm{F}^{-1}(U)$ *is closed in* X *for all weakly closed set* U *in* Y.

Definition 2.5. *Let* Ω *be a nonempty subset of* X, $(u_n)_{n\geq 1} \subseteq \Omega$ *and* $F : \Omega \subset X \to 2^{X^*} \setminus \emptyset$*. Then, the set-valued operator* F *is*

(1) *of type* (S_+)*, if* $u_n \rightharpoonup u$ *in* X *and for each sequence* (h_n) *in* X^* *with* $h_n \in Fu_n$ *such that*

$$\limsup_{n\to\infty} \langle h_n, u_n - u\rangle \leq 0,$$

we get $u_n \to u$ *in* X*;*

(2) *quasi-monotone, if* $u_n \rightharpoonup u$ *in* X *and for each sequence* (w_n) *in* X^* *such that* $w_n \in Fu_n$ *yield*

$$\liminf_{n\to\infty} \langle w_n, u_n - u\rangle \geq 0.$$

Definition 2.6. *Let* Ω *be a nonempty subset of* X *such that* $\Omega \subset \Omega_1$, $(u_n)_{n\geq 1} \subseteq \Omega$ *and* $T : \Omega_1 \subset X \to X^*$ *be a bounded operator. Then, the set-valued operator* $F : \Omega \subset X \to 2^X \setminus \emptyset$ *is of type* $(S_+)_T$*, if*

$$\begin{cases} u_n \rightharpoonup u \text{ in } X, \\ Tu_n \rightharpoonup y \text{ in } X^*, \end{cases}$$

and for any sequence (h_n) *in* X *with* $h_n \in Fu_n$ *such that*

$$\limsup_{n\to\infty} \langle h_n, Tu_n - y\rangle \leq 0,$$

we have $u_n \to u$ *in* X.

Next, we consider the following sets :

$$\mathcal{F}_1(\Omega) := \{F : \Omega \to X^* | F \text{ is bounded, demicontinuous and of type } (S_+)\},$$
$$\mathcal{F}_T(\Omega) := \{F : \Omega \to 2^X | F \text{ is locally bounded, w.u.s.c. and of type } (S_+)_T\},$$

for any $\Omega \subset D_F$ and each bounded operator $T : \Omega \to X^*$, where D_F denotes the domain of F.

Remark 2.7. *We say that the operator* T *is an essential inner map of* F*, if* $T \in \mathcal{F}_1(\overline{G})$.

Lemma 2.8. *([29, Lemma 1.4]) Let* X *be a real reflexive Banach space and* $G \subset X$ *is a bounded open set. Assume that* $T \in \mathcal{F}_1(\overline{G})$ *is continuous and* $S : D_S \subset X^* \to 2^X$ *weakly upper semicontinuous and locally bounded with* $T(\overline{G}) \subset D_s$*. Then the following alternative holds:*

(1) *If S is quasi-monotone, yield $I + S \circ T \in \mathcal{F}_T(\overline{G})$, where I denotes the identity operator.*

(2) *If S is of type (S_+), yield $S \circ T \in \mathcal{F}_T(\overline{G})$.*

Definition 2.9. *([29]) Let $T : \overline{G} \subset X \to X^*$ be a bounded operator, a homotopy $H : [0,1]\times\overline{G} \to 2^X$ is called of type $(S_+)_T$, if for every sequence (t_k, u_k) in $[0,1]\times\overline{G}$ and each sequence (a_k) in X with $a_k \in H(t_k, u_k)$ such that*

$$u_k \rightharpoonup u \ \in X, \quad t_k \to t \in [0,1], \quad Tu_k \rightharpoonup y \quad \textit{in} \quad X^* \textit{ and } \quad \limsup_{k\to\infty}\langle a_k, Tu_k - y\rangle \le 0,$$

we get $u_k \to u$ in X.

Lemma 2.10. *([29]) Let X be a real reflexive Banach space and $G \subset X$ is a bounded open set, $T : \overline{G} \to X^*$ is continuous and bounded. If F, S are bounded and of class $(S_+)_T$, then an affine homotopy $H : [0,1]\times\overline{G} \to 2^X$ given by*

$$H(t,u) := (1-t)Fu + tSu, \quad \textit{for } (t,u) \in [0,1]\times\overline{G},$$

is of type $(S_+)_T$.

Now, we introduce the topological degree for a class of locally bounded, w.u.s.c. and satisfies condition $(S_+)_T$ for more details see [29].

Theorem 2.11. *Let*

$$L = \big\{(F,G,g)|G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_T(\overline{G}),\ g \notin F(\partial G)\big\},$$

where $\mathcal{O}$ denotes the collection of all bounded open sets in X. There exists a unique (Hammerstein type) degree function

$$d : L \longrightarrow \mathbb{Z}$$

such that the following alternative holds:

(1) *(*Normalization*) For each $g \in G$, we have $d(I,G,g) = 1$.*

(2) *(*Domain Additivity*) Let $F \in \mathcal{F}_T(\overline{G})$. We have*

$$d(F,G,g) = d(F,G_1,g) + d(F,G_2,g),$$

with G_1, $G_2 \subseteq G$ disjoint open such that $g \notin F(\overline{G}\backslash(G_1 \cup G_2))$.

(3) *(*Homotopy invariance*) If $H : [0,1]\times\overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and g: $[0,1] \to X$ is a continuous path in X such that $g(t) \notin H(t,\partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,\cdot),G,g(t))$ is constant for any $t \in [0,1]$.*

(4) *(*Solution Property*) If $d(F,G,g) \neq 0$, then the equation $g \in Fu$ has a solution in G.*

3 Proof of Theorem 1.2

In the present section, following compactness methods (see [18, 32]), we prove the existence of weak solutions for the problem (1.1) in fractional Sobolev spaces. In doing so, we transform this elliptic Dirichlet boundary value problem involving the fractional p-Laplacian operator with discontinuous nonlinearities into a new problem governed by a Hammerstein equation. More precisely, by means of the topological degree theory introduced in section 2, we establish the existence of weak solutions to the state problem, which holds under appropriate assumptions. First, we give several lemmas.

Lemma 3.1. *Let $0 < s < 1$ and $1 < p(x,y) < +\infty$, (or $sp_+ < N$) the operator F defined in* (1.5) *is*

(i) bounded and strictly monotone operator.

(ii) of type (S_+).

Proof. *(i)* It is clear that F is a bounded operator. For all $\xi, \eta \in \mathbb{R}^N$, we have the Simon inequality (see [36]) from which we can obtain the strictly monotonicity of F:

$$\begin{cases} |\xi-\eta|^p \le c_p\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)(\xi-\eta); & p \ge 2 \\ \\ |\xi-\eta|^p \le C_p\Big[\Big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\Big)(\xi-\eta)\Big]^{\frac{p}{2}}\big(|\xi|^p+|\eta|^p\big)^{\frac{2-p}{2}}; & 1 < p < 2, \end{cases}$$

where $c_p = \left(\frac{1}{2}\right)^{-p}$ and $C_p = \dfrac{1}{p-1}$.

(ii) Let $(u_n) \in W_0^{s,p(x,y)}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Fu_n - Fu, u_n - u\rangle \le 0$. In view of (i), we get

$$\lim_{n\to\infty}\langle Fu_n - Fu, u_n - u\rangle = 0.$$

Thanks to Proposition 2.1, we obtain

$$u_n(x) \to u(x), \ \text{a.e. } x \in \Omega. \tag{3.1}$$

In the sequel, we denote by $L(x,y) = |x-y|^{-N-sp(x,y)}$.

By Fatou's lemma and (3.1), we get

$$\liminf_{n\to+\infty}\int_{\Omega\times\Omega}|u_n(x)-u_n(y)|^{p(x,y)}L(x,y)dxdy \ge \int_{\Omega\times\Omega}|u(x)-u(y)|^{p(x,y)}L(x,y)dxdy. \tag{3.2}$$

On the other hand, from $u_n \rightharpoonup u$ we have

$$\lim_{n\to+\infty}\langle Fu_n,\ u_n - u\rangle = \lim_{n\to+\infty}\langle Fu_n - Fu, u_n - u\rangle = 0. \tag{3.3}$$

Now, by using Young's inequality, there exists a positive constant c such that

$$\begin{aligned}\langle Fu_n, u_n - u\rangle &= \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)}L(x,y)dxdy \\ &- \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))(u(x)-u(y))L(x,y)dxdy \\ &\geq \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)}L(x,y)dxdy \\ &- \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)-1}|u(x)-u(y)|L(x,y)dxdy \\ &\geq c\int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)}L(x,y)dxdy \\ &- c\int_{\Omega\times\Omega} |u(x)-u(y)|^{p(x,y)}L(x,y)dxdy,\end{aligned} \tag{3.4}$$

combining (3.2), (3.3) and (3.4), we obtain

$$\lim_{n\to+\infty}\int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)}L(x,y)dxdy = \int_{\Omega\times\Omega} |u(x)-u(y)|^{p(x,y)}L(x,y)dxdy. \tag{3.5}$$

According to (3.1), (3.5) and the Brezis-Lieb lemma [13], our result is proved. □

Proposition 3.2. *([16, Proposition 1]) For any fixed $x \in \Omega$, the functions $\overline{\psi}(x,s)$ and $\underline{\psi}(x,s)$ are upper semicontinuous (u.s.c.) functions on $\mathbb{R}^N$.*

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with smooth boundary. The operator $A : W_0^{s,p(x,y)}(\Omega) \to \left(W_0^{s,p(x,y)}(\Omega)\right)^*$ defined by*

$$\langle Au, v\rangle = \int_\Omega (|u(x)|^{q(x)-2}u(x) + \lambda H(x,u))v dx, \quad \forall u, v \in W_0$$

is compact.

Proof. The proof is broken down into three sections.

Step 1. Let $\phi : W_0 \to L^{q'(x)}(\Omega)$ be the operator defined by

$$\phi\, u(x) := -|u(x)|^{q(x)-2}u(x) \quad \text{for} \quad u \in W_0 \quad \text{and} \quad x \in \Omega.$$

It is obvious that ϕ is continuous. Next we show that ϕ is bounded. For every $u \in W_0$, we have by the inequalities (2.1) and (2.2) that

$$\|\phi u\|_{q'(x)} \leq \rho_{q'(\cdot)}(\phi u) + 1 = \int_\Omega \left||u|^{q(x)-1}\right|^{q'(x)} dx + 1 = \rho_{q(\cdot)}(u) \leq \|u\|_{q(x)}^{q^-} + \|u\|_{q(x)}^{q^+} + 1.$$

By the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ we have

$$\|\phi u\|_{q'(x)} \leq \text{const}\left(\|u\|_{W_0}^{q^-} + \|u\|_{W_0}^{q^+}\right) + 1.$$

This implies that ϕ is bounded on W_0.

Step 2. We show that the operator ψ defined from W_0 into $L^{p'(x)}(\Omega)$ by

$$\psi u(x) := -\lambda H(x,u) \quad \text{for} \quad u \in W_0 \quad \text{and} \quad x \in \Omega$$

is bounded and continuous. Let $u \in W_0$, by using the growth condition (H_0) we obtain

$$\begin{aligned}
\|\psi\, u\|_{p'(x)}^{p'(x)} &\leq \int_\Omega |\lambda H(x,u)|^{p'(x)}dx \\
&\leq (\varrho\lambda)^{p'(x)} \int_\Omega \big(|e(x)|^{p'(x)} + |u|^{(q(x)-1)p'(x)}\big)dx \\
&\leq (\varrho\lambda)^{p'(x)} \int_\Omega \big(|e(x)|^{p'(x)} + |u|^{(p(x)-1)p'(x)}\big)dx \\
&\leq (\varrho\lambda)^{p'(x)} \int_\Omega |e(x)|^{p'(x)}dx + (\varrho\lambda)^{p'(x)} \int_\Omega |u|^{p(x)}\, dx \\
&\leq (\varrho\lambda)^{p'(x)}(\|e\|_{p'(x)}^{p'+} + \|e\|_{p'(x)}^{p'-}) + (\varrho\lambda)^{p'(x)}(\|u\|_{p(x)}^{p+} + \|u\|_{p(x)}^{p-}) \\
&\leq C_m(\|u\|_{W_0}^{p+} + \|u\|_{W_0}^{p-} + 1),
\end{aligned} \tag{3.6}$$

where $C_m = \max\big((\varrho\lambda)^{p'(x)}(\|e\|_{p'(x)}^{p'+} + \|e\|_{p'(x)}^{p'-}), (\varrho\lambda)^{p'(x)}\big)$. (Due to $e(x)$ is a positive function in $L^{p'(x)}(\Omega)$).

Therefore ψ is bounded on $W^{s,q(x),p(x,y)}(\Omega)$.

Next, we show that ψ is continuous, let $u_n \to u$ in $W^{s,q(x),p(x,y)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Thus there exists a subsequence still denoted by (u_n) and measurable function φ in $L^{p(x)}(\Omega)$ such that

$$u_n(x) \to u(x),$$
$$|u_n(x)| \leq \varphi(x),$$

for a.e. $x \in \Omega$ and all $n \in \mathbb{N}$. Since H satisfies the Carathéodory condition, we obtain

$$H(x,u_n(x)) \to H(x,u(x)) \quad \text{a.e. } x \in \Omega. \tag{3.7}$$

Thanks to (H_0) we obtain

$$|H(x,u_n(x))| \leq \varrho\big(e(x) + |\varphi(x)|^{q(x)-1}\big)$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Since

$$e(x) + |\varphi(x)|^{p(x)-1} \in L^{p'(x)}(\Omega),$$

and from (3.7), we get

$$\int_\Omega |H(x,u_k(x)) - H(x,u(x))|^{p'(x)}dx \longrightarrow 0,$$

by using the dominated convergence theorem we have

$$\psi u_k \to \psi u \quad \text{in} \quad L^{p'(x)}(\Omega).$$

Thus the entire sequence (ψu_n) converges to ψu in $L^{p'(x)}(\Omega)$ and then ψ is continuous.

Step 3. Since the embedding $I : W_0 \to L^{q(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{q'(x)}(\Omega) \to W_0^*$ is also compact. Therefore, the compositions $I^* \circ \phi$ and $I^* \circ \psi : W_0 \to W_0^*$ are compact. We conclude that $S = I^* \circ \phi + I^* \circ \psi$ is compact. $\square$

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with smooth boundary. If the hypotheses (H_1) and (H_2) hold, then the set-valued operator $\mathcal{N}$ defined above is bounded, upper semicontinuous (u.s.c.) and compact.*

Proof. Let $\Lambda : L^{p(x)}(\Omega) \to 2^{L^{p'(x)}(\Omega)}$ be a set-valued operator defined as follows

$$\Lambda u = \big\{ h \in L^{p'(x)}(\Omega) | \quad \underline{\psi}(x, u(x)) \leq h(x) \leq \overline{\psi}(x, u(x)) \quad \text{a. e.} \quad x \in \Omega \big\}.$$

Let $u \in W_0$, by the assumption (H_2) we obtain

$$\max \big\{ \ |\underline{\psi}(x, s)| \, ; \ |\overline{\psi}(x, s)| \ \big\} \leq b(x) + c(x)|s|^{\gamma(x)-1}.$$

for all $(x, t) \in \Omega \times \mathbb{R}$ where $1 < \gamma(x) < p(x)$ for all $x \in \overline{\mathbb{R}}$.

As a result

$$\int_\Omega |\, \overline{\psi}(x, u(x))|^{p'(x)} dx \leq 2^{p'^{+}+1} \Big(\int_\Omega |b(x)|^{p'(x)} dx + \int_\Omega |c|^{p'(x)} |u(x)|^{p(x)} dx \Big).$$

A same inequality is shown for $\underline{\psi}(x, s)$, it follows that the set-valued operator Λ is bounded on $W_0(\Omega)$. It remains to prove that Λ is upper semi-continuous (u.s.c.), *i. e.*,

$$\forall \varepsilon > 0, \ \exists \delta > 0, \quad \|u - u_0\|_p < \delta \ \Rightarrow \ \Lambda u \subset \Lambda u_0 + B_\varepsilon,$$

where B_ε is the ε-ball in $L^{p'(x)}(\Omega)$.

To come to an end, given $u_0 \in L^{p(x)}(\Omega)$, let us consider the sets

$$G_{m,\varepsilon} = \bigcap_{t \in \mathbb{R}^N} K_t,$$

where

$$K_t = \Big\{ x \in \Omega, \ \text{if } |t - u_0(x)| < \frac{1}{m}, \ \text{then } [\underline{\psi}(x, t), \overline{\psi}(x, t)] \subset \Big] \underline{\psi}(x, u_0(x)) - \frac{\varepsilon}{R}, \overline{\psi}(x, u_0(x)) + \frac{\varepsilon}{R} \Big[\Big\},$$

m being an integer, $|t| = \max\limits_{1 \leq i \leq N} |t_i|$ and R is a constant to be determined in the following pages. In view of Proposition 3.2, we define the sets of points as follows

$$G_{m,\varepsilon} = \bigcap_{r \in \mathbb{R}_a^N} K_r,$$

where $\mathbb{R}_a^N$ denotes the set of all rational grids in $\mathbb{R}^N$. For any $r = (r_1, \ldots, r_N) \in \mathbb{R}_a^N$,

$$K_r = \Big\{ x \in \Omega \mid u_0(x) \in C \prod_{i=1}^N \Big] r_i - \frac{1}{m}, r_i + \frac{1}{m} \Big[\Big\} \cup \Big\{ x \in \Omega \mid u_0(x) \in \prod_{i=1}^N \Big] r_i - \frac{1}{m}, r_i + \frac{1}{m} \Big[\Big\}$$
$$\cap \Big\{ x \in \Omega \mid \overline{\psi}(x, r) < \overline{\psi}(x, u_0(x)) + \frac{\varepsilon}{R} \text{ and } \underline{\psi}(x, r) > \underline{\psi}(x, u_0(x)) - \frac{\varepsilon}{R} \Big\},$$

so that K_r and therefore $G_{m,\varepsilon}$ are measurable. It is obvious that

$$G_{1,\varepsilon} \subset G_{2,\varepsilon} \subset \cdots$$

In light of Proposition 3.2, we have

$$\bigcup_{m=1}^{\infty} G_{m,\varepsilon} = \Omega,$$

therefore there exists $m_0 \in \mathbb{N}$ such that

$$m(G_{m_0,\varepsilon}) > m(\Omega) - \frac{\varepsilon}{R}. \tag{3.8}$$

But for each $\varepsilon > 0$, there is $\eta = \eta(\varepsilon) > 0$, such that $m(T) < \eta$ yields

$$2^{p'^+-1} \int_T 2|b(x)|^{p'(x)} + c^{p'(x)}(x)(2^{p'^+-1}+1)|u_0(x)|^{p(x)}dx < \left(\frac{\varepsilon}{3}\right)^{p'^+}, \tag{3.9}$$

because of $b \in L^{p'(x)}(\Omega)$ and $u_0 \in L^{p(x)}(\Omega)$.
Let now

$$0 < \delta < \min\left\{\frac{1}{m_0}\left(\frac{\eta}{2}\right)^{\frac{1}{p^-}}, \frac{1}{2^{p^+-2}}\left(\frac{\varepsilon}{6C}\right)^{\frac{p'^+}{\theta}}\right\}, \tag{3.10}$$

$$R > \max\left\{\frac{2\varepsilon}{\eta}, 3\left(m(\Omega)\right)^{\frac{1}{p'^-}}\right\}, \tag{3.11}$$

where

$$\theta = \begin{cases} p^+ & \text{if} \quad \|u-u_0\|_{p(x)} \leq 1 \\ p^- & \text{if} \quad \|u-u_0\|_{p(x)} \geq 1. \end{cases}$$

Suppose that $\|u-u_0\|_{p(x)} < \delta$ and define the set $G = \left\{x \in \Omega \setminus |u(x)-u_0(x)| \geq \dfrac{1}{m_0}\right\}$, we get

$$m(G) < (m_0\delta)^{p(x)} < \frac{\eta}{2}. \tag{3.12}$$

If $x \in G_{m_0,\varepsilon} \backslash G$, then, for any $h \in \Lambda u$,

$$|u(x)-u_0(x)| < \frac{1}{m_0}$$

and

$$h(x) \in \left]\underline{\psi}(x,u_0(x)) - \frac{\varepsilon}{R}, \overline{\psi}(x,u_0(x)) + \frac{\varepsilon}{R}\right[.$$

Let

$$\begin{aligned} K^0 &= \Big\{x \in \Omega; \quad h(x) \in \left[\underline{\psi}(x,u_0(x)), \overline{\psi}(x,u_0(x))\right]\Big\}, \\ K^- &= \Big\{x \in \Omega; \quad h(x) < \underline{\psi}(x,u_0(x))\Big\}, \\ K^+ &= \Big\{x \in \Omega; \quad h(x) > \overline{\psi}(x,u_0(x))\Big\}, \end{aligned}$$

and

$$w(x) = \begin{cases} \overline{\psi}(x, u_0(x)), & \text{for} \quad x \in K^+; \\ h(x) \ , & \text{for} \quad x \in K^0; \\ \underline{\psi}(x, u_0(x)), & \text{for} \quad x \in K^-. \end{cases}$$

Hence $w \in \Lambda u_0$ and

$$|w(x) - h(x)| < \frac{\varepsilon}{R} \quad \text{for all} \quad x \in G_{m_0,\varepsilon} \setminus G. \tag{3.13}$$

From (3.11) and (3.13), we have

$$\int_{G_{m_0,\varepsilon} \setminus G} |w(x) - h(x)|^{p'(x)} dx < \Big(\frac{\varepsilon}{R}\Big)^{p'^+} m(\Omega) < \Big(\frac{\varepsilon}{3}\Big)^{p'^+}. \tag{3.14}$$

Assume that V is a coset in Ω of $G_{m_0,\varepsilon} \setminus G$, then $V = (\Omega \setminus G_{m_0,\varepsilon}) \cup (G_{m_0,\varepsilon} \cap G)$ and

$$m(V) \leq m(\Omega \setminus G_{m_0,\varepsilon}) + m(G_{m_0,\varepsilon} \cap G) < \frac{\varepsilon}{R} + m(G) < \eta.$$

According to (3.8), (3.11) and (3.12). From (H_2), (3.9) and (3.10), we obtain

$$\begin{aligned} \int_V |w(x) - h(x)|^{p'(x)} dx &\leq \int_V |w(x)|^{p'(x)} + |h(x)|^{p'(x)} dx \\ &\leq 2^{p'^+ - 1} \left(\int_V |b(x)|^{p'(x)} + c^{p'(x)}(x) |u_0(x)|^{p(x)} + |b(x)|^{p'(x)} + c^{p'}(x) |u(x)|^{p(x)} dx \right) \\ &\leq 2^{p'^+ - 1} \Big(\int_V 2|b(x)|^{p'(x)} + c^{p'(x)}(x)(2^{p^+ - 1} + 1)|u_0(x)|^{p(x)} dx \Big) \\ &\quad + 2^{p'^+ - 1} \Big(\int_V 2^{p^+ - 1} c^{p'(x)}(x) |u(x) - u_0(x)|^{p(x)} dx \Big) \\ &\leq 2^{p'^+ - 1} \int_V 2|b(x)|^{p'(x)} + c^{p'(x)}(x)(2^{p^+ - 1} + 1)|u_0(x)|^{p(x)} dx \\ &\quad + 2^{p^+ + p'^+ - 2} \, \|c^{p'^+}\|_{L^\infty(\Omega)} \int_V |u(x) - u_0(x)|^{p(x)} dx \\ &\leq \Big(\frac{\varepsilon}{3}\Big)^{p'^+} + 2^{p^+ + p'^+ - 2} \, \|c^{p'^+}\|_{L^\infty(\Omega)} \delta^\theta \leq 2\Big(\frac{\varepsilon}{3}\Big)^{p'^+} \leq \varepsilon^{p'^+}. \end{aligned} \tag{3.15}$$

Thanks to (3.14), (3.15) and (2.1), we get $\|w - h\|_{p'(x)} \leq \int_\Omega |w(x) - h(x)|^{p'(x)} dx + 1 < \varepsilon$.

Hence Λ is upper semicontinuous (u.s.c.). Hence $\mathcal{N} = I^* \circ \Lambda \circ I$ is clearly bounded, upper semicontinuous (u.s.c.) and compact. ☐

Next, we give the proof of Theorem 1.2. Let $S := A + \mathcal{N} : W_0^{s,p(x,y)}(\Omega) \to 2^{\left(W_0^{s,p(x,y)}(\Omega)\right)^*}$, where A and $\mathcal{N}$ were defined in Lemma 3.3 and in section 2 respectively. This means that a point $u \in W_0^{s,p(x,y)}(\Omega)$ is a weak solution of (1.1) if and only if

$$Fu \in -Su, \tag{3.16}$$

with F defined in (1.5). By the properties of the operator F given in Lemma 3.1 and the Minty-Browder's Theorem on monotone operators in [45, Theorem 26 A], we guarantee that the inverse operator $T := F^{-1} : \big(W_0^{s,p(x,y)}(\Omega)\big)^* \to W_0^{s,p(x,y)}(\Omega)$ is continuous, of type (S_+) and bounded. Moreover, thanks to Lemma 3.3 the operator S is quasi-monotone, upper semicontinuous (u.s.c.) and bounded. As a result, the equation (3.16) is equivalent to the abstract Hammerstein equation

$$u = Tv \quad \text{and} \quad v \in -S \circ Tv. \tag{3.17}$$

We will apply the theory of degrees introduced in section 3 to solve the equations (3.17). For this, we first show the following Lemma.

Lemma 3.5. *The set*

$$B := \Big\{ v \in \big(W_0\big)^* \ \textit{such that}\ v \in -tS \circ Tv \quad \textit{for some} \quad t \in [0,1] \Big\}$$

is bounded.

Proof. Let $v \in B$, so, $v + ta = 0$ for every $t \in [0,1]$, with $a \in S \circ Tv$. Setting $u := Tv$, we can write $a = Au + \varphi \in Su$, where $\varphi \in \mathcal{N}u$, namely,

$$\langle \varphi, u \rangle = \int_\Omega h(x)u(x)dx,$$

for each $h \in L^{p'(x)}(\Omega)$ with $\underline{\psi}(x, u(x)) \le h(x) \le \overline{\psi}(x, u(x))$ for almost all $x \in \Omega$.

If $\|u\|_{W_0} \le 1$, then $\|Tv\|_{W_0}$ is bounded.

If $\|u\|_{W_0} > 1$, then we get by the implication (i) in Proposition 2.1 and the inequality (2.2) and using (H_0), the Young inequality, the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$, the estimate

$$\begin{aligned}
\|Tv\|_{W_0}^{p^-} &= \|u\|_{W_0}^{p^-} \\
&\le \rho_{p(\cdot,\cdot)}(u) \\
&\le t|\langle a, Tv\rangle| \\
&\le t\int_\Omega |u|^{q(x)}\,dx + t\int_\Omega \lambda|H(x,u)|u dx + t\int_\Omega |hu| dx \\
&\le t\int_\Omega |u|^{q(x)} + tC_{p'}\int_\Omega |\lambda H(x,u)|^{q'(x)}dx + tC_p\int_\Omega |u|^{q(x)}\,dx \\
&\qquad + C_\gamma t\left(\int_\Omega |u|^{\gamma(x)}dx\right) + C_{\gamma'} t\left(\int_\Omega |h|^{\gamma'(x)}dx\right) \\
&\le \text{Const}\Big(\|u\|_{q(x)}^{q^-} + \|u\|_{q(x)}^{q^+} + \|u\|_{\gamma(x)}^{\gamma^-} + \|u\|_{\gamma(x)}^{\gamma+} + 1\Big) \\
&\le \text{Const}\Big(\|u\|_{W_0}^{q^-} + \|u\|_{W_0}^{q^+} + \|u\|_{W_0}^{\gamma^-} + \|u\|_{W_0}^{\gamma+} + 1\Big) \\
&\le \text{Const}\Big(\|Tv\|_{W_0}^{q^+} + \|Tv\|_{W_0}^{\gamma^+} + 1\Big).
\end{aligned}$$

Hence it is obvious that $\{ Tv \mid v \in B \}$ is bounded.

As the operator S is bounded and from (3.17), we deduce the set B is bounded in $\big(W_0\big)^*$. ◻

Thanks to Lemma 3.5, we can find a positive constant R such that

$$\|v\|_{(W_0)^*} < R \quad \text{for any} \quad v \in B.$$

This says that

$$v \in -tS \circ Tv \quad \text{for each} \quad v \in \partial B_R(0) \quad \text{and each} \quad t \in [0,1].$$

Under the Lemma 2.8, we get

$$I + S \circ T \in \mathcal{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = F \circ T \in \mathcal{F}_T(\overline{B_R(0)}).$$

Now, we are in a position to consider the affine homotopy $H : [0,1] \times \overline{B_R(0)} \to 2^{(W_0)^*}$ defined by

$$H(t,v) := (1-t)Iv + t(I + S \circ T)v \quad \text{for} \quad (t,v) \in [0,1] \times \overline{B_R(0)}.$$

By applying the normalization and homotopy invariance property of the degree d fixed in Theorem 2.11, we have

$$d(I + S \circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1.$$

It follows that, we can get a function $v \in B_R(0)$ such that

$$v \in -S \circ Tv.$$

Which implies that $u = Tv$ is a weak solution of (1.1). This completes the proof.

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