

Smooth quotients of abelian surfaces by finite groups that fix the origin

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ABSTRACT

Let A be an abelian surface and let G be a finite group of automorphisms of A fixing the origin. Assume that the analytic representation of G is irreducible. We give a classification of the pairs (A, G) such that the quotient A/G is smooth. In particular, we prove that $A = E^2$ with E an elliptic curve and that $A/G \simeq \mathbb{P}^2$ in all cases. Moreover, for fixed E , there are only finitely many pairs (E^2, G) up to isomorphism. This fills a small gap in the literature and completes the classification of smooth quotients of abelian varieties by finite groups fixing the origin started by the first two authors.

RESUMEN

Sea A una superficie abeliana y sea G un grupo finito de automorfismos de A fijando el origen. Se asume que la representación analítica de G es irreducible. Damos una clasificación de los pares (A, G) tales que el cociente A/G es suave. En particular, probamos que $A = E^2$ con E una curva elíptica y que $A/G \simeq \mathbb{P}^2$ en todos los casos. Más aún, para E fija, hay solo una cantidad finita de pares (E^2, G) , salvo isomorfismo. Esto llena una pequeña brecha en la literatura y completa la clasificación de cocientes suaves de variedades abelianas por grupos finitos fijando el origen comenzado por los dos primeros autores.

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1 Introduction

The purpose of this paper is to give a complete classification of all smooth quotients of abelian surfaces by finite groups that fix the origin, and is to be seen as the completion of the classification given in [2] of smooth quotients of abelian varieties that fix the origin. This kind of quotients of abelian surfaces has already been studied by Tokunaga and Yoshida in [6], where infinite 2-dimensional complex reflection groups, which are extensions of a finite complex reflection group G by a G -invariant lattice, are classified. However, these do not cover all possible G -invariant lattices and hence not all possible group actions on abelian surfaces. Moreover, there seem to be some complex reflection groups that the authors missed, as can be seen by looking at Popov's classification of the same groups in [3].

The techniques used in this paper are similar, but not exactly the same, to the methods used in [2]. Indeed, the ideas used in this last paper have been modified in order to apply them to the two-dimensional case. Moreover our approach is far different from that used in [6].

Our main theorem states the following:

Theorem 1.1. *Let A be an abelian surface and let G be a (non-trivial) finite group of automorphisms of A that fix the origin. Then the following conditions are equivalent:*

- (1) A/G is smooth and the analytic representation of G is irreducible.
- (2) $A/G \simeq \mathbb{P}^2$.
- (3) *There exists an elliptic curve E such that $A \simeq E^2$ and (A, G) satisfies exactly one of the following:*
 - (a) $G \simeq C^2 \rtimes S_2$ where C is a non-trivial (cyclic) subgroup of automorphisms of E that fix the origin; here the action of C^2 is coordinate-wise and S_2 permutes the coordinates.
 - (b) $G \simeq S_3$ and acts on

$$A \simeq \{(x_1, x_2, x_3) \in E^3 : x_1 + x_2 + x_3 = 0\},$$

by permutations.

- (c) $E = \mathbb{C}/\mathbb{Z}[i]$ and G is the order 16 subgroup of $\mathrm{GL}_2(\mathbb{Z}[i])$ generated by:

$$\left\{ \begin{pmatrix} -1 & 1+i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & i-1 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ i-1 & 1 \end{pmatrix} \right\},$$

acting on $A \simeq E^2$ in the obvious way.

The first two cases found in item (3) of the above theorem were studied in detail in [1] (in arbitrary dimension), where it was proven that both examples give the projective plane as a quotient.

Throughout the paper we will refer to these two examples as Example (a) and Example (b), respectively. The equivalent assertion for Example (c) is Proposition 4.1 in this paper. Note that, aside from Examples (a) and (b) which belong to infinite families, Example (c) is the only new case of an action of G on an abelian variety satisfying condition (1) from Theorem 1.1, cf. [2, Thm. 1.1].

Remark 1.2. *If A/G is smooth and the analytic representation of G is reducible, then the results in [2] imply that A is isogenous to a product of two elliptic curves. The quotient is then either $\mathbb{P}^1 \times \mathbb{P}^1$ (in which case $A = E_1 \times E_2$) or a bielliptic surface.*

In [7], Yoshihara introduces the notion of a Galois embedding of a smooth projective variety. If X is a smooth projective variety of dimension n and D is a very ample divisor that induces an embedding $X \hookrightarrow \mathbb{P}^N$, then the embedding is said to be *Galois* if there exists an $(N - n - 1)$ -dimensional linear subspace W of \mathbb{P}^N such that $X \cap W = \emptyset$ and the restriction of the linear projection $\pi_W : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ to X is Galois. Yoshihara specifically studies when abelian surfaces have a Galois embedding. He gives a classification of abelian surfaces having a Galois embedding, along with their Galois groups, and proves that after taking the quotient of the original abelian variety by the translations of the Galois group, the abelian variety must be isomorphic to the self-product of an elliptic curve. Unfortunately, his results were incomplete since they depended on a classification of smooth quotients like the one given in this paper, which Yoshihara attributed to Tokunaga and Yoshida in [6]. But as stated before, Tokunaga and Yoshida's results do not imply such a classification. Nevertheless, we can now safely say, thanks to Theorem 1.1, that Yoshihara's results remain correct.

The structure of this paper is as follows: in Section 2 we fix notations and give some preliminary results that will be needed in the proofs of Theorem 1.1. The implication (2) \Rightarrow (1) is obvious and (3) \Rightarrow (2) was already treated in [1] in the case of Examples (a) and (b). Thus, we are mainly concerned with (1) \Rightarrow (3), which we treat in Section 3. Finally, in Section 4 we treat (3) \Rightarrow (2) for Example (c), which is a construction of a different nature that only exists in the 2-dimensional case.

2 Preliminaries on group actions on abelian varieties

We recall here some elementary results that were proved in [2]. Let A be an abelian surface and let G be a group of automorphisms of A that fix the origin, such that the quotient variety A/G is smooth. By the Chevalley-Shephard-Todd Theorem, the stabilizer in G of each point in A must be generated by pseudoreflections; that is, elements that fix pointwise a divisor (i.e. a curve) containing the point. In particular, $G = \text{Stab}_G(0)$ is generated by pseudoreflections and G acts on the tangent space at the origin $T_0(A)$ (this is the analytic representation). In this context, a pseudoreflection is an element that fixes a line pointwise. We will often abuse notation and display

G as either acting on A or $T_0(A)$; it will be clear from the context which action we are considering. In what follows, let \mathcal{L} be a fixed G -invariant polarization on A (take the pullback of an ample class on A/G , for example). For σ a pseudoreflection in G of order r , define

$$D_\sigma := \text{im}(1 + \sigma + \cdots + \sigma^{r-1}), \quad E_\sigma := \text{im}(1 - \sigma).$$

These are both abelian subvarieties of A . The following result corresponds to [2, Lem. 2.1].

Lemma 2.1. *We have the following:*

1. D_σ is the connected component of $\ker(1 - \sigma)$ that contains 0 and E_σ is the complementary abelian subvariety of D_σ with respect to \mathcal{L} . In particular, D_σ and E_σ are elliptic curves.
2. $F_\sigma := D_\sigma \cap E_\sigma$ consists of 2-torsion points for $r = 2, 4$, of 3-torsion points for $r = 3$ and $D_\sigma \cap E_\sigma = 0$ for $r = 6$.

We will consider now a new abelian surface B equipped with a G -equivariant isogeny to A , which we will call G -isogeny from now on. Let Λ_A denote the lattice in \mathbb{C}^2 such that $A = \mathbb{C}^2/\Lambda_A$. Let $\Lambda_B \subset \Lambda_A$ be a G -invariant sublattice, and let $B := \mathbb{C}^2/\Lambda_B$ be the induced abelian surface, along with the G -isogeny

$$\pi : B \rightarrow A,$$

whose analytic representation is the identity. Note that this implies that $\sigma \in G$ is a pseudoreflection of B if and only if it is a pseudoreflection of A . We may then consider the subvarieties $E_\sigma, D_\sigma, F_\sigma \subset A$ defined as above, which we will denote by $E_{\sigma,A}, D_{\sigma,A}$ and $F_{\sigma,A}$. We do similarly for B . Note that, by definition, π sends $E_{\sigma,B}$ to $E_{\sigma,A}$ and $D_{\sigma,B}$ to $D_{\sigma,A}$, hence $F_{\sigma,B}$ to $F_{\sigma,A}$. The following result was proved in [2, Prop. 2.4].

Proposition 2.2. *Let $\sigma \in G$ be a pseudoreflection and let L be the line defining both $E_{\sigma,A}$ and $E_{\sigma,B}$. Assume that the map $F_{\sigma,B} \rightarrow F_{\sigma,A}$ is surjective and that $\Lambda_A \cap L = \Lambda_B \cap L$. Then $\ker(\pi)$ is contained in $D_{\tau\sigma\tau^{-1},B}$ for every $\tau \in G$. \square*

Define $\Delta := \ker(\pi)$. Since π is G -equivariant, G acts on Δ and hence we may consider the group $\Delta \rtimes G$. This group acts on B in the obvious way: Δ acts by translations and G by automorphisms that fix the origin. In particular, we see that the quotient $B/(\Delta \rtimes G)$ is isomorphic to A/G . We conclude this section by recalling a result on pseudoreflections in $\Delta \rtimes G$ (cf. [2, Lem. 2.5]).

Lemma 2.3. *Let $\sigma \in \Delta \rtimes G$ be a pseudoreflection. Then $\sigma = (t, \tau)$ with $\tau \in G$ a pseudoreflection and $t \in \Delta \cap E_{\tau,B}$.*

3 Proof of (1) \Rightarrow (3)

Assume (1), that is, we have an abelian surface A with an action of a finite group G that fixes the origin and such that A/G is smooth and the analytic representation of G is irreducible. Under these conditions, we see that G is an irreducible finite complex reflection group in the sense of Shephard-Todd [4]. These groups were completely classified by Shephard and Todd in [4]. In the particular case of dimension 2, we get that G is either one of 19 sporadic cases or it is isomorphic to a semidirect product $G(m, p) := H(m, p) \rtimes S_2$, where $p|m$, $m \geq 2$, and

$$H = H(m, p) = \{(\zeta_m^{a_1}, \zeta_m^{a_2}) \mid a_1 + a_2 \equiv 0 \pmod{p}\} \subset \mu_m^2,$$

with ζ_m denoting a primitive m -th root of unity. The action of S_2 on H is the obvious one. The case $G = G(2, 2)$ is excluded since G is then a Klein group and thus the representation is not irreducible. The action of G on \mathbb{C}^2 is given as follows: H acts on \mathbb{C}^2 coordinate-wise while S_2 permutes the coordinates.

Emulating the work done in [2], we wish to describe which of these actions actually appear on abelian surfaces and give smooth quotients. The sporadic cases were already treated in [2] and were proven not to give a smooth quotient (cf. [2, §3.3]), so we may and will assume henceforth that $G = G(m, p)$ as above. This fixes a G -equivariant isomorphism of $T_0(A)$ with \mathbb{C}^2 . We denote by e_1 and e_2 the canonical basis of $T_0(A)$ thus obtained.

Lemma 3.1. *Assume that G acts on A as above. Then $m \in \{2, 3, 4, 6\}$.*

Proof. We have that (ζ_m, ζ_m^{-1}) acts on A , and so the characteristic polynomial of $(\zeta_m, \zeta_m^{-1}) \oplus \overline{(\zeta_m, \zeta_m^{-1})}$ must have integer coefficients, and so must be the k -th power of the m -th cyclotomic polynomial Φ_m , where $k = 2$ if $m \geq 2$ and $k = 4$ if $m = 2$. Looking at the degrees, we get that $4 = k\varphi(m)$, where φ is Euler's totient function. Therefore, if $m \neq 2$ then $\varphi(m) = 2$ and so $m \in \{3, 4, 6\}$. □

Having proved this result, we see that there is a finite list of cases to be analyzed, that is:

$$(m, p) \in \{(2, 1), (2, 2), (3, 1), (4, 1), (6, 1), (3, 3), (4, 2), (4, 4), (6, 2), (6, 3), (6, 6)\}.$$

Recall that we have already eliminated the case $(2, 2)$ since the analytic representation of $G(2, 2)$ is not irreducible. Moreover, it is well-known that there is an exceptional isomorphism of complex reflection groups between $G(4, 4)$ and $G(2, 1)$. We will prove then the following:

- If $G = G(m, 1)$ and A/G is smooth, then the pair (A, G) corresponds to Example (a) (Sections 3.1, 3.3, 3.4, 3.5);
- If $G = G(3, 3)$ and A/G is smooth, then the pair (A, G) corresponds to Example (b) (Section 3.8);

- If $G = G(4, 2)$ and A/G is smooth, then the pair (A, G) corresponds to Example (c) (Section 3.2);
- If $G = G(6, p)$ with $p \geq 2$, then A/G cannot be smooth (Sections 3.6, 3.7, 3.9).

In order to do this, we will construct a G -isogeny $B \rightarrow A$ such that the action of G on B is “well-known”. Let us concentrate first on the cases where $m \neq p$. Then we obtain B as follows:

Let E_i be the image of $\mathbb{C}e_i$ in A via the exponential map. We claim that it corresponds to an elliptic curve. Indeed, consider the non-trivial element $\tau = (\zeta_m^p, 1) \in H$. Then a direct computation shows that $\text{im}(1 - \tau) = \mathbb{C}e_1$. This tells us that $E_1 = (1 - \tau)(A)$ and hence it corresponds to an elliptic curve. The same proof works for E_2 .

Now, let Λ_A be a lattice for A in \mathbb{C}^2 . Then $\mathbb{C}e_i \cap \Lambda_A$ corresponds to the lattice of E_i in $\mathbb{C} = \mathbb{C}e_i$. We can thus define the G -stable sublattice of Λ_A

$$\Lambda_B := (\mathbb{C}e_1 \cap \Lambda_A) \oplus (\mathbb{C}e_2 \cap \Lambda_A).$$

As in Section 2, this defines a G -isogeny $\pi : B \rightarrow A$. Moreover, we see that $B \simeq E_1 \times E_2 \simeq E^2$ and that $\pi|_{E_i}$ is an injection. Let Δ be the kernel of π . We will study the different possible quotients A/G by studying the possible quotients $B/(\Delta \rtimes G)$ and thus by studying the possible Δ 's. Our first result is the following:

Lemma 3.2. *Assume that $m \neq p$. Then the coordinates of every element in Δ are invariant by ζ_m^p , so in particular these elements are*

- 2-torsion if $(m, p) \in \{(2, 1), (4, 1), (4, 2), (6, 3)\}$;
- 3-torsion if $(m, p) \in \{(3, 1), (6, 2)\}$;
- trivial if $(m, p) = (6, 1)$.

Proof. Let $\bar{t} = (t_1, t_2) \in \Delta$. Then, since Δ is G -stable, we have that, for $\tau_1 = (\zeta_m^p, 1) \in H$,

$$(1 - \tau_1)(\bar{t}) = ((1 - \zeta_m^p)t_1, 0) \in \Delta.$$

But, by construction, there are no elements of the form $(x, 0)$ in Δ . We deduce then that t_1 is ζ_m^p -invariant. The same proof works for t_2 . The assertion on the torsion of t_1 and t_2 follows immediately. \square

Let us study now pseudoreflections in $\Delta \rtimes G$. Define the elements

$$\rho := (\zeta_m, \zeta_m^{-1}) \in H \subset G; \quad \sigma := (12) \in S_2 \subset G; \quad \tau := (\zeta_m^p, 1) \in H \subset G.$$

Then there are two types of pseudoreflections in G :

- conjugates of $\rho^a \sigma$ for $0 \leq a < \frac{m}{p}$;
- conjugates of powers of τ ;

and the corresponding elliptic curves in B are respectively:

$$E_{\rho^a \sigma} = \{(x, -\zeta_m^a x) \mid x \in E\}; \quad E_\tau = \{(x, 0) \mid x \in E\}.$$

Recall that elements of the form $(x, 0)$ are not in Δ by construction of the isogeny $\pi : B \rightarrow A$. Using Lemmas 2.3 and 3.2, we obtain immediately the following result:

Lemma 3.3. *Every pseudoreflection in $\Delta \rtimes G$ that is not in G is a conjugate of $(\bar{t}, \rho^a \sigma)$, where $0 \leq a < \frac{m}{p}$, $\bar{t} = (t, -\zeta_m^a t) \in \Delta$ and t is ζ_m^p -invariant. \square*

With these considerations, we can start a case by case study of the non-trivial Δ 's. We recall that the main tool will be the Chevalley-Shephard-Todd Theorem, which states that $A/G = B/(\Delta \rtimes G)$ is smooth if and only if the stabilizer in $\Delta \rtimes G$ of each point in B is generated by pseudoreflections.

3.1 The case $G = G(2, 1)$

By Lemma 3.2, we know that Δ is 2-torsion. Since we also know that there are no elements of the form $(t, 0)$ for $t \in E$, we get the following possible options for Δ :

- (1) $\Delta = \{0\}$;
- (2) $\Delta = \langle (t, t) \rangle$ with $t \in E[2]$;
- (3) $\Delta = \{(t, t) \mid t \in E[2]\}$;
- (4) $\Delta = \{(0, 0), (t_1, t_2), (t_2, t_1), (t_1 + t_2, t_1 + t_2)\}$ with $t_1, t_2 \in E[2]$, $t_1 \neq t_2$.

Case (1) clearly corresponds to Example (a) (which gives a smooth quotient, cf. [2, Prop. 3.4]).

Case (2) cannot give a smooth quotient and this follows directly from [2, Prop. 3.7].¹

In case (3), we claim that the pair (A, G) is isomorphic to the pair (B, G) . This will reduce us to the case with trivial Δ , which was already dealt with. To prove the claim, consider the canonical basis of $T_0(A) = T_0(B) = \mathbb{C}^2$. Then the analytic representation of G is given by the following values in its generators:

$$\rho_a((1, -1)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_a((1\ 2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

¹The proof of this proposition only uses two variables and thus it works in dimension 2 as well.

Now, with this basis and this Δ , we can view the G -isogeny $B \rightarrow A$ as the morphism $E^2 \rightarrow E^2$ given by the following matrix:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (*)$$

for which one can check that its kernel is precisely the elements in Δ . In order to prove that the pairs (A, G) and (B, G) are isomorphic, it suffices thus to prove that the image of this representation of G under conjugation by M is G once again. Direct computations give:

$$M\rho_a((1, -1))M^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho_a((12)), \quad M\rho_a((12))M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho_a((1, -1)).$$

And these clearly generate the same group G .

In case (4), consider the element $\bar{t} = (t'_1, t'_2)$ where $2t'_i = t_i$. Note that G cannot fix \bar{t} as t'_1 and t'_2 lie in different orbits by the action of μ_2 . Now, it is easy to see that there is no way the action of Δ can compensate the action of G except in the case when we add the element (t_1, t_2) . A direct computation tells us then that the only element fixing \bar{t} is $((t_1, t_2), (-1, -1)) \in \Delta \times G$ and since this stabilizer is not generated by pseudoreflections by Lemma 3.3, we see that A/G is not smooth.

3.2 The case $G = G(4, 2)$

Since $G(4, 2)$ contains $G(2, 1)$, we may start from the precedent list of possible non-trivial Δ 's. However, these must also be stable by the new element $(i, i) \in H(4, 2)$ (where $i = \zeta_4$). Note that such an element acts on each component E of B by multiplication by i , which implies in particular that $E = \mathbb{C}/\mathbb{Z}[i]$. Defining by t_0 the only non-trivial i -invariant element in E , we get the following possibilities:

- (1) $\Delta = \{0\}$;
- (2) $\Delta = \langle (t_0, t_0) \rangle$;
- (3) $\Delta = \{(t, t) \mid t \in E[2]\}$;
- (4) $\Delta = \{(0, 0), (t, t + t_0), (t + t_0, t), (t_0, t_0)\}$ with $t \in E[2]$, $t \neq t_0$.

Case (1) does not give a smooth quotient A/G , cf. [2, Prop. 3.4]. Case (2) corresponds to Example (c) (and it actually gives a smooth quotient A/G as we prove in section 4). Indeed, the G -isogeny $B \rightarrow A$ corresponds in this case to the morphism $E^2 \rightarrow E^2$ with $E = \mathbb{C}/\mathbb{Z}[i]$ given by the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & i - 1 \end{pmatrix},$$

and the generators given in Example (c) correspond to the conjugates by this matrix of the following respective matrices:

$$\left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

But these are clearly the matrix expressions of the generators $(-1, 1), (-i, i) \in H$ and $(12) \in S_2$ of $G = H \rtimes S_2$.

Remark 3.4. *Since the first and third matrices above generate the subgroup $G(2, 1) \subset G(4, 2)$, we see that if we take F to be the subgroup of G spanned by the pseudoreflections*

$$\begin{pmatrix} -1 & i+1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ i-1 & 1 \end{pmatrix},$$

then F is isomorphic to $G(2, 1)$ and $A/F \simeq \mathbb{P}^2$. In particular, the pair (A, F) is isomorphic to Example (a) with C cyclic of order 2.

In cases (3) and (4), we claim that the pair (A, G) is isomorphic to the pair (B, G) . This will reduce us to the case with trivial Δ , which was already dealt with. To prove the claim, we consider as for $G = G(2, 1)$ the canonical basis of $T_0(A) = T_0(B) = \mathbb{C}^2$. Then the analytic representation of G is given by the following values in its generators:

$$\rho_a((i, -i)) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho_a((-1, 1)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_a((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now, with this basis and the Δ from case (2), we already know that $B \rightarrow A$ looks like $E^2 \rightarrow E^2$ with matrix M from (*). It suffices to check then that the new generator $\rho_a((i, -i))$ falls into $\rho_a(G)$ after conjugation by M . And indeed we have that $M\rho_a((i, -i))M^{-1} = \rho_a((i, i))\rho_a((12))$.

With the Δ from case (3), the corresponding matrix for $B \rightarrow A$ is:

$$N = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

And once again, direct computations give:

$$\begin{aligned} N\rho_a((i, -i))N^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \rho_a((-1, 1))\rho_a((12)), \\ N\rho_a((-1, 1))N^{-1} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \rho_a((12))\rho_a((i, -i)), \\ N\rho_a((12))N^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \rho_a((12)). \end{aligned}$$

And these clearly generate the same group G .

3.3 The case $G = G(4, 1)$

Since $G(4, 1)$ contains $G(4, 2)$, we may start from the precedent list of possible non-trivial Δ 's. Now, by Lemma 3.2, we know that the coordinates of the elements in Δ are i -invariant. We get then that there are only two options for Δ , that is the trivial case and $\Delta = \langle (t_0, t_0) \rangle$.

In the trivial case, we immediately see that (A, G) corresponds to Example (a). Assume then that Δ is non-trivial and consider the element $(s, t) \in B$ with $s \in E[2]$, s not i -invariant and $2t = t_0$. Since clearly these elements have different order, we see that the orbits of these elements by the action of $\langle t_0 \rangle \times \mu_4$ are different. Thus no action of an element in $\Delta \times H \subset \Delta \rtimes G$ can compensate the action of $(12) \in G$ in order to fix (s, t) . In other words, the stabilizer of \bar{t} must be contained in $\Delta \times H$. It is easy to see then that it corresponds to $\langle ((t_0, t_0), (i, -1)) \rangle$. By Lemma 3.3, this stabilizer is not generated by pseudoreflections and hence A/G is not smooth in this case.

3.4 The case $G = G(3, 1)$

By Lemma 3.2, we know that the coordinates of the elements in Δ are ζ_3 -invariant. Now, there are only two such non-trivial elements that we will denote by s_0 and $-s_0$. Since we also know that there are no elements of the form $(t, 0)$ for $t \in E$, we get the following possible options for a non-trivial Δ :

- (1) $\Delta = \{0\}$;
- (2) $\Delta = \langle (s_0, s_0) \rangle$;
- (3) $\Delta = \langle (s_0, -s_0) \rangle$.

We immediately see that the trivial case gives us Example (a). In case (2), Lemma 3.3 tells us that the only pseudoreflections in $\Delta \rtimes G$ are those coming from G . In particular, in order to prove that A/G cannot be smooth, it suffices to exhibit an element in B such that its stabilizer in $\Delta \rtimes G$ has elements that are not in G . Let $\tau = (\zeta_3, \zeta_3) \in H \subset G$, then $1 - \tau$ is surjective. Then there exists an element $\bar{z} \in B$ such that $\bar{z} - \tau(\bar{z}) = (s_0, s_0)$. This implies that $((s_0, s_0), \tau) \in \Delta \rtimes G$ stabilizes z , proving thus that A/G is not smooth in this case.

In case (3), consider the element $\bar{s} = (s, -s) \in B$ with $s \in E[3]$ and s not ζ_3 -invariant. Note that $\langle s_0 \rangle \times \mu_3$ acts on $E[3]$ and a direct computation tells us that the orbit of s is $\{s, s + s_0, s - s_0\}$. In particular, we see that s and $-s$ lie in different orbits for this action. The same argument used in the case of $G(4, 1)$ tells us then that the stabilizer of \bar{s} must be contained in $\Delta \times H$. It is easy to see then that, up to changing \bar{s} by $-\bar{s}$, it corresponds to $\langle ((s_0, -s_0), (\zeta_3, \zeta_3)) \rangle$. Since this stabilizer is not generated by pseudoreflections by Lemma 3.3, we see that A/G is not smooth in this case as well.

3.5 The case $G = G(6, 1)$

By Lemma 3.2, we know that the only possibility is a trivial Δ . This clearly corresponds to Example (a).

3.6 The case $G = G(6, 2)$

Since $G(6, 2)$ contains $G(3, 1)$, we may start from the possible non-trivial Δ 's for that case. Note that these are all 3-torsion subgroups. Thus, if $\bar{x} \in B$ denotes a 2-torsion element, we see that its stabilizer in $\Delta \rtimes G$ can only contain elements in G . Consider then the element $\bar{t} = (t, 0)$ where t is a non-trivial 2-torsion element. As it is proven in [2, Prop. 3.4], the stabilizer of this element in G is not generated by pseudoreflections. This implies that $A/G = B/(\Delta \rtimes G)$ cannot be smooth regardless of the choice of possible Δ .

3.7 The case $G = G(6, 3)$

Since $G(6, 3)$ contains $G(2, 1)$, we may start from the possible non-trivial Δ 's for that case. Note that these are all 2-torsion subgroups. Thus, like we noticed in the previous case, if $\bar{x} \in B$ denotes a 3-torsion element, its stabilizer in $\Delta \rtimes G$ only contains elements in G . Consider then the element $\bar{s} = (s_0, 0)$ where s_0 is a ζ_3 -invariant element (hence 3-torsion). Once again, as proven in [2, Prop. 3.4], the stabilizer of this element in G is not generated by pseudoreflections, which implies that A/G cannot be smooth in any case of Δ .

This finishes the study of the cases where $m \neq p$. We are left thus with the cases $G(3, 3)$ and $G(6, 6)$. In these particular cases we forget all the constructions done before and start from scratch.

3.8 The case $G = G(3, 3)$

The group $G(3, 3)$ is easily seen to be isomorphic *as a complex reflection group* to S_3 acting on \mathbb{C}^2 via the standard representation. As such, it has already been treated by the first two authors in [2, §3.1] and we know that in that case we get a smooth quotient if and only if we are in Example (b).

3.9 The case $G = G(6, 6)$

Note that $G(6, 6)$ is isomorphic to the direct product $G(3, 3) \times \{\pm 1\}$. Since the actions of S_3 and $\mu_2 = \{\pm 1\}$ commute, we may follow the approach taken by [2] for S_3 and we will prove the following:

Proposition 3.5. *Let $G(6, 6) = S_3 \times \mu_2$ act on an abelian surface A in such a way that its action on $T_0(A)$ is the standard one for S_3 and the obvious one for μ_2 . Then A/G is not smooth.*

Proof. Let $\sigma = (12) \in S_3$ and $E = E_\sigma$ be induced by a line $L_\sigma \subset T_0(A)$, and define the lattice

$$\Lambda_B := \sum_{\tau \in S_3} \tau(L_\sigma \cap \Lambda_A).$$

Since clearly all lattices are μ_2 -invariant, this gives us a G -invariant sublattice of Λ_A . Therefore, we get a G -equivariant isogeny $\pi : B \rightarrow A$ with kernel Δ . Applying this construction to Example (b), to which we can naturally add the action of μ_2 in order to get an action of G , we see that it gives the whole lattice. We can thus see B as

$$B = \{(x_1, x_2, x_3) \in E^3 \mid x_1 + x_2 + x_3 = 0\},$$

where S_3 and μ_2 act in their respective natural ways. Using the notations from Section 2, we see by inspection that $F_{\sigma,B} = E_{\sigma,B}[2] \simeq E[2]$, hence the map $\pi : F_{\sigma,B} \rightarrow F_{\sigma,A}$ is surjective since by Lemma 2.1, case 2., we have $F_{\sigma,A} \subset E_{\sigma,A}[2] \simeq E[2]$. By Proposition 2.2, we have that Δ is contained in the fixed locus of all the conjugates of σ , which clearly generate S_3 . Thus, Δ consists of elements of the form $(x, x, x) \in E^3$ such that $3x = 0$. In particular, Δ is isomorphic to a subgroup of $E[3]$ and hence of order 1, 3 or 9.

Assume that Δ is trivial, that is, that $A = B$. Then the action of $G = S_3 \times \mu_2$ on $B \simeq E^2$ induces an action of μ_2 on $B/S_3 \simeq \mathbb{P}^2$ (recall that the action of S_3 on B is that of Example (b)). We only need to notice then that any quotient of \mathbb{P}^2 by a non trivial action of the group μ_2 is not smooth. This is well-known.

Assume now that Δ has order 3 and let $\bar{t} = (t, t, t) \in \Delta$ be a non-trivial element (thus $t \in E[3]$). Let $x \in E[3]$ be a non-trivial element different from $\pm t$ and consider $\bar{x} = (x, x + t, x - t)$. It is then easy to see that the element $(\bar{t}, (123)) \in \Delta \times G$ fixes \bar{x} and that $\text{Stab}_G(\bar{x}) = \{1\}$, so that every pseudoreflection fixing \bar{x} must lie outside G . Let (\bar{s}, σ) be such a pseudoreflection. Using Lemma 2.3, we see that $\sigma \in \{-(12), -(23), -(13)\}$, where $-\tau$ denotes $(\tau, -1) \in S_3 \times \mu_2 = G$. Now, for any such σ , direct computations tell us that (\bar{s}, σ) fixes \bar{x} if and only if $\bar{s} = (s, s, s)$ with $s = a_\sigma x + b_\sigma t$ for some $a_\sigma \neq 0$. Since $x \notin \langle t \rangle \subset E[3]$, we see that $\bar{s} \notin \Delta$ and hence these pseudoreflections do not exist. We get then that $\text{Stab}_{\Delta \times G}(\bar{x})$ is not generated by pseudoreflections and hence A/G cannot be smooth.

Assume finally that Δ has order 9. We claim that in this case the pair (A, G) is isomorphic to the pair (B, G) . This will reduce us to the case with trivial Δ , which was already dealt with. To prove the claim, fix the basis $\{(1, 0, -1), (0, 1, -1)\}$ of $T_0(B) = T_0(A) \subset \mathbb{C}^3$. Then the analytic representation of G is given by the following values in its generators:

$$\rho_a((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_a(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_a((123)) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

Now, with this basis and this Δ , the analytic representation of $B \rightarrow A$ is given by the inverse of the following matrix:

$$M = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Indeed, this corresponds to the morphism that sends $(x, y, -x - y) \in B \subset E^3$ to $(-x - 2y, 2x + y, -x + y) \in A \subset E^3$ and thus its kernel is precisely the elements of the form $(x, x, x) \in E[3]^3 \subset B$, that is, Δ . In order to prove that the pairs (A, G) and (B, G) are isomorphic, it suffices thus to prove that the image of this representation of G under conjugation by M is G once again. Direct computations give:

$$\begin{aligned} M\rho_a(-1)M^{-1} &= \rho_a(-1), & M\rho_a((1\ 2\ 3))M^{-1} &= \rho_a((1\ 2\ 3)), \\ M\rho_a((1\ 2))M^{-1} &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \rho_a((1\ 2))\rho_a(-1). \end{aligned}$$

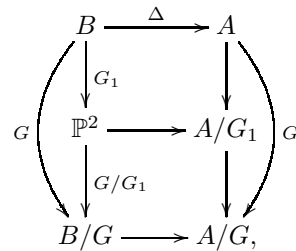
And these clearly generate the same group G . □

4 Proof of (3) \Rightarrow (2)

The only thing left to prove is that Example (c) satisfies property (2) from Theorem 1.1 (the other two are proved in [1]). Let us then study this example in detail.

Recall that in section 3.2 we proved that the pair (A, G) from Example (c) can be obtained as follows. Let $G = G(4, 2)$ and let $B = E^2$ with $E = \mathbb{C}^2/\mathbb{Z}[i]$. Denote by t_0 the i -invariant element in E and denote by q_0 the quotient morphism $E \rightarrow E/\langle t_0 \rangle \simeq E$. Then $A = B/\Delta$ with $\Delta = \langle (t_0, t_0) \rangle \in E^2 = B$ and the action of G on A is the one induced by $B \rightarrow A$.

Note now that G has an index 2 subgroup $G_1 := G(2, 1) = H_1 \times S_2$, which is thus normal in G (here, $H_1 = \{\pm 1\}^2$). Moreover, the pair (B, G_1) corresponds to Example (a), so that $B/G_1 \simeq \mathbb{P}^2$. Finally, note that Δ is an order 2 subgroup of B and thus G acts *trivially* on it. In particular, the actions of G and Δ on B commute and hence we have a commutative diagram of Galois covers



where parallel arrows have the same Galois group. Since Δ and G/G_1 have both order 2, we see then that A/G is a quotient of \mathbb{P}^2 by the action of a Klein group.

Proposition 4.1. *The quotient A/G is isomorphic to \mathbb{P}^2 .*

This proposition finishes the proof of (3) \Rightarrow (2) in Theorem 1.1.

Remark 4.2. *This example was already known to Tokunaga and Yoshida (cf. [6, §5, Table II]). However, in order to prove that $A/G \simeq \mathbb{P}^2$, they cite an article by Švarcman which contains no proofs (cf. [5]).*

Proof. Since A/G is a quotient of \mathbb{P}^2 by the action of a Klein group K , the only thing we need to check is that this action gives \mathbb{P}^2 as a quotient. Note first that the action is faithful since it comes from the faithful action of $G \times \Delta$ on B . Consider then K as a subgroup of $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$ and let K_1 be its preimage in SL_3 . This is an order 12 group and hence any 2-Sylow subgroup of K_1 gives a lift of K to a subgroup of GL_3 . This implies that the action lifts to \mathbb{C}^3 and it can thus be seen as a linear representation of K . Since there are exactly four irreducible representations of K , all of dimension 1, a direct check tells us that any choice of three different representations gives the same faithful action on \mathbb{P}^2 up to conjugation, whereas any other choice gives a non-faithful action. We may assume then that the nontrivial elements $x_i \in K$ for $i = 1, 2, 3$ act on \mathbb{P}^2 via the diagonal matrices with 1 on the i -th coordinate and -1 elsewhere. The quotient of \mathbb{P}^2 by such a group is the weighted projective space $\mathbb{P}(2, 2, 2)$, which is well-known to be isomorphic to $\mathbb{P}(1, 1, 1) = \mathbb{P}^2$. This concludes the proof. \square

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