

# Foundations of generalized Prabhakar-Hilfer fractional calculus with applications

GEORGE A. ANASTASSIOU<sup>1</sup> 

<sup>1</sup> *Department of Mathematical Sciences*  
*University of Memphis, Memphis,*  
*TN 38152, U.S.A.*  
*ganastss@memphis.edu*

## ABSTRACT

Here we introduce the generalized Prabhakar fractional calculus and we also combine it with the generalized Hilfer calculus. We prove that the generalized left and right side Prabhakar fractional integrals preserve continuity and we find tight upper bounds for them. We present several left and right side generalized Prabhakar fractional inequalities of Hardy, Opial and Hilbert-Pachpatte types. We apply these in the setting of generalized Hilfer calculus.

## RESUMEN

Introducimos el cálculo fraccionario generalizado de Prabhakar y también lo combinamos con el cálculo generalizado de Hilfer. Demostramos que las integrales fraccionarias generalizadas de Prabhakar izquierda y derecha preservan la continuidad y encontramos cotas superiores ajustadas para ellas. Presentamos diversas desigualdades fraccionarias generalizadas de Prabhakar izquierda y derecha de tipos Hardy, Opial y Hilbert-Pachpatte. Aplicamos estos resultados en el contexto del cálculo generalizado de Hilfer.

**Keywords and Phrases:** Prabhakar fractional calculus, Hilfer fractional calculus, fractional Hardy, Opial and Hilbert-Pachpatte inequalities.

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## 1 Background

During the last 50 years fractional calculus due to its wide applications to many applied sciences has become a main trend in mathematics. Its predominant kinds are the old Riemann-Liouville fractional calculus and the newer one of Caputo type. Around these two versions have been built a plethora of other variants and all of these involve singular kernels. More recently researchers presented also new fractional calculi of non singular kernels.

The recent Hilfer fractional calculus unifies the Riemann-Liouville and Caputo fractional calculi and the Prabhakar fractional calculus unifies both singular and non-singular kernel fractional calculi.

Finally the newer Hilfer-Prabhakar fractional calculus is the most general one unifying all trends and for different values of its parameters we get the particular fractional calculi. In this article we present and employ unifying advanced and generalized versions of Prabhakar and Hilfer-Prabhakar fractional calculi and we establish related unifying fractional integral inequalities of the following types: Hardy, Opial and Hilbert-Pachpatte. The advantage of this unification is the uniform action taken in describing the various natural phenomena.

We are inspired by [7], [6] and [1]. We start by introducing our own generalized  $\psi$ -Prabhakar type of fractional calculus, then mixing it with the  $\psi$ -Hilfer fractional calculus. Then, we prove a variety of generalized Hardy, Opial and Hilbert-Pachpatte type left and right fractional integral inequalities related to  $\psi$ -Hilfer ([8]) and  $\psi$ -Prabhakar fractional calculi. We involve several functions.

We consider the Prabhakar function (also known as the three parameter Mittag-Leffler function), (see [4, p. 97]; [3])

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k! \Gamma(\alpha k + \beta)} z^k, \quad (1.1)$$

where  $\Gamma$  is the gamma function;  $\alpha, \beta, \gamma \in \mathbb{R} : \alpha, \beta > 0, z \in \mathbb{R}$ , and  $(\gamma)_k = \gamma(\gamma + 1) \cdots (\gamma + k - 1)$ . It is  $E_{\alpha,\beta}^0(z) = \frac{1}{\Gamma(\beta)}$ .

Let  $a, b \in \mathbb{R}, a < b$  and  $x \in [a, b]$ ;  $f \in C([a, b])$ . Let also  $\psi \in C^1([a, b])$  which is increasing. The left and right Prabhakar fractional integrals with respect to  $\psi$  are defined as follows:

$$\left( e_{\rho,\mu,\omega,a+}^{\gamma;\psi} f \right) (x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu-1} E_{\rho,\mu}^{\gamma} [\omega (\psi(x) - \psi(t))^{\rho}] f(t) dt, \quad (1.2)$$

and

$$\left( e_{\rho,\mu,\omega,b-}^{\gamma;\psi} f \right) (x) = \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\mu-1} E_{\rho,\mu}^{\gamma} [\omega (\psi(t) - \psi(x))^{\rho}] f(t) dt, \quad (1.3)$$

where  $\rho, \mu > 0; \gamma, \omega \in \mathbb{R}$ .

Functions (1.2) and (1.3) are continuous, see Theorem 3.1.

Next, additionally, assume that  $\psi'(x) \neq 0$  over  $[a, b]$ .

Let  $\psi, f \in C^N([a, b])$ , where  $N = \lceil \mu \rceil$ , ( $\lceil \cdot \rceil$  is the ceiling of the number),  $0 < \mu \notin \mathbb{N}$ . We define the

$\psi$ -Prabhakar-Caputo left and right fractional derivatives of order  $\mu$  as follows ( $x \in [a, b]$ ):

$$\begin{aligned} \left({}^C D_{\rho, \mu, \omega, a+}^{\gamma; \psi} f\right)(x) &= \int_a^x \psi'(t) (\psi(x) - \psi(t))^{N-\mu-1} \\ &E_{\rho, N-\mu}^{-\gamma} [\omega (\psi(x) - \psi(t))^\rho] \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^N f(t) dt, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \left({}^C D_{\rho, \mu, \omega, b-}^{\gamma; \psi} f\right)(x) &= (-1)^N \int_x^b \psi'(t) (\psi(t) - \psi(x))^{N-\mu-1} \\ &E_{\rho, N-\mu}^{-\gamma} [\omega (\psi(t) - \psi(x))^\rho] \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^N f(t) dt. \end{aligned} \tag{1.5}$$

One can write (see (1.4), (1.5))

$$\left({}^C D_{\rho, \mu, \omega, a+}^{\gamma; \psi} f\right)(x) = \left(e_{\rho, N-\mu, \omega, a+}^{-\gamma; \psi} f_\psi^{[N]}\right)(x), \tag{1.6}$$

and

$$\left({}^C D_{\rho, \mu, \omega, b-}^{\gamma; \psi} f\right)(x) = (-1)^N \left(e_{\rho, N-\mu, \omega, b-}^{-\gamma; \psi} f_\psi^{[N]}\right)(x), \tag{1.7}$$

where

$$f_\psi^{[N]}(x) = f_\psi^{(N)} f(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^N f(x), \tag{1.8}$$

$\forall x \in [a, b]$ .

Functions (1.6) and (1.7) are continuous on  $[a, b]$ .

Next we define the  $\psi$ -Prabhakar-Riemann-Liouville left and right fractional derivatives of order  $\mu$  as follows ( $x \in [a, b]$ ):

$$\begin{aligned} \left({}^{RL} D_{\rho, \mu, \omega, a+}^{\gamma; \psi} f\right)(x) &= \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^N \int_a^x \psi'(t) (\psi(x) - \psi(t))^{N-\mu-1} \\ &E_{\rho, N-\mu}^{-\gamma} [\omega (\psi(x) - \psi(t))^\rho] f(t) dt, \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} \left({}^{RL} D_{\rho, \mu, \omega, b-}^{\gamma; \psi} f\right)(x) &= \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^N \int_x^b \psi'(t) (\psi(t) - \psi(x))^{N-\mu-1} \\ &E_{\rho, N-\mu}^{-\gamma} [\omega (\psi(t) - \psi(x))^\rho] f(t) dt. \end{aligned} \tag{1.10}$$

That is we have

$$\left({}^{RL} D_{\rho, \mu, \omega, a+}^{\gamma; \psi} f\right)(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^N \left(e_{\rho, N-\mu, \omega, a+}^{-\gamma; \psi} f\right)(x), \tag{1.11}$$

and

$$\left({}^{RL} D_{\rho, \mu, \omega, b-}^{\gamma; \psi} f\right)(x) = \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^N \left(e_{\rho, N-\mu, \omega, b-}^{-\gamma; \psi} f\right)(x), \tag{1.12}$$

$\forall x \in [a, b]$ .

We define also the  $\psi$ -Hilfer-Prabhakar left and right fractional derivatives of order  $\mu$  and type  $0 \leq \beta \leq 1$ , as follows

$$\left( {}^H\mathbb{D}_{\rho,\mu,\omega,a+}^{\gamma,\beta;\psi} f \right) (x) = e^{-\gamma\beta;\psi}_{\rho,\beta(N-\mu),\omega,a+} \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)^N e^{-\gamma(1-\beta);\psi}_{\rho,(1-\beta)(N-\mu),\omega,a+} f(x), \quad (1.13)$$

and

$$\left( {}^H\mathbb{D}_{\rho,\mu,\omega,b-}^{\gamma,\beta;\psi} f \right) (x) = e^{-\gamma\beta;\psi}_{\rho,\beta(N-\mu),\omega,b-} \left( -\frac{1}{\psi'(x)} \frac{d}{dx} \right)^N e^{-\gamma(1-\beta);\psi}_{\rho,(1-\beta)(N-\mu),\omega,b-} f(x), \quad (1.14)$$

$\forall x \in [a, b]$ .

When  $\beta = 0$ , we get the Riemann-Liouville version, and when  $\beta = 1$ , we get the Caputo version.

We call  $\xi = \mu + \beta(N - \mu)$ , we have that  $N - 1 < \mu \leq \mu + \beta(N - \mu) \leq \mu + N - \mu = N$ , hence  $[\xi] = N$ .

We can easily write that

$$\left( {}^H\mathbb{D}_{\rho,\mu,\omega,a+}^{\gamma,\beta;\psi} f \right) (x) = e^{-\gamma\beta;\psi}_{\rho,\xi-\mu,\omega,a+} {}^{RL}D_{\rho,\xi,\omega,a+}^{\gamma(1-\beta);\psi} f(x), \quad (1.15)$$

and

$$\left( {}^H\mathbb{D}_{\rho,\mu,\omega,b-}^{\gamma,\beta;\psi} f \right) (x) = e^{-\gamma\beta;\psi}_{\rho,\xi-\mu,\omega,b-} {}^{RL}D_{\rho,\xi,\omega,b-}^{\gamma(1-\beta);\psi} f(x), \quad (1.16)$$

$\forall x \in [a, b]$ .

## 2 Main results

We start with a left  $\psi$ -Prabhakar fractional Hardy type integral inequality involving several functions.

**Theorem 2.1.** Here  $i = 1, \dots, m$ ;  $f_i \in C([a, b])$ ,  $\psi \in C^1([a, b])$  and  $\psi$  is increasing. Let  $\rho_i, \mu_i > 0$ ,  $\gamma_i, \omega_i \in \mathbb{R}$ . Also let  $r_1, r_2, r_3 > 1$ :  $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$ , and assume that  $\mu_i > \frac{1}{r_2} + \frac{1}{r_3}$ , for all  $i = 1, \dots, m$ .

Then

$$\begin{aligned} & \left\| \prod_{i=1}^m e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_i \right\|_{L_{r_1}([a,b],\psi)} \leq \\ & \frac{(\psi(b) - \psi(a))^{\left[ \sum_{i=1}^m \mu_i - m + \frac{m}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2} \right]}}{\left( r_1 r_3 \left( \sum_{i=1}^m \mu_i - m \right) + m r_3 + 1 \right)^{\frac{1}{r_1 r_3}} \left( \prod_{i=1}^m (r_1 (\mu_i - 1) + 1) \right)^{\frac{1}{r_1}}} \\ & \left\{ \int_a^b \left[ \prod_{i=1}^m \left( \int_a^x |E_{\rho_i,\mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}]|^{r_2} d\psi(t) \right)^{r_1} d\psi(x) \right]^{\frac{1}{r_1 r_2}} \right. \\ & \left. \left( \prod_{i=1}^m \|f_i\|_{L_{r_3}([a,b],\psi)} \right) \right\}. \end{aligned} \quad (2.1)$$

*Proof.* By (1.2) we have

$$\left( e_{\rho_i, \mu_i, \omega_i, a+}^{\gamma_i; \psi} f_i \right) (x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu_i - 1} E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] f_i(t) dt, \quad (2.2)$$

$i = 1, \dots, m; \forall x \in [a, b]$ .

By Hölder's inequality and (2.2) we obtain

$$\begin{aligned} & \left| \left( e_{\rho_i, \mu_i, \omega_i, a+}^{\gamma_i; \psi} f_i \right) (x) \right| \leq \\ & \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu_i - 1} \left| E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] \right| |f_i(t)| dt \leq \\ & \left( \int_a^x (\psi(x) - \psi(t))^{r_1(\mu_i - 1)} d\psi(t) \right)^{\frac{1}{r_1}} \\ & \left( \int_a^x \left| E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] \right|^{r_2} d\psi(t) \right)^{\frac{1}{r_2}} \left( \int_a^x |f_i(t)|^{r_3} d\psi(t) \right)^{\frac{1}{r_3}} \leq \\ & \frac{(\psi(x) - \psi(a))^{\mu_i - 1 + \frac{1}{r_1}}}{(r_1(\mu_i - 1) + 1)^{\frac{1}{r_1}}} \\ & \left( \int_a^x \left| E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] \right|^{r_2} d\psi(t) \right)^{\frac{1}{r_2}} \|f_i\|_{L_{r_3}([a, b], \psi)}. \end{aligned} \quad (2.3)$$

So far we have

$$\begin{aligned} & \left| \left( e_{\rho_i, \mu_i, \omega_i, a+}^{\gamma_i; \psi} f_i \right) (x) \right| \leq \frac{(\psi(x) - \psi(a))^{\mu_i - 1 + \frac{1}{r_1}}}{(r_1(\mu_i - 1) + 1)^{\frac{1}{r_1}}} \\ & \left( \int_a^x \left| E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] \right|^{r_2} d\psi(t) \right)^{\frac{1}{r_2}} \|f_i\|_{L_{r_3}([a, b], \psi)}, \end{aligned} \quad (2.4)$$

$\forall x \in [a, b]$ , with  $\mu_i > \frac{1}{r_2} + \frac{1}{r_3}$ , for any  $i = 1, \dots, m$ .

Hence it holds

$$\begin{aligned} & \left( \prod_{i=1}^m \left| \left( e_{\rho_i, \mu_i, \omega_i, a+}^{\gamma_i; \psi} f_i \right) (x) \right| \right)^{r_1} \leq \frac{(\psi(x) - \psi(a))^{r_1 \sum_{i=1}^m \mu_i - mr_1 + m}}{\left( \prod_{i=1}^m (r_1(\mu_i - 1) + 1) \right)} \\ & \left[ \prod_{i=1}^m \left( \int_a^x \left| E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] \right|^{r_2} d\psi(t) \right) \right]^{\frac{r_1}{r_2}} \left( \prod_{i=1}^m \|f_i\|_{L_{r_3}([a, b], \psi)} \right)^{r_1}, \end{aligned} \quad (2.5)$$

$\forall x \in [a, b]$ .

Therefore we obtain

$$\int_a^b \left( \prod_{i=1}^m \left| \left( e_{\rho_i, \mu_i, \omega_i, a+}^{\gamma_i; \psi} f_i \right) (x) \right| \right)^{r_1} d\psi(x) \leq \frac{\left( \prod_{i=1}^m \|f_i\|_{L_{r_3}([a, b], \psi)} \right)^{r_1}}{\left( \prod_{i=1}^m (r_1(\mu_i - 1) + 1) \right)} \quad (2.6)$$

$$\left[ \int_a^b (\psi(x) - \psi(a))^{r_1 \sum_{i=1}^m \mu_i - mr_1 + m}$$

$$\left[ \prod_{i=1}^m \left( \int_a^x |E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}]|^{r_2} d\psi(t) \right) \right]^{\frac{r_1}{r_2}} d\psi(x)$$

(again by Hölder’s inequality)

$$\begin{aligned} &\leq \frac{\left( \prod_{i=1}^m \|f_i\|_{L_{r_3}([a,b],\psi)} \right)^{r_1}}{\left( \prod_{i=1}^m (r_1(\mu_i - 1) + 1) \right)} \left( \int_a^b (\psi(x) - \psi(a))^{r_1 r_3 \sum_{i=1}^m \mu_i - m r_1 r_3 + m r_3} d\psi(x) \right)^{\frac{1}{r_3}} \\ &\left\{ \int_a^b \left[ \prod_{i=1}^m \left( \int_a^x |E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}]|^{r_2} d\psi(t) \right) \right]^{r_1} d\psi(x) \right\}^{\frac{1}{r_2}} (\psi(b) - \psi(a))^{\frac{1}{r_1}} = \\ &\frac{\left( \prod_{i=1}^m \|f_i\|_{L_{r_3}([a,b],\psi)} \right)^{r_1} (\psi(b) - \psi(a))^{r_1 \sum_{i=1}^m \mu_i - m r_1 + m + 1 - \frac{1}{r_2}}}{\left( \prod_{i=1}^m (r_1(\mu_i - 1) + 1) \right) \left( r_1 r_3 \sum_{i=1}^m \mu_i - m r_1 r_3 + m r_3 + 1 \right)^{\frac{1}{r_3}}} \tag{2.7} \\ &\left\{ \int_a^b \left[ \prod_{i=1}^m \left( \int_a^x |E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}]|^{r_2} d\psi(t) \right) \right]^{r_1} d\psi(x) \right\}^{\frac{1}{r_2}}, \end{aligned}$$

where  $\mu_i > \frac{1}{r_2} + \frac{1}{r_3}$ ,  $i = 1, \dots, m$ .

The claim is proved. □

We continue with a right  $\psi$ -Prabhakar fractional Hardy type integral inequality involving several functions.

**Theorem 2.2.** *All as in Theorem 2.1. It holds*

$$\begin{aligned} &\left\| \prod_{i=1}^m e_{\rho_i, \mu_i, \omega_i, b-}^{\gamma_i; \psi} f_i \right\|_{L_{r_1}([a,b],\psi)} \leq \\ &\frac{(\psi(b) - \psi(a))^{\left[ \sum_{i=1}^m \mu_i - m + \frac{m}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2} \right]}}{\left( r_1 r_3 \left( \sum_{i=1}^m \mu_i - m \right) + m r_3 + 1 \right)^{\frac{1}{r_1 r_3}} \left( \prod_{i=1}^m (r_1(\mu_i - 1) + 1) \right)^{\frac{1}{r_1}}} \\ &\left\{ \int_a^b \left[ \prod_{i=1}^m \left( \int_x^b |E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi(t) - \psi(x))^{\rho_i}]|^{r_2} d\psi(t) \right) \right]^{r_1} d\psi(x) \right\}^{\frac{1}{r_1 r_2}} \\ &\left( \prod_{i=1}^m \|f_i\|_{L_{r_3}([a,b],\psi)} \right). \tag{2.8} \end{aligned}$$

*Proof.* Similar to the proof of Theorem 2.1 and omitted. □

Next we apply Theorems 2.1, 2.2.

We give the related Hardy type inequalities:

**Theorem 2.3.** Here  $i = 1, \dots, m$ ;  $f_i \in C^{N_i}([a, b])$ , where  $N_i = [\mu_i]$ ,  $0 < \mu_i \notin \mathbb{N}$ ;  $\theta := \max\{N_1, \dots, N_m\}$ ,  $\psi \in C^\theta([a, b])$  with  $\psi' \neq 0$  and increasing. Let  $\rho_i > 0$ ,  $\gamma_i, \omega_i \in \mathbb{R}$ . Also let  $r_1, r_2, r_3 > 1 : \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$ , and assume that  $N_i - \mu_i > \frac{1}{r_2} + \frac{1}{r_3}$ , for all  $i = 1, \dots, m$ . Then

i)

$$\begin{aligned} & \left\| \prod_{i=1}^m {}^C D_{\rho_i, \mu_i, \omega_i, a+}^{\gamma_i; \psi} f_i \right\|_{L_{r_1}([a, b], \psi)} \leq \\ & \frac{(\psi(b) - \psi(a))^{\left[ \sum_{i=1}^m (N_i - \mu_i) - m + \frac{1}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2} \right]}}{\left( r_1 r_3 \left( \sum_{i=1}^m (N_i - \mu_i) - m \right) + m r_3 + 1 \right)^{\frac{1}{r_1 r_3}} \left( \prod_{i=1}^m (r_1 (N_i - \mu_i - 1) + 1) \right)^{\frac{1}{r_1}}} \\ & \left\{ \int_a^b \left[ \prod_{i=1}^m \left( \int_a^x \left| E_{\rho_i, N_i - \mu_i}^{-\gamma_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] \right|^{r_2} d\psi(t) \right)^{r_1} d\psi(x) \right]^{\frac{1}{r_1 r_2}} \\ & \left( \prod_{i=1}^m \|f_i^{[N_i]}\|_{L_{r_3}([a, b], \psi)} \right), \end{aligned} \tag{2.9}$$

and

ii)

$$\begin{aligned} & \left\| \prod_{i=1}^m {}^C D_{\rho_i, \mu_i, \omega_i, b-}^{\gamma_i; \psi} f_i \right\|_{L_{r_1}([a, b], \psi)} \leq \\ & \frac{(\psi(b) - \psi(a))^{\left[ \sum_{i=1}^m (N_i - \mu_i) - m + \frac{1}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2} \right]}}{\left( r_1 r_3 \left( \sum_{i=1}^m (N_i - \mu_i) - m \right) + m r_3 + 1 \right)^{\frac{1}{r_1 r_3}} \left( \prod_{i=1}^m (r_1 (N_i - \mu_i - 1) + 1) \right)^{\frac{1}{r_1}}} \\ & \left\{ \int_a^b \left[ \prod_{i=1}^m \left( \int_x^b \left| E_{\rho_i, N_i - \mu_i}^{-\gamma_i} [\omega_i (\psi(t) - \psi(x))^{\rho_i}] \right|^{r_2} d\psi(t) \right)^{r_1} d\psi(x) \right]^{\frac{1}{r_1 r_2}} \\ & \left( \prod_{i=1}^m \|f_i^{[N_i]}\|_{L_{r_3}([a, b], \psi)} \right). \end{aligned} \tag{2.10}$$

*Proof.* By (1.6), (1.7) and Theorems 2.1, 2.2. □

We also present other Hardy type related inequalities:

**Theorem 2.4.** Here  $i = 1, \dots, m$ ;  $f_i \in C^{N_i}([a, b])$ , where  $N_i = [\mu_i]$ ,  $0 < \mu_i \notin \mathbb{N}$ ;  $\theta := \max\{N_1, \dots, N_m\}$ ,  $\psi \in C^\theta([a, b])$ ,  $\psi' \neq 0$ , and  $\psi$  is increasing. Let  $\rho_i > 0$ ,  $\gamma_i, \omega_i \in \mathbb{R}$ ,  $0 \leq \beta_i \leq 1$ ,  $\xi_i = \mu_i + \beta_i (N_i - \mu_i)$ . Also let  $r_1, r_2, r_3 > 1 : \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$ , and assume that  $\xi_i - \mu_i > \frac{1}{r_2} + \frac{1}{r_3}$ , for all  $i = 1, \dots, m$ .

Also assume that  ${}^{RL} D_{\rho_i, \xi_i, \omega_i, a+}^{\gamma_i(1-\beta_i); \psi} f_i, {}^{RL} D_{\rho_i, \xi_i, \omega_i, b-}^{\gamma_i(1-\beta_i); \psi} f_i \in C([a, b])$ ,  $i = 1, \dots, m$ .

Then

i)

$$\begin{aligned} & \left\| \prod_{i=1}^m H\mathbb{D}_{\rho_i, \mu_i, \omega_i, a+}^{\gamma_i; \psi} f_i \right\|_{L_{r_1}([a, b], \psi)} \leq \\ & \frac{(\psi(b) - \psi(a))^{\left[ \sum_{i=1}^m (\xi_i - \mu_i) - m + \frac{m}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2} \right]}}{\left( r_1 r_3 \left( \sum_{i=1}^m (\xi_i - \mu_i) - m \right) + m r_3 + 1 \right)^{\frac{1}{r_1 r_3}} \left( \prod_{i=1}^m (r_1 (\xi_i - \mu_i - 1) + 1) \right)^{\frac{1}{r_1}}} \\ & \left\{ \int_a^b \left[ \prod_{i=1}^m \left( \int_a^x \left| E_{\rho_i, \xi_i - \mu_i}^{-\gamma_i \beta_i} [\omega_i (\psi(x) - \psi(t))^{\rho_i}] \right|^{r_2} d\psi(t) \right) \right]^{r_1} d\psi(x) \right\}^{\frac{1}{r_1 r_2}} \\ & \left( \prod_{i=1}^m \left\| {}^{RL}D_{\rho_i, \xi_i, \omega_i, a+}^{\gamma_i (1 - \beta_i); \psi} f_i \right\|_{L_{r_3}([a, b], \psi)} \right), \end{aligned} \tag{2.11}$$

and

ii)

$$\begin{aligned} & \left\| \prod_{i=1}^m H\mathbb{D}_{\rho_i, \mu_i, \omega_i, b-}^{\gamma_i; \psi} f_i \right\|_{L_{r_1}([a, b], \psi)} \leq \\ & \frac{(\psi(b) - \psi(a))^{\left[ \sum_{i=1}^m (\xi_i - \mu_i) - m + \frac{m}{r_1} + \frac{1}{r_1} - \frac{1}{r_1 r_2} \right]}}{\left( r_1 r_3 \left( \sum_{i=1}^m (\xi_i - \mu_i) - m \right) + m r_3 + 1 \right)^{\frac{1}{r_1 r_3}} \left( \prod_{i=1}^m (r_1 (\xi_i - \mu_i - 1) + 1) \right)^{\frac{1}{r_1}}} \\ & \left\{ \int_a^b \left[ \prod_{i=1}^m \left( \int_x^b \left| E_{\rho_i, \xi_i - \mu_i}^{-\gamma_i \beta_i} [\omega_i (\psi(t) - \psi(x))^{\rho_i}] \right|^{r_2} d\psi(t) \right) \right]^{r_1} d\psi(x) \right\}^{\frac{1}{r_1 r_2}} \\ & \left( \prod_{i=1}^m \left\| {}^{RL}D_{\rho_i, \xi_i, \omega_i, b-}^{\gamma_i (1 - \beta_i); \psi} f_i \right\|_{L_{r_3}([a, b], \psi)} \right). \end{aligned} \tag{2.12}$$

*Proof.* By (1.15), (1.16) and Theorems 2.1, 2.2. □

From now on all entities are according and respectively to Section 1. Background.

Next we give Opial type inequalities related to Prabhakar fractional calculus.

A left side one follows:

**Theorem 2.5.** Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\begin{aligned} & \int_a^x \left| \left( e_{\rho, \mu, \omega, a+}^{\gamma; \psi} f \right) (w) \right| |f(w)| \psi'(w) dw \leq 2^{-\frac{1}{q}} \\ & \left[ \int_a^x \left\{ \int_a^w (\psi(w) - \psi(t))^{p(\mu-1)} \left| E_{\rho, \mu}^{\gamma} [\omega (\psi(w) - \psi(t))^{\rho}] \right|^p dt \right\} dw \right]^{\frac{1}{p}} \\ & \left( \int_a^x |f(w)|^q (\psi'(w))^q dw \right)^{\frac{2}{q}}, \end{aligned} \tag{2.13}$$

$\forall x \in [a, b]$ .



*Proof.* By (1.2), using Hölder's inequality, we have

$$\begin{aligned} \left| \left( e^{\gamma;\psi}_{\rho,\mu,\omega,a+} f \right) (x) \right| &\leq \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu-1} |E_{\rho,\mu}^\gamma [\omega (\psi(x) - \psi(t))^\rho]| |f(t)| dt \\ &\leq \left( \int_a^x (\psi(x) - \psi(t))^{p(\mu-1)} |E_{\rho,\mu}^\gamma [\omega (\psi(x) - \psi(t))^\rho]|^p dt \right)^{\frac{1}{p}} \\ &\quad \left( \int_a^x (\psi'(t) |f(t)|)^q dt \right)^{\frac{1}{q}}. \end{aligned} \tag{2.14}$$

Call

$$\phi(x) = \int_a^x (\psi'(t) |f(t)|)^q dt, \quad \phi(a) = 0. \tag{2.15}$$

Thus

$$\phi'(x) = (\psi'(x) |f(x)|)^q \geq 0, \tag{2.16}$$

and

$$(\phi'(x))^{\frac{1}{q}} = \psi'(x) |f(x)| \geq 0, \quad \forall x \in [a, b].$$

Consequently, we get

$$\begin{aligned} \left| \left( e^{\gamma;\psi}_{\rho,\mu,\omega,a+} f \right) (w) \right| \psi'(w) |f(w)| &\leq \\ \left( \int_a^w (\psi(w) - \psi(t))^{p(\mu-1)} |E_{\rho,\mu}^\gamma [\omega (\psi(w) - \psi(t))^\rho]|^p dt \right)^{\frac{1}{p}} \\ &\quad (\phi(w) \phi'(w))^{\frac{1}{q}}, \quad \forall w \in [a, b]. \end{aligned} \tag{2.17}$$

Then, by applying again Hölder's inequality:

$$\begin{aligned} \int_a^x \left| \left( e^{\gamma;\psi}_{\rho,\mu,\omega,a+} f \right) (w) \right| |f(w)| \psi'(w) dw &\leq \\ \int_a^x \left\{ \int_a^w (\psi(w) - \psi(t))^{p(\mu-1)} |E_{\rho,\mu}^\gamma [\omega (\psi(w) - \psi(t))^\rho]|^p dt \right\}^{\frac{1}{p}} \\ &\quad (\phi(w) \phi'(w))^{\frac{1}{q}} dw \leq \\ \left[ \int_a^x \left\{ \int_a^w (\psi(w) - \psi(t))^{p(\mu-1)} |E_{\rho,\mu}^\gamma [\omega (\psi(w) - \psi(t))^\rho]|^p dt \right\} dw \right]^{\frac{1}{p}} \\ &\quad \left( \int_a^x \phi(w) d\phi(w) \right)^{\frac{1}{q}} = \\ \left[ \int_a^x \left\{ \int_a^w (\psi(w) - \psi(t))^{p(\mu-1)} |E_{\rho,\mu}^\gamma [\omega (\psi(w) - \psi(t))^\rho]|^p dt \right\} dw \right]^{\frac{1}{p}} \\ &\quad \left( \frac{\phi^2(x)}{2} \right)^{\frac{1}{q}} = 2^{-\frac{1}{q}} \\ \left[ \int_a^x \left\{ \int_a^w (\psi(w) - \psi(t))^{p(\mu-1)} |E_{\rho,\mu}^\gamma [\omega (\psi(w) - \psi(t))^\rho]|^p dt \right\} dw \right]^{\frac{1}{p}} \\ &\quad \left( \int_a^x (\psi'(w) |f(w)|)^q dw \right)^{\frac{2}{q}}. \end{aligned} \tag{2.19}$$

The theorem is proved. □

The right side Opial inequality follows:

**Theorem 2.6.** Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\int_x^b \left| \left( e^{\gamma; \psi}_{\rho, \mu, \omega, b-} f \right) (w) \right| |f(w)| \psi'(w) dw \leq 2^{-\frac{1}{q}}$$

$$\left[ \int_x^b \left\{ \int_w^b (\psi(t) - \psi(w))^{p(\mu-1)} \left| E_{\rho, \mu}^{\gamma} [\omega (\psi(t) - \psi(w))^{\rho}] \right|^p dt \right\} dw \right]^{\frac{1}{p}}$$

$$\left( \int_x^b |f(w)|^q (\psi'(w))^q dw \right)^{\frac{2}{q}}, \quad (2.20)$$

$\forall x \in [a, b]$ .

*Proof.* As it is similar to the proof of Theorem 2.5, is omitted.  $\square$

We continue with more interesting Opial type Prabhakar-Caputo fractional inequalities:

**Theorem 2.7.** Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ . Then

i)

$$\int_a^x \left| \left( {}^C D_{\rho, \mu, \omega, a+}^{\gamma; \psi} f \right) (w) \right| \left| f_{\psi}^{[N]}(w) \right| \psi'(w) dw \leq 2^{-\frac{1}{q}}$$

$$\left[ \int_a^x \left\{ \int_a^w (\psi(w) - \psi(t))^{p(N-\mu-1)} \left| E_{\rho, N-\mu}^{-\gamma} [\omega (\psi(w) - \psi(t))^{\rho}] \right|^p dt \right\} dw \right]^{\frac{1}{p}}$$

$$\left( \int_a^x \left| f_{\psi}^{[N]}(w) \right|^q (\psi'(w))^q dw \right)^{\frac{2}{q}}, \quad (2.21)$$

and

ii)

$$\int_x^b \left| \left( {}^C D_{\rho, \mu, \omega, b-}^{\gamma; \psi} f \right) (w) \right| \left| f_{\psi}^{[N]}(w) \right| \psi'(w) dw \leq 2^{-\frac{1}{q}}$$

$$\left[ \int_x^b \left\{ \int_w^b (\psi(t) - \psi(w))^{p(N-\mu-1)} \left| E_{\rho, N-\mu}^{-\gamma} [\omega (\psi(t) - \psi(w))^{\rho}] \right|^p dt \right\} dw \right]^{\frac{1}{p}}$$

$$\left( \int_x^b \left| f_{\psi}^{[N]}(w) \right|^q (\psi'(w))^q dw \right)^{\frac{2}{q}}, \quad (2.22)$$

$\forall x \in [a, b]$ .

*Proof.* By Theorems 2.5, 2.6 and (1.6)-(1.8).  $\square$

Next come  $\psi$ -Hilfer-Prabhakar left and right Opial type fractional inequalities:

**Theorem 2.8.** Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ . Additionally here assume that

$${}^{RL}D_{\rho, \xi, \omega, a+}^{\gamma(1-\beta); \psi} f, {}^{RL}D_{\rho, \xi, \omega, b-}^{\gamma(1-\beta); \psi} f \in C([a, b]).$$

Then

$$\begin{aligned}
 i) \quad & \int_a^x \left| \left( {}^H\mathbb{D}_{\rho, \mu, \omega, a+}^{\gamma, \beta; \psi} f \right) (w) \right| \left| \left( {}^{RL}D_{\rho, \xi, \omega, a+}^{\gamma(1-\beta); \psi} f \right) (w) \right| \psi'(w) dw \leq 2^{-\frac{1}{q}} \\
 & \left[ \int_a^x \left\{ \int_a^w (\psi(w) - \psi(t))^{p(\xi - \mu - 1)} \left| E_{\rho, \xi - \mu}^{-\gamma\beta} [\omega (\psi(w) - \psi(t))^{\rho}] \right|^p dt \right\} dw \right]^{\frac{1}{p}} \\
 & \left( \int_a^x \left| \left( {}^{RL}D_{\rho, \xi, \omega, a+}^{\gamma(1-\beta); \psi} f \right) (w) \right|^q (\psi'(w))^q dw \right)^{\frac{2}{q}}, \tag{2.23}
 \end{aligned}$$

and

$$\begin{aligned}
 ii) \quad & \int_x^b \left| \left( {}^H\mathbb{D}_{\rho, \mu, \omega, b-}^{\gamma, \beta; \psi} f \right) (w) \right| \left| \left( {}^{RL}D_{\rho, \xi, \omega, b-}^{\gamma(1-\beta); \psi} f \right) (w) \right| \psi'(w) dw \leq 2^{-\frac{1}{q}} \\
 & \left[ \int_x^b \left\{ \int_w^b (\psi(t) - \psi(w))^{p(\xi - \mu - 1)} \left| E_{\rho, \xi - \mu}^{-\gamma\beta} [\omega (\psi(t) - \psi(w))^{\rho}] \right|^p dt \right\} dw \right]^{\frac{1}{p}} \\
 & \left( \int_x^b \left| \left( {}^{RL}D_{\rho, \xi, \omega, b-}^{\gamma(1-\beta); \psi} f \right) (w) \right|^q (\psi'(w))^q dw \right)^{\frac{2}{q}}, \tag{2.24}
 \end{aligned}$$

$$\forall x \in [a, b].$$

*Proof.* By Theorems 2.5, 2.6 and (1.15), (1.16). □

Next we give several Prabhakar Hilbert-Pachpatte fractional inequalities. We start with a left side one.

**Theorem 2.9.** Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; i = 1, 2$ . Let  $[a_i, b_i] \subset \mathbb{R}, \psi_i \in C^1([a_i, b_i])$  and strictly increasing,  $f_i \in C([a_i, b_i]); \rho_i, \mu_i > 0, \gamma_i, \omega_i \in \mathbb{R}$ . Then

$$\begin{aligned}
 & \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left| \left( e_{\rho_1, \mu_1, \omega_1, a_1+}^{\gamma_1; \psi_1} f_1 \right) (x_1) \right| \left| \left( e_{\rho_2, \mu_2, \omega_2, a_2+}^{\gamma_2; \psi_2} f_2 \right) (x_2) \right| dx_1 dx_2}{\left\{ \frac{\left[ \int_{a_1}^{x_1} \{ (\psi_1(x_1) - \psi_1(t_1))^{\mu_1 - 1} \left| E_{\rho_1, \mu_1}^{\gamma_1} [\omega_1 (\psi_1(x_1) - \psi_1(t_1))^{\rho_1}] \right|^p dt_1 \right]}{p} + \frac{\left[ \int_{a_2}^{x_2} \{ (\psi_2(x_2) - \psi_2(t_2))^{\mu_2 - 1} \left| E_{\rho_2, \mu_2}^{\gamma_2} [\omega_2 (\psi_2(x_2) - \psi_2(t_2))^{\rho_2}] \right|^q dt_2 \right]}{q} \right\}}{q} \right\}} \\
 & \leq (b_1 - a_1) (b_2 - a_2) \|\psi_1' f_1\|_q \|\psi_2' f_2\|_p. \tag{2.25}
 \end{aligned}$$

*Proof.* We have that ( $i = 1, 2$ )

$$\begin{aligned}
 & \left( e_{\rho_i, \mu_i, \omega_i, a_i+}^{\gamma_i; \psi_i} f_i \right) (x_i) \stackrel{(1.2)}{=} \\
 & \int_{a_i}^{x_i} \psi_i'(t_i) (\psi_i(x_i) - \psi_i(t_i))^{\mu_i - 1} E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi_i(x_i) - \psi_i(t_i))^{\rho_i}] f_i(t_i) dt_i, \tag{2.26}
 \end{aligned}$$

$\forall x_i \in [a_i, b_i]$ , where  $\rho_i, \mu_i > 0$ ;  $\gamma_i, \omega_i \in \mathbb{R}$ .

Then

$$\left| \left( e_{\rho_i, \mu_i, \omega_i, a_i}^{\gamma_i; \psi_i} f_i \right) (x_i) \right| \leq \int_{a_i}^{x_i} \psi_i'(t_i) (\psi_i(x_i) - \psi_i(t_i))^{\mu_i - 1} \left| E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i (\psi_i(x_i) - \psi_i(t_i))^{\rho_i}] \right| |f_i(t_i)| dt_i, \quad (2.27)$$

$i = 1, 2, \forall x_i \in [a_i, b_i]$ .

By applying Hölder's inequality twice we get:

$$\left| \left( e_{\rho_1, \mu_1, \omega_1, a_1}^{\gamma_1; \psi_1} f_1 \right) (x_1) \right| \leq \left[ \int_{a_1}^{x_1} \left\{ (\psi_1(x_1) - \psi_1(t_1))^{\mu_1 - 1} \left| E_{\rho_1, \mu_1}^{\gamma_1} [\omega_1 (\psi_1(x_1) - \psi_1(t_1))^{\rho_1}] \right| \right\}^p dt_1 \right]^{\frac{1}{p}} \left( \int_{a_1}^{x_1} (\psi_1'(t_1) |f_1(t_1)|)^q dt_1 \right)^{\frac{1}{q}}, \quad (2.28)$$

$\forall x_1 \in [a_1, b_1]$ , and

$$\left| \left( e_{\rho_2, \mu_2, \omega_2, a_2}^{\gamma_2; \psi_2} f_2 \right) (x_2) \right| \leq \left[ \int_{a_2}^{x_2} \left\{ (\psi_2(x_2) - \psi_2(t_2))^{\mu_2 - 1} \left| E_{\rho_2, \mu_2}^{\gamma_2} [\omega_2 (\psi_2(x_2) - \psi_2(t_2))^{\rho_2}] \right| \right\}^q dt_2 \right]^{\frac{1}{q}} \left( \int_{a_2}^{x_2} (\psi_2'(t_2) |f_2(t_2)|)^p dt_2 \right)^{\frac{1}{p}}, \quad (2.29)$$

$\forall x_2 \in [a_2, b_2]$ .

Hence we have (by (2.28), (2.29))

$$\left| \left( e_{\rho_1, \mu_1, \omega_1, a_1}^{\gamma_1; \psi_1} f_1 \right) (x_1) \right| \left| \left( e_{\rho_2, \mu_2, \omega_2, a_2}^{\gamma_2; \psi_2} f_2 \right) (x_2) \right| \leq \left[ \int_{a_1}^{x_1} \left\{ (\psi_1(x_1) - \psi_1(t_1))^{\mu_1 - 1} \left| E_{\rho_1, \mu_1}^{\gamma_1} [\omega_1 (\psi_1(x_1) - \psi_1(t_1))^{\rho_1}] \right| \right\}^p dt_1 \right]^{\frac{1}{p}} \left[ \int_{a_2}^{x_2} \left\{ (\psi_2(x_2) - \psi_2(t_2))^{\mu_2 - 1} \left| E_{\rho_2, \mu_2}^{\gamma_2} [\omega_2 (\psi_2(x_2) - \psi_2(t_2))^{\rho_2}] \right| \right\}^q dt_2 \right]^{\frac{1}{q}} \|\psi_1' f_1\|_q \|\psi_2' f_2\|_p \leq \quad (2.30)$$

(using Young's inequality for  $a, b \geq 0$ ,  $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$ )

$$\left\{ \frac{\left[ \int_{a_1}^{x_1} \left\{ (\psi_1(x_1) - \psi_1(t_1))^{\mu_1 - 1} \left| E_{\rho_1, \mu_1}^{\gamma_1} [\omega_1 (\psi_1(x_1) - \psi_1(t_1))^{\rho_1}] \right| \right\}^p dt_1 \right]^{\frac{1}{p}}}{p} + \frac{\left[ \int_{a_2}^{x_2} \left\{ (\psi_2(x_2) - \psi_2(t_2))^{\mu_2 - 1} \left| E_{\rho_2, \mu_2}^{\gamma_2} [\omega_2 (\psi_2(x_2) - \psi_2(t_2))^{\rho_2}] \right| \right\}^q dt_2 \right]^{\frac{1}{q}}}{q} \right\} \|\psi_1' f_1\|_q \|\psi_2' f_2\|_p,$$

$\forall x_i \in [a_i, b_i], i = 1, 2.$

So far we have

$$\frac{\left| \left( e^{\gamma_1; \psi_1}_{\rho_1, \mu_1, \omega_1, a_1 + f_1} \right) (x_1) \right| \left| \left( e^{\gamma_2; \psi_2}_{\rho_2, \mu_2, \omega_2, a_2 + f_2} \right) (x_2) \right|}{\left\{ \frac{\left[ \int_{a_1}^{x_1} \left\{ (\psi_1(x_1) - \psi_1(t_1))^{\mu_1 - 1} \left| E_{\rho_1, \mu_1}^{\gamma_1} [\omega_1(\psi_1(x_1) - \psi_1(t_1))^{\rho_1}] \right| \right\}^p dt_1 \right]_+}{p} + \frac{\left[ \int_{a_2}^{x_2} \left\{ (\psi_2(x_2) - \psi_2(t_2))^{\mu_2 - 1} \left| E_{\rho_2, \mu_2}^{\gamma_2} [\omega_2(\psi_2(x_2) - \psi_2(t_2))^{\rho_2}] \right| \right\}^q dt_2 \right]}{q} \right\}} \leq \|\psi'_1 f_1\|_q \|\psi'_2 f_2\|_p, \quad (2.31)$$

$\forall x_i \in [a_i, b_i], i = 1, 2.$

The denominator in (2.31) can be zero only when  $x_1 = a_1$  and  $x_2 = a_2$ . Therefore we obtain (2.25) by integrating (2.31) over  $[a_1, b_1] \times [a_2, b_2]$ .  $\square$

It follows the corresponding to (2.25) right side inequality.

**Theorem 2.10.** *All as in Theorem 2.9. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left| \left( e^{\gamma_1; \psi_1}_{\rho_1, \mu_1, \omega_1, b_1 - f_1} \right) (x_1) \right| \left| \left( e^{\gamma_2; \psi_2}_{\rho_2, \mu_2, \omega_2, b_2 - f_2} \right) (x_2) \right| dx_1 dx_2}{\left\{ \frac{\left[ \int_{x_1}^{b_1} \left\{ (\psi_1(t_1) - \psi_1(x_1))^{\mu_1 - 1} \left| E_{\rho_1, \mu_1}^{\gamma_1} [\omega_1(\psi_1(t_1) - \psi_1(x_1))^{\rho_1}] \right| \right\}^p dt_1 \right]_+}{p} + \frac{\left[ \int_{x_2}^{b_2} \left\{ (\psi_2(t_2) - \psi_2(x_2))^{\mu_2 - 1} \left| E_{\rho_2, \mu_2}^{\gamma_2} [\omega_2(\psi_2(t_2) - \psi_2(x_2))^{\rho_2}] \right| \right\}^q dt_2 \right]}{q} \right\}} \leq (b_1 - a_1) (b_2 - a_2) \|\psi'_1 f_1\|_q \|\psi'_2 f_2\|_p. \quad (2.32)$$

*Proof.* As similar to the proof of Theorem 2.9 is omitted.  $\square$

We continue with applications of Theorems 2.9, 2.10.

**Theorem 2.11.** *Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; i = 1, 2.$  Let  $[a_i, b_i] \subset \mathbb{R}, \psi_i \in C^{\max(N_1, N_2)}([a_i, b_i]), \psi'_i \neq 0,$  and strictly increasing;  $f_i \in C^{N_i}([a_i, b_i]),$  where  $N_i = \lceil \mu_i \rceil, 0 < \mu_i \notin \mathbb{N}.$  Here  $\rho_i > 0; \gamma_i, \omega_i \in \mathbb{R}.$  Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left| \left( {}^C D_{\rho_1, \mu_1, \omega_1, a_1 + f_1}^{\gamma_1; \psi_1} \right) (x_1) \right| \left| \left( {}^C D_{\rho_2, \mu_2, \omega_2, a_2 + f_2}^{\gamma_2; \psi_2} \right) (x_2) \right| dx_1 dx_2}{\left\{ \frac{\left[ \int_{a_1}^{x_1} \left\{ (\psi_1(x_1) - \psi_1(t_1))^{N_1 - \mu_1 - 1} \left| E_{\rho_1, N_1 - \mu_1}^{-\gamma_1} [\omega_1(\psi_1(x_1) - \psi_1(t_1))^{\rho_1}] \right| \right\}^p dt_1 \right]_+}{p} + \frac{\left[ \int_{a_2}^{x_2} \left\{ (\psi_2(x_2) - \psi_2(t_2))^{N_2 - \mu_2 - 1} \left| E_{\rho_2, N_2 - \mu_2}^{-\gamma_2} [\omega_2(\psi_2(x_2) - \psi_2(t_2))^{\rho_2}] \right| \right\}^q dt_2 \right]}{q} \right\}} \leq (b_1 - a_1) (b_2 - a_2) \left\| \psi'_1 f_1^{[N_1]} \right\|_q \left\| \psi'_2 f_2^{[N_2]} \right\|_p. \quad (2.33)$$

*Proof.* By Theorem 2.9 and (1.2), (1.6).  $\square$

We also give

**Theorem 2.12.** *All as in Theorem 2.11. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left| \left( {}^C D_{\rho_1, \mu_1, \omega_1, b_1}^{\gamma_1; \psi_1} f_1 \right) (x_1) \right| \left| \left( {}^C D_{\rho_2, \mu_2, \omega_2, b_2}^{\gamma_2; \psi_2} f_2 \right) (x_2) \right| dx_1 dx_2}{\left\{ \frac{\left[ \int_{x_1}^{b_1} \left\{ (\psi_1(t_1) - \psi_1(x_1))^{N_1 - \mu_1 - 1} \left| E_{\rho_1, N_1 - \mu_1}^{-\gamma_1} [\omega_1 (\psi_1(t_1) - \psi_1(x_1))^{\rho_1}] \right\}^p dt_1 \right]_+}{p} + \frac{\left[ \int_{x_2}^{b_2} \left\{ (\psi_2(t_2) - \psi_2(x_2))^{N_2 - \mu_2 - 1} \left| E_{\rho_2, N_2 - \mu_2}^{-\gamma_2} [\omega_2 (\psi_2(t_2) - \psi_2(x_2))^{\rho_2}] \right\}^q dt_2 \right]_+}{q} \right\}} \right\}} \leq (b_1 - a_1) (b_2 - a_2) \left\| \psi'_1 f_1^{[N_1]} \right\|_q \left\| \psi'_2 f_2^{[N_2]} \right\|_p. \tag{2.34}$$

*Proof.* By Theorem 2.10 and (1.3), (1.7). □

We present

**Theorem 2.13.** *Let  $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; i = 1, 2$ . Let  $[a_i, b_i] \subset \mathbb{R}, \psi_i \in C^{\max(N_i, N_2)}([a_i, b_i]), \psi'_i \neq 0$ , and strictly increasing;  $f_i \in C^{N_i}([a_i, b_i])$ , where  $N_i = [\mu_i], 0 < \mu_i \notin \mathbb{N}$ . Here  $\rho_i > 0; \gamma_i, \omega_i \in \mathbb{R}$  and  $\xi_i = \mu_i + \beta_i (N_i - \mu_i), i = 1, 2$ , where  $0 \leq \beta_i \leq 1$ . Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left| \left( {}^H \mathbb{D}_{\rho_1, \mu_1, \omega_1, a_1}^{\gamma_1, \beta_1; \psi_1} f_1 \right) (x_1) \right| \left| \left( {}^H \mathbb{D}_{\rho_2, \mu_2, \omega_2, a_2}^{\gamma_2, \beta_2; \psi_2} f_2 \right) (x_2) \right| dx_1 dx_2}{\left\{ \frac{\left[ \int_{a_1}^{x_1} \left\{ (\psi_1(x_1) - \psi_1(t_1))^{\xi_1 - \mu_1 - 1} \left| E_{\rho_1, \xi_1 - \mu_1}^{-\gamma_1 \beta_1} [\omega_1 (\psi_1(x_1) - \psi_1(t_1))^{\rho_1}] \right\}^p dt_1 \right]_+}{p} + \frac{\left[ \int_{a_2}^{x_2} \left\{ (\psi_2(x_2) - \psi_2(t_2))^{\xi_2 - \mu_2 - 1} \left| E_{\rho_2, \xi_2 - \mu_2}^{-\gamma_2 \beta_2} [\omega_2 (\psi_2(x_2) - \psi_2(t_2))^{\rho_2}] \right\}^q dt_2 \right]_+}{q} \right\}} \right\}} \leq (b_1 - a_1) (b_2 - a_2) \left\| \psi'_1 {}^{RL} D_{\rho_1, \xi_1, \omega_1, a_1}^{\gamma_1(1-\beta_1); \psi_1} f_1 \right\|_q \left\| \psi'_2 {}^{RL} D_{\rho_2, \xi_2, \omega_2, a_2}^{\gamma_2(1-\beta_2); \psi_2} f_2 \right\|_p. \tag{2.35}$$

*Proof.* By Theorem 2.9 and (1.15). □

We also give

**Theorem 2.14.** *All as in Theorem 2.13. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left| \left( {}^H \mathbb{D}_{\rho_1, \mu_1, \omega_1, b_1}^{\gamma_1, \beta_1; \psi_1} f_1 \right) (x_1) \right| \left| \left( {}^H \mathbb{D}_{\rho_2, \mu_2, \omega_2, b_2}^{\gamma_2, \beta_2; \psi_2} f_2 \right) (x_2) \right| dx_1 dx_2}{\left\{ \frac{\left[ \int_{a_1}^{b_1} \left\{ (\psi_1(t_1) - \psi_1(x_1))^{\xi_1 - \mu_1 - 1} \left| E_{\rho_1, \xi_1 - \mu_1}^{-\gamma_1 \beta_1} [\omega_1 (\psi_1(t_1) - \psi_1(x_1))^{\rho_1}] \right\}^p dt_1 \right]_+}{p} + \frac{\left[ \int_{a_2}^{b_2} \left\{ (\psi_2(t_2) - \psi_2(x_2))^{\xi_2 - \mu_2 - 1} \left| E_{\rho_2, \xi_2 - \mu_2}^{-\gamma_2 \beta_2} [\omega_2 (\psi_2(t_2) - \psi_2(x_2))^{\rho_2}] \right\}^q dt_2 \right]_+}{q} \right\}} \right\}} \leq (b_1 - a_1) (b_2 - a_2) \left\| \psi'_1 {}^{RL} D_{\rho_1, \xi_1, \omega_1, b_1}^{\gamma_1(1-\beta_1); \psi_1} f_1 \right\|_q \left\| \psi'_2 {}^{RL} D_{\rho_2, \xi_2, \omega_2, b_2}^{\gamma_2(1-\beta_2); \psi_2} f_2 \right\|_p. \tag{2.36}$$

*Proof.* By Theorem 2.10 and (1.16). □

### 3 Appendix

We give the following important fundamental results:

**Theorem 3.1.** *Let  $\rho, \mu > 0$ ;  $\gamma, \omega \in \mathbb{R}$ ; and  $\psi \in C^1([a, b])$  increasing,  $f \in C([a, b])$ . Then  $(e^{\gamma; \psi}_{\rho, \mu, \omega, a+} f), (e^{\gamma; \psi}_{\rho, \mu, \omega, b-} f) \in C([a, b])$ .*

*Proof.* We only prove  $(e^{\gamma; \psi}_{\rho, \mu, \omega, a+} f) \in C([a, b])$ . We skip the proof for the other is similar.

We consider the power series

$$\bar{E}_{\rho, \mu}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{|\gamma|_k}{k! \Gamma(\rho k + \mu) (\rho k + \mu)} z^k, \quad z \in \mathbb{R}. \tag{3.1}$$

We form

$$\bar{R}^{-1} := \lim_{k \rightarrow \infty} \frac{\frac{|\gamma|_{k+1}}{(k+1)! \Gamma(\rho(k+1)+\mu) (\rho(k+1)+\mu)}}{\frac{|\gamma|_k}{k! \Gamma(\rho k + \mu) (\rho k + \mu)}} = \lim_{k \rightarrow \infty} \frac{\frac{|\gamma+k|}{(k+1) \Gamma(\rho(k+1)+\mu) (\rho(k+1)+\mu)}}{\frac{1}{\Gamma(\rho k + \mu) (\rho k + \mu)}} = \tag{3.2}$$

$$\lim_{k \rightarrow \infty} \frac{|\gamma+k| \Gamma(\rho k + \mu) (\rho k + \mu)}{(k+1) \Gamma(\rho(k+1)+\mu) (\rho(k+1)+\mu)} = \lim_{k \rightarrow \infty} \left( \frac{|\gamma+k| \Gamma(\rho k + \mu)}{(k+1) \Gamma(\rho(k+1)+\mu)} \right) \lim_{k \rightarrow \infty} \left( \frac{\rho k + \mu}{(\rho k + \mu) + \rho} \right) =: (\Xi). \tag{3.3}$$

Notice that

$$\lim_{k \rightarrow \infty} \left( \frac{\rho k + \mu}{(\rho k + \mu) + \rho} \right) = 1. \tag{3.4}$$

From (1.1) we have that its radius of convergence is

$$R = \lim_{k \rightarrow \infty} \frac{\frac{|\gamma|_k}{k! \Gamma(\rho k + \mu)}}{\frac{|\gamma|_{k+1}}{(k+1)! \Gamma(\rho(k+1)+\mu)}} = \lim_{k \rightarrow \infty} \frac{\frac{1}{\Gamma(\rho k + \mu)}}{\frac{|\gamma+k|}{(k+1) \Gamma(\rho(k+1)+\mu)}} = \lim_{k \rightarrow \infty} \frac{(k+1) \Gamma(\rho(k+1)+\mu)}{|\gamma+k| \Gamma(\rho k + \mu)} = \infty,$$

because (1.1) is an entire function.

Therefore, we have that

$$\lim_{k \rightarrow \infty} \frac{|\gamma+k| \Gamma(\rho k + \mu)}{(k+1) \Gamma(\rho(k+1)+\mu)} = 0.$$

Consequently by (3.3), (3.4), we get that  $(\Xi) = 0$ . Thus  $\bar{R}^{-1} = 0$  and the radius of convergence of  $\bar{E}_{\rho, \mu}^{\gamma}(z)$ , see (3.1), is  $\bar{R} = \infty$ , hence (3.1) is convergent everywhere.

Consequently it holds

$$\sum_{k=0}^{\infty} \frac{|\gamma|_k (|\omega| (\psi(x) - \psi(a))^\rho)^k}{k! \Gamma(\rho k + \mu) (\rho k + \mu)} < \infty, \tag{3.5}$$

$\forall x \in [a, b]$ .

We notice that

$$\sum_{k=0}^{\infty} \frac{|\gamma|_k |\omega|^k}{k! \Gamma(\rho k + \mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\rho k + \mu) - 1} |f(t)| dt \leq$$

$$\|f\|_\infty \sum_{k=0}^\infty \frac{|\gamma)_k| |\omega|^k}{k! \Gamma(\rho k + \mu)} \frac{(\psi(x) - \psi(a))^{\rho k + \mu}}{\rho k + \mu} \leq \tag{3.6}$$

$$\|f\|_\infty (\psi(b) - \psi(a))^\mu \sum_{k=0}^\infty \frac{|\gamma)_k| (|\omega| (\psi(x) - \psi(a))^\rho)^k}{k! \Gamma(\rho k + \mu) (\rho k + \mu)} \stackrel{(3.5)}{<} \infty.$$

Consequently, by [5, p. 175], we derive

$$\begin{aligned} \left( e_{\rho, \mu, \omega, a+}^{\gamma; \psi} f \right) (x) &\stackrel{(1.2)}{=} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\mu-1} \left( \sum_{k=0}^\infty \frac{(\gamma)_k}{k! \Gamma(\rho k + \mu)} (\omega (\psi(x) - \psi(t))^\rho)^k \right) f(t) dt \\ &= \sum_{k=0}^\infty \frac{(\gamma)_k \omega^k}{k! \Gamma(\rho k + \mu)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\rho k + \mu) - 1} f(t) dt, \end{aligned} \tag{3.7}$$

$\forall x \in [a, b]$ .

By [2, p. 98], we obtain that the function

$$\lambda_{\rho, \mu}^{(k)}(f, x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\rho k + \mu) - 1} f(t) dt,$$

$x \in [a, b]$ , is absolutely continuous for  $\rho k + \mu \geq 1$  and continuous for  $\rho k + \mu \in (0, 1)$ ;  $\psi \in C^1([a, b])$  and increasing.

That is always  $\lambda_{\rho, \mu}^{(k)}(|f|, x) \in C([a, b])$ , for all  $k = 0, 1, \dots$

By (3.5), one can derive that

$$\sum_{k=0}^\infty \frac{|\gamma)_k| |\omega|^k}{k! \Gamma(\rho k + \mu)} \lambda_{\rho, \mu}^{(k)}(|f|, x) \leq$$

$$\|f\|_\infty (\psi(b) - \psi(a))^\mu \sum_{k=0}^\infty \frac{|\gamma)_k| (|\omega| (\psi(b) - \psi(a))^\rho)^k}{k! \Gamma(\rho k + \mu) (\rho k + \mu)} < \infty. \tag{3.8}$$

Notice that

$$\begin{aligned} \left| \lambda_{\rho, \mu}^{(k)}(f, x) \right| &\leq \lambda_{\rho, \mu}^{(k)}(|f|, x) = \int_a^x \psi'(t) (\psi(x) - \psi(t))^{(\rho k + \mu) - 1} |f(t)| dt \\ &\leq \|f\|_\infty \frac{(\psi(b) - \psi(a))^{(\rho k + \mu)}}{(\rho k + \mu)}, \quad k = 0, 1, \dots \end{aligned} \tag{3.9}$$

And even more we get:

$$\begin{aligned} \frac{|\gamma)_k| |\omega|^k}{k! \Gamma(\rho k + \mu)} \left| \lambda_{\rho, \mu}^{(k)}(f, x) \right| &\leq \frac{|\gamma)_k| |\omega|^k}{k! \Gamma(\rho k + \mu)} \lambda_{\rho, \mu}^{(k)}(|f|, x) \leq \\ \left( \frac{|\gamma)_k| |\omega|^k}{k! \Gamma(\rho k + \mu)} \right) \frac{\|f\|_\infty (\psi(b) - \psi(a))^{(\rho k + \mu)}}{(\rho k + \mu)} &=: M_k, \quad k = 0, 1, \dots; \end{aligned} \tag{3.10}$$

and by (3.8) that  $\sum_{k=0}^\infty M_k < \infty$ , converges.

By Weierstrass  $M$ -test we get that  $\sum_{k=0}^\infty \frac{(\gamma)_k \omega^k}{k! \Gamma(\rho k + \mu)} \lambda_{\rho, \mu}^{(k)}(f, x)$  is uniformly and absolutely convergent for  $x \in [a, b]$ .

Consequently by (3.7) we derive that  $\left( e_{\rho, \mu, \omega, a+}^{\gamma; \psi} f \right) \in C([a, b])$ . The proof is completed. □



We finish with

**Corollary 3.2.** *All as in Theorem 3.1. We have that*

$$\left\| e_{\rho, \mu, \omega, a+(b-)}^{\gamma; \psi} f \right\|_{\infty} \leq \left( \sum_{k=0}^{\infty} \frac{|(\gamma)_k| |\omega|^k (\psi(b) - \psi(a))^{\rho k + \mu}}{k! \Gamma(\rho k + \mu + 1)} \right) \|f\|_{\infty} < +\infty. \quad (3.11)$$

*That is  $e_{\rho, \mu, \omega, a+(b-)}^{\gamma; \psi}$  are bounded linear operators and positive operators if  $\gamma, \omega > 0$ .*

*Proof.* By (3.7), (3.8). □

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