CUBO, A Mathematical Journal
Vol. 23, no. 02, pp. 207–224, August 2021
DOI: 10.4067/S0719-06462021000200207

Coincidence point results of nonlinear contractive mappings in partially ordered metric spaces

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ABSTRACT

In this paper, we proved some coincidence point results for f-nondecreasing self-mapping satisfying certain rational type contractions in the context of a metric space endowed with a partial order. Moreover, some consequences of the main result are given by involving integral type contractions in the space. Some numerical examples are illustrated to support our results. As an application, we have discussed the existence of a unique solution of integral equation.

RESUMEN

En este artículo, probamos algunos resultados sobre puntos de coincidencia para un auto-mapeo no decreciente f satisfaciendo ciertas contracciones de tipo racional en el contexto de un espacio métrico dotado de un orden parcial. Más aún, se entregan algunas consecuencias del resultado principal que involucran contracciones de tipo integral en el espacio. Se ilustran algunos ejemplos numéricos en apoyo a nuestros resultados. Como una aplicación, discutimos la existencia de una única solución de una ecuación integral.

Keywords and Phrases: Ordered metric spaces; rational contractions; compatible mappings; weakly compatible mappings; coupled fixed point; common fixed point.

2020 AMS Mathematics Subject Classification: 41A50, 47H10.

Accepted: 01 April, 2021
Received: 25 Nov, 2020

1 Introduction

A remarkable fixed point theorem was first introduced by Banach [4] in 1922, which is one of the most influential results in analysis. It is being used widely in many different areas of mathematics and its applications. It needs the structure of complete metric spaces together with a contractive condition on the self map which is easy to test in many circumstances. Basically this principle gives a sequence of approximate solutions and also give a valuable information about the convergence rate of a fixed point. This kind of iteration process has been used both in mathematics and computer science. In particular, fixed point iterations together with monotone iterative techniques are the central methods when solving a large class of problems in theoretical and applied mathematics and play an important role in many algorithms. Many authors have extended this theorem by introducing more generalized contractive conditions, which imply the existence of a fixed point [6, 7, 8, 9, 11, 12, 13, 14, 15, 16].

The existence of fixed point results for self-mappings in partially ordered sets have been considered first by Ran and Reurings [36] and presented some applications to matrix equations therein. These results were again generalized and extended by Nieto et al. [32, 33] in partially ordered sets and applied their results to study the ordinary differential equations. Prominent works on various existence and uniqueness theorems on fixed point and common fixed point for monotone mappings in cone metric spaces, partially ordered metric spaces and others spaces, refer the readers to [5, 10, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45], which generate natural interest to establish usable fixed point theorems by weakening its hypothesis. Various types of contraction conditions have been used to find a fixed point of a single and multivalued mappings on metric spaces by Altun et al. [1], Aslantas et al. [2, 3], and Sahin et al. [37]. It is well known that a powerful technique for proving existence results for nonlinear problems is the method of upper and lower solutions. In many cases it is possible to find a minimal and a maximal solution between the lower and the upper solution by an iterative scheme: the monotone iterative technique. This method provides a constructive procedure for the solutions and it is also useful for the investigation of qualitative properties of solutions. This method has been used to acquire the unique solution of periodic boundary value problems of ordinary and partial differential equations, integro ordinary and partial differential equations by several authors, some of which are in [23, 32, 33].

The aim of this paper is to prove the coincidence point and common fixed point results for f-nondecreasing self-mapping satisfying generalized contractive conditions of rational type in the context of partially ordered metric spaces. These results generalize and extend the result of [7, 12, 14, 25, 26] in partially ordered metric spaces. Some consequences of the main results are given in terms of integral type contractions in the same space. Further, some examples and an application for the existence of the unique solution for an integral equation are presented at the end.

2 Preliminaries

The following definitions are frequently used in our study.

Definition 2.1. *[40] The triple $(X, d, \leq)$ is called a partially ordered metric space, if $(X, \leq)$ is a partially ordered set together with (X, d) is a metric space.*

Definition 2.2. *[40] If (X, d) is a complete metric space, then the triple $(X, d, \leq)$ is called a complete partially ordered metric space.*

Definition 2.3. *[38] Let $(X, \leq)$ be a partially ordered set. A mapping $f : X \to X$ is said to be strictly increasing (strictly decreasing), if $f(x) < f(y)$ $(f(x) > f(y))$ for all $x, y \in X$ with $x < y$.*

Definition 2.4. *[40] A point $x \in A$, where A is a non-empty subset of a partially ordered set $(X, \leq)$ is called a common fixed (coincidence) point of two self-mappings f and T, if $fx = Tx = x$ $(fx = Tx)$.*

Definition 2.5. *[39] The two self-mappings f and T defined over a subset A of a partially ordered metric space $(X, d, \leq)$ are called commuting, if $fTx = Tfx$ for all $x \in A$.*

Definition 2.6. *[39] Two self-mappings f and T defined over $A \subset X$ are compatible, if for any sequence $\{x_n\}$ with $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} Tx_n = \mu$ for some $\mu \in A$, then $\lim_{n\to+\infty} d(Tfx_n, fTx_n) = 0$.*

Definition 2.7. *[40] Two self-mappings f and T defined over $A \subset X$ are said to be weakly compatible, if they commute only at their coincidence points (i.e., if $fx = Tx$, then $fTx = Tfx$).*

Definition 2.8. *[40] Let f and T be two self-mappings defined over a partially ordered set $(X, \leq)$. A mapping T is called monotone f-nondecreasing, if*

$$fx \leq fy \text{ implies } Tx \leq Ty, \text{ for all } x, y \in X.$$

Definition 2.9. *[38] Let A be a non-empty subset of a partially ordered set $(X, \leq)$. If every two elements of A are comparable, then it is called a well ordered set.*

Definition 2.10. *[39] A partially ordered metric space $(X, d, \leq)$ is called an ordered complete, if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, one of the following conditions holds:*

- *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \leq x$, for all $n \in \mathbb{N}$ that is, $x = \sup\{x_n\}$ or,*
- *if $\{x_n\}$ is a non-increasing sequence in X such that $x_n \to x$ implies $x \leq x_n$, for all $n \in \mathbb{N}$ that is, $x = \inf\{x_n\}$.*

3 Main Results

We start this section with the following coincidence point theorem in the context of a partially ordered metric space.

Theorem 3.1. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following condition*

$$\begin{aligned} d(Tx,Ty) \leq{}& \alpha\frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \beta\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} \\ &+ \gamma\,[d(fx,Tx)+d(fy,Ty)] + \delta\,[d(fx,Ty)+d(fy,Tx)] \\ &+ \lambda d(fx,fy), \end{aligned} \tag{3.1}$$

for all x, y in X for which $fx \neq fy$ are comparable, and for some $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leq \alpha + \beta + 2(\gamma + \delta) + \lambda < 1$. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings f and T are compatible, then f and T have a coincidence point in X.

Proof. Suppose for some $x_0 \in X$ such that $fx_0 \leq Tx_0$. From the hypothesis, we have $T(X) \subseteq f(X)$, then choose a point $x_1 \in X$ such that $fx_1 = Tx_0$. But $Tx_1 \in f(X)$, then there exists another point $x_2 \in X$ such that $fx_2 = Tx_1$. As by a similar argument above, we obtain a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$ for all $n \geq 0$.

Since, $fx_0 \leq Tx_0 = fx_1$ and T is monotone f-nondecreasing mapping, then we have that $Tx_0 \leq Tx_1$. Similarly, we get $Tx_1 \leq Tx_2$ as $fx_1 \leq fx_2$. Continuing the same process, we obtain that

$$Tx_0 \leq Tx_1 \leq ... \leq Tx_n \leq Tx_{n+1} \leq ... \,.$$

Now, we discuss the following two cases.

Case 1: If $d(Tx_{n_0}, Tx_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$, then $Tx_{n_0+1} = Tx_{n_0}$ and by the above argument, we have $Tx_{n_0+1} = Tx_{n_0} = fx_{n_0+1}$. Therefore, x_{n_0+1} is a coincidence point of T and f, and so we have the result.

Case 2: If $d(Tx_n, Tx_{n+1}) > 0$ for all $n \in \mathbb{N}$, then from contraction condition (3.1), we have

$$\begin{aligned} d(Tx_{n+1},Tx_n) \leq{}& \alpha\frac{d(fx_{n+1},Tx_{n+1})\,[1+d(fx_n,Tx_n)]}{1+d(fx_{n+1},fx_n)} + \beta\frac{d(fx_{n+1},Tx_{n+1})\ d(fx_n,Tx_n)}{d(fx_{n+1},fx_n)} \\ &+ \gamma\,[d(fx_{n+1},Tx_{n+1})+d(fx_n,Tx_n)] + \delta\,[d(fx_{n+1},Tx_n)+d(fx_n,Tx_{n+1})] \\ &+ \lambda d(fx_{n+1},fx_n), \end{aligned}$$

which implies that

$$\begin{aligned} d(Tx_{n+1},Tx_n) \leq{}& \alpha d(Tx_n,Tx_{n+1}) + \beta d(Tx_n,Tx_{n+1}) \\ &+ \gamma\,[d(Tx_n,Tx_{n+1})+d(Tx_{n-1},Tx_n)] \\ &+ \delta\,[d(Tx_n,Tx_n)+d(Tx_{n-1},Tx_{n+1})] + \lambda d(Tx_n,Tx_{n-1}). \end{aligned}$$

Therefore, we arrive at

$$d(Tx_{n+1}, Tx_n) \leq \left(\frac{\gamma+\delta+\lambda}{1-\alpha-\beta-\gamma-\delta}\right) d(Tx_n, Tx_{n-1}).$$

Continuing the same process up to n times, we obtain that

$$d(Tx_{n+1}, Tx_n) \leq \left(\frac{\gamma+\delta+\lambda}{1-\alpha-\beta-\gamma-\delta}\right)^n d(Tx_1, Tx_0).$$

Let $k = \frac{\gamma+\delta+\lambda}{1-\alpha-\beta-\gamma-\delta} < 1$. Moreover, from the triangular inequality for $m \geq n$, we have

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + ... + d(Tx_{n+1}, Tx_n) \\ &\leq \left(k^{m-1} + k^{m-2} + ... + k^n\right) d(Tx_1, Tx_0) \\ &\leq \frac{k^n}{1-k}\, d(Tx_1, Tx_0), \end{aligned}$$

as $m, n \to +\infty$, $d(Tx_m, Tx_n) \to 0$, this shows that the sequences $\{Tx_n\}$ is a Cauchy sequence in X. So, by the completeness of X, there exists a point $\mu \in X$ such that $Tx_n \to \mu$ as $n \to +\infty$.

The continuity of T implies that

$$\lim_{n\to+\infty} T(Tx_n) = T\left(\lim_{n\to+\infty} Tx_n\right) = T\mu.$$

Since, $fx_{n+1} = Tx_n$ then $fx_{n+1} \to \mu$ as $n \to +\infty$. Further, the compatibility of T and f, we have

$$\lim_{n\to+\infty} d(Tfx_n, fTx_n) = 0.$$

From the triangular inequality of a metric d, we have

$$d(T\mu, f\mu) = d(T\mu, Tfx_n) + d(Tfx_n, fTx_n) + d(fTx_n, f\mu),$$

on taking limit as $n \to +\infty$ in the above inequality and using the fact that T and f are continuous, we obtain that $d(T\mu, f\mu) = 0$. Thus, $T\mu = f\mu$. Hence, μ is a coincidence point of T and f in X. □

We obtain the following consequences from Theorem 3.1 on taking zero value to $\alpha, \beta, \gamma, \delta$ and λ as special cases.

Corollary 3.2. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the self-mappings f and T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following contraction conditions*

(a)

$$\begin{aligned} d(Tx, Ty) \leq{}& \alpha \frac{d(fx, Tx)\left[1 + d(fy, Ty)\right]}{1 + d(fx, fy)} + \gamma\left[d(fx, Tx) + d(fy, Ty)\right] \\ &+ \delta\left[d(fx, Ty) + d(fy, Tx)\right] + \lambda d(fx, fy), \end{aligned} \tag{3.2}$$

for some $\alpha, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leq \alpha + 2(\gamma + \delta) + \lambda < 1$,

(b)

$$d(Tx,Ty) \leq \alpha \frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \gamma\,[d(fx,Tx)+d(fy,Ty)] + \lambda d(fx,fy), \tag{3.3}$$

where $\alpha, \gamma, \lambda \in [0,1)$ *such that* $0 \leq \alpha + 2\gamma + \lambda < 1$,

(c)

$$d(Tx,Ty) \leq \alpha \frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \delta\,[d(fx,Ty)+d(fy,Tx)] + \lambda d(fx,fy), \tag{3.4}$$

there exist $\alpha, \delta, \lambda \in [0,1)$ *such that* $0 \leq \alpha + 2\delta + \lambda < 1$,

(d)

$$d(Tx,Ty) \leq \gamma\,[d(fx,Tx)+d(fy,Ty)] + \delta\,[d(fx,Ty)+d(fy,Tx)] + \lambda d(fx,fy), \tag{3.5}$$

for some $\gamma, \delta, \lambda \in [0,1)$ *with* $0 \leq 2(\gamma+\delta) + \lambda < 1$,

for all x, y *in* X *for which* $fx \neq fy$ *are comparable. If there exists a point* $x_0 \in X$ *such that* $fx_0 \leq Tx_0$ *and the mappings* T *and* f *are compatible, then* T *and* f *have a coincidence point in* X.

Corollary 3.3. *Let* $(X, d, \leq)$ *be a complete partially ordered metric space. Suppose that the mappings* $f, T : X \to X$ *are continuous,* T *is a monotone* f*-nondecreasing,* $T(X) \subseteq f(X)$ *and satisfying the following contraction conditions*

(i)

$$\begin{aligned} d(Tx,Ty) \leq \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} &+ \gamma\,[d(fx,Tx)+d(fy,Ty)] \\ &+\delta\,[d(fx,Ty)+d(fy,Tx)] + \lambda d(fx,fy), \end{aligned} \tag{3.6}$$

where $\beta, \gamma, \delta, \lambda \in [0,1)$ *such that* $0 \leq \beta + 2(\gamma+\delta) + \lambda < 1$,

(ii)

$$d(Tx,Ty) \leq \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \gamma\,[d(fx,Tx)+d(fy,Ty)] + \lambda d(fx,fy), \tag{3.7}$$

for some $\beta, \gamma, \lambda \in [0,1)$ *with* $0 \leq \beta + 2\gamma + \lambda < 1$,

(iii)

$$d(Tx,Ty) \leq \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \delta\,[d(fx,Ty)+d(fy,Tx)] + \lambda d(fx,fy), \tag{3.8}$$

there exist $\beta, \delta, \lambda \in [0,1)$ *such that* $0 \leq \beta + 2\delta + \lambda < 1$,

(iv)

$$d(Tx,Ty) \leq \alpha \frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \lambda d(fx,fy), \tag{3.9}$$

where $\alpha, \beta, \lambda \in [0,1)$ *such that* $0 \leq \alpha + \beta + \lambda < 1$,

for all x, y *in* X *for which* $fx \neq fy$ *are comparable. If there exists a point* $x_0 \in X$ *such that* $fx_0 \leq Tx_0$ *and the mappings* T *and* f *are compatible, then* T *and* f *have a coincidence point in* X.

Corollary 3.4. *Let* $(X, d, \leq)$ *be a complete partially ordered metric space. Suppose that* $T : X \to X$ *is a mapping such that for all comparable* $x, y \in X$, *the contraction condition(s) in Theorem 3.1 (or Corollaries 3.2 and 3.3) is satisfied. Assume that* T *satisfies the following hypotheses:*

(i). T *is continuous,*

(ii). $T(Tx) \leq Tx$ *for all* $x \in X$.

If there exists a point $x_0 \in X$ *such that* $x_0 \leq Tx_0$, *then* T *has a fixed point in* X.

Proof. Follow from Theorem 3.1 by taking $f = I_X$ (the identity map). □

We may remove the continuity criteria of T in Theorem 3.1, is still valid by assuming the following hypothesis in X:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Theorem 3.5. *Let* $(X, d, \leq)$ *be a complete partially ordered metric space. Suppose that* $T, f : X \to X$ *are two mappings such that* T *is a monotone* f*-nondecreasing,* $T(X) \subseteq f(X)$ *and satisfying*

$$\begin{aligned} d(Tx,Ty) \leq{}& \alpha \frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \beta \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} \\ &+ \gamma\,[d(fx,Tx)+d(fy,Ty)] + \delta\,[d(fx,Ty)+d(fy,Tx)] \\ &+ \lambda d(fx,fy), \end{aligned} \tag{3.10}$$

for all x, y *in* X *for which* $fx \neq fy$ *are comparable and where* $\alpha, \beta, \gamma, \delta, \lambda \in [0,1)$ *such that* $0 \leq \alpha + \beta + 2(\gamma + \delta) + \lambda < 1$. *Assume that there exists* $x_0 \in X$ *such that* $fx_0 \leq Tx_0$ *and* $\{x_n\}$ *is a non-decreasing sequence in* X *such that* $x_n \to x$, *then* $x_n \leq x$ *for all* $n \in \mathbb{N}$. *If* $f(X)$ *is a complete subset of* X, *then* T *and* f *have a coincidence point in* X.

Further, if T *and* f *are weakly compatible then* T *and* f *have a common fixed point in* X. *Moreover, the set of common fixed points of* T *and* f *are well ordered if and only if* T *and* f *have one and only one common fixed point in* X.

Proof. Suppose $f(X)$ is a complete subset of X. As we know from Theorem 3.1, the sequence $\{Tx_n\}$ is a Cauchy sequence and hence, $\{fx_n\}$ is also a Cauchy sequence in $(f(X), d)$ as $fx_{n+1} = Tx_n$ and $T(X) \subseteq f(X)$. Since $f(X)$ is complete then there exists $fu \in f(X)$ such that

$$\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} fx_n = fu. \tag{3.11}$$

Also note that the sequences $\{Tx_n\}$ and $\{fx_n\}$ are nondecreasing and from the hypothesis, we have $Tx_n \leq fu$ and $fx_n \leq fu$ for all $n \in \mathbb{N}$. Since T is a monotone f-nondecreasing, we get $Tx_n \leq Tu$ for all n. Letting $n \to +\infty$, we obtain $fu \leq Tu$.

Suppose that $fu < Tu$, define a sequence $\{u_n\}$ by $u_0 = u$ and $fu_{n+1} = Tu_n$ for all $n \in \mathbb{N}$. An argument similar to that in the proof of Theorem 3.1 yields that $\{fu_n\}$ is a nondecreasing sequence and

$$\lim_{n\to+\infty} fu_n = \lim_{n\to+\infty} Tu_n = fv \text{ for some } v \in X. \tag{3.12}$$

So from the hypothesis, we have that $\sup_{n\in N} fu_n \leq fv$ and $\sup_{n\in N} Tu_n \leq fv$.

Notice that

$$fx_n \leq fu \leq fu_1 \leq fu_2 \leq ... \leq fu_n \leq ... \leq fv.$$

Now, we discuss the following two cases:

Case 1: If there exists some $n_0 \geq 1$ with $fx_{n_0} = fu_{n_0}$, then we have

$$fx_{n_0} = fu = fu_{n_0} = fu_1 = Tu,$$

this is a contradiction to $fu < Tu$. Thus, $fu = Tu$, that is, u is a coincidence point of T and f in X.

Case 2: Suppose $fx_n \neq fu_{n+1}$ for all n. Then from condition (3.10), we have

$$\begin{aligned} d(fx_{n+1}, fu_{n+1}) &= d(Tx_n, Tu_n) \\ &\leq \alpha \frac{d(fx_n, Tx_n)\,[1 + d(fu_n, Tu_n)]}{1 + d(fx_n, fu_n)} + \beta \frac{d(fx_n, Tx_n)\; d(fu_n, Tu_n)}{d(fx_n, fu_n)} \\ &\quad + \gamma\,[d(fx_n, Tx_n) + d(fu_n, Tu_n)] + \delta\,[d(fx_n, Tu_n) + d(fu_n, Tx_n)] \\ &\quad + \lambda d(fx_n, fu_n). \end{aligned}$$

On taking limit as $n \to +\infty$ in the above inequality and from equations (3.11) and (3.12), we get

$$\begin{aligned} d(fu, fv) &\leq (2\delta + \lambda)\; d(fu, fv) \\ &< \; d(fu, fv), \text{ since } 2\delta + \lambda < 1. \end{aligned}$$

So, we have

$$fu = fv = fu_1 = Tu,$$

this is again a contradiction to $fu < Tu$. Hence, we conclude that u is a coincidence point of T and f in X.

Now, we suppose that T and f are weakly compatible. Let w be the coincidence point then

$$Tw = Tfz = fTz = fw, \text{ since } w = Tz = fz, \text{ for some } z \in X.$$

Now from (3.10), we have

$$\begin{aligned} d(Tz, Tw) &\leq \alpha \frac{d(fz, Tz)\,[1 + d(fw, Tw)]}{1 + d(fz, fw)} + \beta \frac{d(fz, Tz)\; d(fw, Tw)}{d(fz, fw)} \\ &\quad + \gamma\,[d(fz, Tz) + d(fw, Tw)] + \delta\,[d(fz, Tw) + d(fw, Tz)] + \lambda d(fz, fw) \\ &\leq (2\gamma + 2\delta + \lambda)\; d(Tz, Tw). \end{aligned}$$

As $2\gamma + 2\delta + \lambda < 1$, then $d(Tz, Tw) = 0$. Therefore, $Tz = Tw = fw = w$. Hence, w is a common fixed point of T and f in X.

Now, suppose that the set of common fixed points of T and f is well ordered, we have to show that the common fixed point of T and f is unique. Let u and v be two common fixed points of T and f such that $u \neq v$, then from condition (3.10), we have

$$\begin{aligned} d(u, v) &\leq \alpha \frac{d(fu, Tu)\,[1 + d(fv, Tv)]}{1 + d(fu, fv)} + \beta \frac{d(fu, Tu)\; d(fv, Tv)}{d(fu, fv)} \\ &\quad + \gamma\,[d(fu, Tu) + d(fv, Tv)] + \delta\,[d(fu, Tv) + d(fv, Tu)] + \lambda d(fu, fv) \\ &\leq (2\gamma + 2\delta + \lambda)\; d(u, v) \\ &< d(u, v), \text{ since } 2\gamma + 2\delta + \lambda < 1, \end{aligned}$$

which is a contradiction and hence, $u = v$. Conversely, suppose T and f have only one common fixed point, then the set of common fixed points of T and f being a singleton is well ordered. □

Besides, in Corollary 3.2 and Corollary 3.3 by relaxing the continuity criteria on T and satisfying the hypotheses given in Theorem 3.5, then T and f have a coincidence point, a common fixed point and its uniqueness in X.

Corollary 3.6. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a mapping such that for all comparable $x, y \in X$, the contraction condition* (3.10) *is satisfied. Suppose that the following hypotheses are satisfied*

(i). if $\{x_n\}$ is a non-decreasing sequence in X with respect to $\leq$ such that $x_n \to x \in X$ as $n \to +\infty$, then $x_n \leq x$, for all $n \in \mathbb{N}$ and

(ii). $T(Tx) \leq Tx$ for all $x \in X$.

If there exists a point $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point in X.

Proof. Follow from Theorem 3.5 by taking $f = I_X$ (the identity map). $\square$

Remark 3.7. *(i). If $\alpha = \gamma = \delta = 0$ in Theorem 3.1 and 3.5, we obtain Theorem 2.1 and 2.3 of Chandok [25].*

(ii). If $f = I$ and $\alpha = \gamma = \delta = 0$ in Theorem 3.1 and 3.5, then we get Theorem 2.1 and 2.3 of Harjani et al. [26].

Some other consequences of the main result for the self mappings involving the integral type contractions are as follows.

Let χ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(a) each φ is Lebesgue integrable function on every compact subset of $[0, +\infty)$ and

(b) for any $\epsilon > 0$, we have $\int_0^\epsilon \varphi(t)dt > 0$, for $t \in [0, +\infty)$.

Corollary 3.8. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the mappings $T, f : X \to X$ are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying*

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt \\ + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt \\ + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.13}$$

for all x, y in X for which $fx \neq fy$ are comparable, $\varphi \in \chi$ and where $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1)$ such that $0 \leq \alpha + \beta + 2(\gamma + \delta) + \lambda < 1$. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Similarly, we obtain the following results from Corollaries 3.2 and 3.3 in a complete partially ordered metric space.

Corollary 3.9. *Let $(X, d, \leq)$ be a complete partially ordered metric space. Suppose that the self-mappings f, T on X are continuous, T is a monotone f-nondecreasing, $T(X) \subseteq f(X)$ satisfying the following contraction conditions*

(a)

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt \\ + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.14}$$

for some $\alpha, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leq \alpha + 2(\gamma + \delta) + \lambda < 1$,

(b)

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt \\ + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.15}$$

where $\alpha, \gamma, \lambda \in [0,1)$ *with* $0 \leq \alpha + 2\gamma + \lambda < 1$,

(c)

$$\begin{aligned}\int_0^{d(Tx,Ty)} \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt \\ + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.16}$$

where $\alpha, \delta, \lambda \in [0,1)$ *such that* $0 \leq \alpha + 2\delta + \lambda < 1$,

(d)

$$\begin{aligned}\int_0^{d(Tx,Ty)} \leq \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt \\ + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.17}$$

there exist $\gamma, \delta, \lambda \in [0,1)$ *such that* $0 \leq 2(\gamma + \delta) + \lambda < 1$,

for all x, y *in* X *for which* $fx \neq fy$ *are comparable, and where* $\varphi \in \chi$. *If there exists a point* $x_0 \in X$ *such that* $fx_0 \leq Tx_0$ *and the mappings* T *and* f *are compatible, then* T *and* f *have a coincidence point in* X.

Corollary 3.10. *Let* $(X, d, \leq)$ *be a complete partially ordered metric space. Suppose that the mappings* $f, T : X \to X$ *are continuous,* T *is a monotone* f*-nondecreasing,* $T(X) \subseteq f(X)$ *and satisfying the following integral type contraction conditions:*

(i)

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt \\ + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.18}$$

for some $\beta, \gamma, \delta, \lambda \in [0,1)$ *with* $0 \leq \beta + 2(\gamma + \delta) + \lambda < 1$,

(ii)

$$\begin{aligned}\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \gamma \int_0^{d(fx,Tx)+d(fy,Ty)} \varphi(t)dt \\ + \lambda \int_0^{d(fx,fy)} \varphi(t)dt,\end{aligned} \tag{3.19}$$

where $\beta, \gamma, \lambda \in [0,1)$ *such that* $0 \leq \beta + 2\gamma + \lambda < 1$,

(iii)

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \delta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.20}$$

there exist $\beta, \delta, \lambda \in [0,1)$ such that $0 \leq \beta + 2\delta + \lambda < 1$,

(iv)

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(fx,Tx)[1+d(fy,Ty)]}{1+d(fx,fy)}} \varphi(t)dt + \beta \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \lambda \int_0^{d(fx,fy)} \varphi(t)dt, \tag{3.21}$$

where $\alpha, \beta, \lambda \in [0,1)$ with $0 \leq \alpha + \beta + \lambda < 1$,

for all x, y in X for which $fx \neq fy$ are comparable, and where $\varphi \in \chi$. If there exists a point $x_0 \in X$ such that $fx_0 \leq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Remark 3.11. *If $\alpha = \gamma = \delta = 0$ in Corollary 3.8, then we obtain the Corollary 2.5 of Chandok [25].*

Now, we give the examples for the main Theorem 3.1.

Example 3.12. *Define a metric $d : X \times X \to [0, +\infty)$ by $d(x,y) = |x - y|$, where $X = [0,1]$ with usual order $\leq$. Let T and f be two self mappings on X such that $Tx = \frac{x^2}{2}$ and $fx = \frac{2x^2}{1+x}$, then T and f have a coincidence point in X.*

Proof. Note that (X,d) is a complete metric space and thus, $(X,d,\leq)$ be a complete partially ordered metric space with respect to usual order $\leq$. Let $x_0 = 0 \in X$ then $fx_0 \leq Tx_0$ and also note that T and f are continuous, T is a monotone f-nondecreasing and $T(X) \subseteq f(X)$.

Now consider the following for any x, y in X with $x < y$,

$$\begin{aligned} d(Tx,Ty) &= \frac{1}{2}|x^2 - y^2| = \frac{1}{2}(x+y)|x-y| \leq \frac{2(x+y+xy)}{(1+x)(1+y)}|x-y| \\ &\leq \alpha \frac{2x^2|3-x|\left[(1+y)+y^2|3-y|\right]}{4(1+x)(1+y)+2|x-y|(x+y+xy)} + \frac{\beta}{4}\frac{x^2y^2}{(x+y+xy)}\frac{|x-3||y-3|}{|x-y|} \\ &\quad + \frac{\gamma}{2}\frac{x^2(1+y)|x-3| + y^2(1+x)|y-3|}{(1+x)(1+y)} \\ &\quad + \delta \frac{(1+y)|4x^2 - y^2(1+x)| + (1+x)|4y^2 - x^2(1+y)|}{2(1+x)(1+y)} + \lambda \frac{2(x+y+xy)}{(1+x)(1+y)}|x-y| \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha \frac{\frac{x^2|x-3|}{2(1+x)} \cdot \frac{2(1+y)+y^2|3-y|}{2(1+y)}}{1+\frac{2|x-y|(x+y+xy)}{(1+x)(1+y)}} + \beta \frac{\frac{x^2|x-3|}{2(1+x)} \cdot \frac{y^2|y-3|}{2(1+y)}}{2|x-y|\frac{x+y+xy}{(1+x)(1+y)}} + \gamma \left[\frac{x^2|x-3|}{2(1+x)} + \frac{y^2|y-3|}{2(1+y)}\right] \\
&\quad + \delta \left[\left|\frac{x^2}{(1+x)} - \frac{y^2}{2}\right| + \left|\frac{2y^2}{(1+y)} - \frac{x^2}{2}\right|\right] + \lambda \frac{2(x+y+xy)}{(1+x)(1+y)}|x-y| \\
&\leq \alpha \frac{d(fx,Tx)\,[1+d(fy,Ty)]}{1+d(fx,fy)} + \beta \frac{d(fx,Tx)\; d(fy,Ty)}{d(fx,fy)} + \gamma\,[d(fx,Tx)+d(fy,Ty)] \\
&\quad + \delta\,[d(fx,Ty)+d(fy,Tx)] + \lambda d(fx,fy).
\end{aligned}
$$

Then, the contraction condition in Theorem 3.1 holds by selecting proper values of $\alpha, \beta, \gamma, \delta, \lambda$ in $[0,1)$ such that $0 \leq \alpha + \beta + 2(\gamma+\delta) + \lambda < 1$. Therefore, T and f have a coincidence point $0 \in X$. $\square$

Example 3.13. *Define a distance function $d : X \times X \to [0,+\infty)$ by $d(x,y) = |x-y|$, where $X = [0,1]$ with usual order $\leq$. Let T and f be two self mappings on X such that $Tx = x^3$ and $fx = x^4$, then T and f have two coincidence points 0, 1 in X with $x_0 = \frac{1}{4}$.*

4 Applications

Now our aim is to give an existence theorem for a solution of the following integral equation.

$$h(x) = \int_0^M \mu(x,y,h(y))dy + g(x), \quad x \in [0,M], \tag{4.1}$$

where $M > 0$. Let $X = C[0,M]$ be the set of all continuous functions defined on $[0,M]$. Now, define $d : X \times X \to \mathbb{R}^+$ by

$$d(u,v) = \sup_{x \in [0,M]} \{|u(x)-v(x)|\}$$

then, $(X, \leq)$ is a partially ordered set. Now, we prove the following result.

Theorem 4.1. *Suppose the following hypotheses holds:*

(i) $\mu : [0,M] \times [0,M] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous,*

(ii) for each $x, y \in [0,M]$, we have

$$\mu\left(x, y, \int_0^M \mu(x,z,h(z))dz + g(x)\right) \leq \mu(x,y,h(y)),$$

(iii) there exists a continuous function $N : [0,M] \times [0,M] \to [0,+\infty]$ such that

$$|\mu(x,y,a) - \mu(x,y,b)| \leq N(x,y)|a-b| \text{ and}$$

(iv)

$$\sup_{x\in[0,M]}\int_0^M N(x,y)dy \le \gamma$$

for some $\gamma < 1$. Then, the integral equation (4.1) has a solution $a \in C[0,M]$.

Proof. Define $T : C[0,M] \to C[0,M]$ by

$$Tw(x) = \int_0^M \mu(x,y,w(x))dx + g(x),\ \ x \in [0,M].$$

Now, we have

$$\begin{aligned} T(Tw(x)) &= \int_0^M \mu(x,y,Tw(x))dx + g(x) \\ &= \int_0^M \mu\left(x,y,\int_0^M \mu(x,z,w(z))dz + g(x)\right)dx + g(x) \\ &\le \int_0^M \mu(x,y,w(z))dz + g(x) \\ &= Tw(x) \end{aligned}$$

Thus, we have $T(Tx) \le Tx$ for all $x \in C[0,M]$. For any $x^*, y^* \in C[0,M]$ with $x \le y$, we have

$$\begin{aligned} d(Tx^*,Ty^*) &= \sup_{x\in[0,M]} |Tx^*(x) - Ty^*(x)| \\ &= \sup_{x\in[0,M]} \left|\int_0^M \mu(x,y,x^*(x)) - \mu(x,y,y^*(x))dx\right| \\ &\le \sup_{x\in[0,M]} \int_0^M |\mu(x,y,x^*(x)) - \mu(x,y,y^*(x))|\,dx \\ &\le \sup_{x\in[0,M]} \int_0^M N(x,y)|x^*(x) - y^*(x)|dx \\ &\le \sup_{x\in[0,M]} |x^*(x) - y^*(x)| \sup_{x\in[0,M]} \int_0^M N(x,y)dx \\ &= d(x^*,y^*) \sup_{x\in[0,M]} \int_0^M N(x,y)dx \\ &\le \gamma d(x^*,y^*). \end{aligned}$$

Moreover, $\{x_n^*\}$ is a nondecreasing sequence in $C[0,M]$ such that $x_n^* \to x^*$, then $x_n^* \le x^*$ for all $n \in \mathbb{N}$. Thus all the required hypotheses of Corollary 3.6 are satisfied. Thus, there exists a solution $a \in C[0,M]$ of the integral equation (4.1). $\square$

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