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Curves in low dimensional projective spaces with the lowest ranks

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ABSTRACT

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. For each $q \in \mathbb{P}^r$ the X-rank $r_X(q)$ of q is the minimal number of points of X spanning q. A general point of $\mathbb{P}^r$ has X-rank $\lceil (r+1)/2 \rceil$. For $r = 3$ (resp. $r = 4$) we construct many smooth curves such that $r_X(q) \le 2$ (resp. $r_X(q) \le 3$) for all $q \in \mathbb{P}^r$ (the best possible upper bound). We also construct nodal curves with the same properties and almost all geometric genera allowed by Castelnuovo's upper bound for the arithmetic genus.

RESUMEN

Sea $X \subset \mathbb{P}^r$ una curva integral y no-degenerada. Para cada $q \in \mathbb{P}^r$ el X-rango $r_X(q)$ de q es el mínimo número de puntos de X que generan q. Un punto general de $\mathbb{P}^r$ tiene X-rango $\lceil (r+1)/2 \rceil$. Para $r = 3$ (resp. $r = 4$) construimos muchas curvas suaves tales que $r_X(q) \le 2$ (resp. $r_X(q) \le 3$) para todo $q \in \mathbb{P}^r$ (la mejor cota superior posible). También construimos curvas nodales con las mismas propiedades y casi todos los géneros geométricos permitidos por la cota superior de Castelnuovo para el género aritmético.

Keywords and Phrases: X-rank, projective curve, space curve, curve in projective spaces.

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1 Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field with characteristic 0. For each $q \in \mathbb{P}^r$ the *X-rank* $r_X(q)$ of q is the minimal cardinality of a finite set $S \subset X$ such that $q \in \langle S\rangle$, where $\langle\ \ \rangle$ denotes the linear span. An interesting problem is the maximum of all integers $r_X(q)$, $q \in \mathbb{P}^r$ ([2, 8]). An obvious lower bound for this integer is the *generic X-rank* $r_{\mathrm{gen}}(X)$, i.e. the only integer such there is a non-empty Zariski open subset $U \subset \mathbb{P}^r$ such that $r_X(q) = r_{\mathrm{gen}}(X)$ for all $q \in U$. For each positive integer t set $W^0_t(X) := \{q \in \mathbb{P}^r \mid r_X(q) = t\}$. Let $W_t(X)$ denote the closure of $W^0_t(X)$ in $\mathbb{P}^r$. If $t \le r_{\mathrm{gen}}(X)$ the algebraic set $W_t(X)$ is the t-secant variety $\sigma_t(X)$ of X. Hence if $1 \le t \le r_{\mathrm{gen}}(X)$ the algebraic set $W_t(X)$ is non-empty, irreducible and $\dim W_t(X) \le \min\{r, t(\dim X + 1) - 1\}$ with equality if $\dim X = 1$ ([1, Remark 1.6]). Thus $r_{\mathrm{gen}}(X) = \lceil (r + 1)/2\rceil$ if $\dim X = 1$. For $t > r_{\mathrm{gen}}(X)$ the geometry of $W_t(X)$ is described in [3, Theorem 3.1], assuming of course $W_t(X) \neq \emptyset$, i.e. $W^0_t(X) \neq \emptyset$.

We prove the following results.

Theorem 1.1. *Fix integers $b \ge a > 0$ such that $a + b \ge 5$. Set $d := a + b$ and $\gamma := ab - a - b + 1$. Then there exists an integral and non-degenerate nodal curve $X \subset \mathbb{P}^3$ with geometric genus g, $\deg(X) = d$, exactly $\gamma - g$ ordinary nodes and $W^0_3(X) = \emptyset$.*

Theorem 1.2. *Fix integers a, b such that $a \ge 2$ and $b \ge 2a + 3$. Set $d := a + b$ and $\gamma := 1 + ab - a(a + 1)/2 - b$. Fix an integer g such that $0 \le g \le \gamma$. Then there is an integral nodal curve $X \subset \mathbb{P}^4$ with degree d, geometric genus g, exactly $\gamma - g$ ordinary nodes and with $W^0_4(X) = \emptyset$.*

Question 1.1. *Is there an integral and non-degenerate curve $X \subset \mathbb{P}^5$ with $W^0_4(X) = \emptyset$? Take an odd integer $r > 5$. Is there an integral and non-degenerate curve $X \subset \mathbb{P}^r$ with $W^0_{(r+3)/2}(X) = \emptyset$?*

By [9, Theorem 1] $W^0_3(X) \neq \emptyset$ for X as in Theorem 1.1, but with $(a, b) \in \{(1, 2), (1, 3), (2, 2)\}$. The case $(a, b) = (3, 3)$ of Theorem 1.1 is [9, Theorem 2]. When $a \le b \le a + 1$ the integer γ appearing in Theorem 1.1 is the maximal arithmetic genus of all non-degenerate space curves ([6, Ch. III]).

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2 Preliminaries

Notation 2.1. *For any $q \in \mathbb{P}^r$ let $\ell_q : \mathbb{P}^r \setminus \{q\} \longrightarrow \mathbb{P}^{r-1}$ denote the linear projection from q.*

Let M be a projective scheme. Let $D \subset M$ be an effective Cartier divisor of M. For any zero-dimensional scheme $Z \subset M$ the *residual scheme* $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of M with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and hence $\mathrm{Res}_D(Z)$ is a zero-dimensional scheme. We have $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$ and for any line bundle $\mathcal{L}$ on M we have an exact sequence of coherent sheaves on M:

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \longrightarrow \mathcal{I}_Z \otimes \mathcal{L} \longrightarrow \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D} \longrightarrow 0 \tag{2.1}$$

We will call (2.1) the *residual exact sequence of* D or the *residual exact sequence of* D *in* M.

Remark 2.1. *Let M be a smooth, projective and rational surface. Thus $h^1(\mathcal{O}_M) = 0$. Assume that $\omega_M^\vee$ is ample. This will be true in the cases in which we apply this remark, i.e. the case in which M is the smooth quadric surface and the case in which M is the Hirzebruch surface F_1. Fix an integer $e \ge 2$, a very ample line bundle $\mathcal{L}$ on M and a nodal curve $D = D_1 \cup \cdots \cup D_e \in |\mathcal{L}|$ with each D_i a smooth and connected curve. Note that $p_a(D) = \sum_{i=1}^e p_a(D_i) + \sharp(\mathrm{Sing}(D)) + 1 - e$. Since $\mathcal{L}$ is very ample, D is connected. Since $\omega_M^\vee$ is ample, we have $D_i \cdot \omega_M < 0$ (intersection number) for all i. A subset $A \subseteq \mathrm{Sing}(D)$ is said to be a* disconnecting *set of nodes if $D \setminus A$ is not connected. Fix a set $A \subset \mathrm{Sing}(D)$ which is not disconnecting and set $g := p_a(D) - \sharp(A)$. With the terminology of [10] we will say that A is the set of* assigned nodes, *while the set $\mathrm{Sing}(D) \setminus A$ is the set of* unassigned nodes. *By [10, Corollary 2.14] there are an affine smooth and connected curve Δ, $o \in \Delta$, and a flat family $\{Y_t\}_{t\in\Delta}$ of elements of $|\mathcal{L}|$ such that $Y_o = D$ and Y_t is integral, nodal and with geometric genus g for all $t \in \Delta \setminus \{o\}$. Moreover, the sets $\{\mathrm{Sing}(Y_t)\}_{t\in\Delta\setminus\{o\}}$ have A as a limit. Thus $p_a(D) = \sharp(\mathrm{Sing}(D)) + 1 - e$. We do not impose (or claim) that all Y_t are singular at the points of A, because it would require very strong restrictions on the integer $\sharp(A)$, only that the nodes of the curves Y_t near D are near A and that Y_t has only $\sharp(A)$ nodes. The quoted result [10, Corollary 2.14] with movable assigned nodes is optimal, as shown by following particular case, the only one we will use. Assume that each D_i is rational. In this case for each integer g with $0 \le g \le p_a(D)$ there is a set of assigned nodes $A \subset \mathrm{Sing}(D)$ such that the corresponding family of nodal curves has as a general member an integral nodal curve with geometric genus g.*

Remark 2.2. *Let X be a smooth projective curve, $\mathcal{L}$ a line bundle on X and $V \subseteq H^0(\mathcal{L})$ a linear subspace. Set $g := p_a(X)$, $d := \deg(\mathcal{L})$ and $n := \dim V - 1$. Assume $n \ge 1$. For each $p \in X$ and each integer $t > 0$ set $V(-tp) := V \cap H^0(\mathcal{I}_{tp} \otimes \mathcal{L})$. We get $n+1$ integers $\dim V(-tp)$, $1 \le t \le n+1$ ([5, pp. 264–277]). This is also done in details in [9]. The point p is said to be an osculating point of the pair $(\mathcal{L}, V)$ (or of the linear system $\mathbb{P}V$) if $\dim(V(-(n+1)p)) > 0$. Since we are in characteristic zero, there are only finitely many osculating points of $(\mathcal{L}, V)$, say $p_1, \dots, p_s$, and at each point p_i one can associate a positive integer $w(p_i)$ (the weight of p_i), only depending on the $n+1$ integers $\dim V(-tp)$, $1 \le t \le n+1$. Moreover, there is an integer δ only depending on g, d and n such that $w(p_1) + \cdots + w(p_s) = \delta$. We have $w(p_i) = 1$ if and only if $\dim V(-np_i) = \dim V(-(n+1)p_i) = 1$. Suppose for instance that $\mathbb{P}V$ induces an embedding of X into $\mathbb{P}^n$ and see X has a curve of $\mathbb{P}^n$. Since $V \subseteq H^0(\mathcal{L})$, X is non-degenerate. The point $p \in X$ is an osculating point if and only if there is a hyperplane $H \subset \mathbb{P}^n$ such that the connected component Z of the scheme $H \cap X$ with p has its reduction has degree $\ge n+1$, i.e. H contains the divisor*

$(n+1)p$. The integer $\deg(Z)$ is the order of contact of the osculating hyperplane H with X at p. The integer $\deg(Z)-n$ is a lower bound for the weight of p. All non-osculating points have weight 0.

3 Proof of Theorem 1.1

In this section we fix a smooth quadric surface $Q \subset \mathbb{P}^3$. For any irreducible curve $Y \subset \mathbb{P}^3$, Y not a line, let $\tau(Y)$ denote the tangential surface of Y, i.e. the closure in $\mathbb{P}^3$ of the union of all tangent lines of Y at its smooth points. $\tau(Y)$ is a plane if and only if $\langle Y \rangle$ is a plane.

Notation 3.1. *For any reduced curve $X \subset \mathbb{P}^3$ with no irreducible component contained in a plane let $T(X)$ be the set of all pairs (H,p), where $H \subset \mathbb{P}^3$ is a plane, $p \in H \cap X$ and the connected component of the scheme $H \cap X$ with p as its reduction has degree at least 5.*

Remark 3.1. *Let Δ a quasi-projective variety and $\mathcal{X} \subset \mathbb{P}^3 \times \Delta$ a closed algebraic set such that the restriction $u : \mathcal{X} \longrightarrow \Delta$ to $\mathcal{X}$ of the projection $\mathbb{P}^3 \times \Delta \longrightarrow \Delta$ is proper and flat. Assume that all fibers of u are reduced curves with no irreducible component contained in a plane. Let $T(\mathcal{X})$ or $T(u)$ denote the set of all triples (s,H,p), where $s \in \Delta$ and $(H,p) \in T(u^{-1}(s))$. The map $u_{T(\mathcal{X})} : T(\mathcal{X}) \longrightarrow \Delta$ is proper. Thus if Δ is irreducible and if $T(u^{-1}(s_0)) = \emptyset$ for some $s_0 \in \Delta$, then $T(u^{-1}(s)) = \emptyset$ for a general $s \in \Delta$.*

Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Fix $p \in X_{\mathrm{reg}}$. We say that p is a *flex point* of X or a *flex* of X or that the tangent line T_pX is a *flex tangent* of X if the connected component of the zero-dimensional scheme $T_pX \cap X$ with p as its reduction has degree at least 3. We say that p is a *stall point* of X or that T_pX is a *stall* of X if T_pX is not a flex tangent, but the osculating plane $O_p(X)$ of X at p has order of contact at least 4 with X at p. Thus a stall point is an osculating point which is not a flex point.

Remark 3.2. *Fix a smooth element Y either of $|\mathcal{O}_Q(1,1)|$ or of $|\mathcal{O}_Q(2,1)|$ or of $|\mathcal{O}_Q(1,2)|$. Since Y is a rational normal curve in its linear span, it is easy to check that $T(Y) = \emptyset$ and that each $q \in \tau(Y) \setminus Y$ is contained in at most 2 tangent lines of Y.*

We collect in the next remark some standard tools and ideas which are used in the proofs of Lemmas 3.1, 3.2 and 3.3 and which may be used in several other cases. In section 4 we will use this set-up for the Hirzebruch surface F_1 and the line bundle $\mathcal{O}_{F_1}(ah+bf)$.

Remark 3.3. *Fix positive integers a, b and an integral quasi-projective family $\mathcal{F}$ of zero-dimensional subschemes of the smooth quadric Q. Suppose you want to compute the dimension of the family Ψ of all $C \in |\mathcal{O}_Q(a,b)|$ containing at least one $Z \in \mathcal{F}$ or of the family Φ of all smooth $C \in |\mathcal{O}_Q(a,b)|$ containing at least one $Z \in \mathcal{F}$. In most lemmas we will need to check that*

$\dim\Phi < \dim|\mathcal{O}_Q(a,b)|$, i.e. that a general $C \in |\mathcal{O}_Q(a,b)|$ contains no $Z \in \mathcal{F}$. Consider the incidence variety $\mathbb{I} := \{(Z,C) \in \mathcal{F} \times |\mathcal{O}_Q(a,b)| : Z \in C\}$. Let $\pi_1 : \mathbb{I} \longrightarrow \mathcal{F}$ and $\pi_2 : \mathbb{I} \longrightarrow |\mathcal{O}_Q(a,b)|$ denote the restriction to $\mathbb{I}$ of the projections of $\mathcal{F} \times |\mathcal{O}_Q(a,b)|$ onto its factors. Note that $\Psi = \pi_2(\mathbb{I})$. The algebraic set $\mathbb{I}$ is a closed subset of $\mathcal{F} \times |\mathcal{O}_Q(a,b)|$. Thus by Chevalley's theorem Ψ is a constructible set ([7, ex. II.3.18 and II.3.19]). If $\mathbb{I}$ is irreducible, then Ψ is irreducible. Obviously $\Phi = \emptyset$, unless at least some $Z \in \mathcal{F}$ is curvilinear. Call $\mathcal{U}$ the set of all smooth $C \in |\mathcal{O}_Q(a,b)|$. Assume that at least some $Z \in \mathcal{F}$ is curvilinear and let $\mathcal{G}$ denote the set of all curvilinear $Z \in \mathcal{F}$. The set $\mathcal{G}$ is an open subset of $\mathcal{F}$. Since $\mathcal{F}$ is assumed to be irreducible, $\mathcal{G}$ is irreducible. Set $\mathbb{J} := \mathbb{I} \cap \mathcal{G} \times \mathcal{U}$. Usually, if we are only interested in smooth curves $C \in |\mathcal{O}_Q(a,b)|$ it is better to start with $\mathcal{G}$, i.e. take an irreducible family of curvilinear schemes. Thus from now on we assume $\mathcal{F} = \mathcal{G}$, but we use $\mathbb{I}$, i.e. we also consider singular curves, to quote below [7, III.9.3, III.9.6, III.9.7]. Suppose there is an integer $z > 0$ such that $h^0(Q, \mathcal{I}_Z(a,b)) = z$ for all $Z \in \mathcal{G}$. With this assumption all fibers of π_1 are projective spaces of dimension $z-1$. Hence π_1 is a proper flat map. Since $\mathcal{G}$ is assumed to be irreducible, $\mathbb{I}$ is irreducible and $\dim\mathbb{I} = \dim\mathcal{G} + z$ ([7, III.9.3, III.9.6, III.9.7]). Since $\mathbb{J}$ is a non-empty open subset of $\mathbb{I}$, $\mathbb{J}$ is irreducible and $\dim\mathbb{J} = \dim\mathbb{I} = \dim\mathcal{G} + z$. Thus Φ is irreducible and $\dim\Phi \le \dim\mathcal{G} + z$. If this inequality is not sufficient to conclude, one should look at a general $C \in \Phi$ and try to compute $\dim(\mathbb{J} \cap \pi_2^{-1}(C))$. Suppose $\dim(\mathbb{J} \cap \pi_2^{-1}(C)) = x$ for a general $C \in |\mathcal{O}_Q(a,b)|$. Then $\dim\Phi = \dim\mathcal{G} + z - x$. Since C is smooth and $\dim C = 1$, $\dim(\mathbb{J} \cap \pi_2^{-1}(C)) \le x$ if $\sharp(Z_{\mathrm{red}}) \le x$ for all $Z \in \mathbb{J} \cap \pi_2^{-1}(C)$. Moreover, $\dim(\mathbb{J} \cap \pi_2^{-1}(C)) = x$ if varying $Z \in \mathbb{J} \cap \pi_2^{-1}(C)$ the sets Z_{red} form an x-dimensional family of x distinct points of C. This set-up is classically summarized by the words " A dimensional count shows that Φ has dimension $\dim\mathcal{G} + z - x$ ". If our family $\mathcal{G}$ is not irreducible, we try to study separately each of its irreducible components. Now we drop the assumption that all integers $h^0(Q, \mathcal{I}_Z(a,b))$ are the same. There are a non-empty open subset $\mathcal{G}'$ of $\mathcal{G}$ and an integer z such that $h^0(Q, \mathcal{I}_Z(a,b)) = z$ for all $Z \in \mathcal{G}'$. Moreover, there are a positive integer s and integers $z_i \ge z$, $1 \le i \le s$, such that $\mathcal{G} \setminus \mathcal{G}'$ is the union of finitely many irreducible quasi-projective varieties, say $\mathcal{G} \setminus \mathcal{G}' = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_s$, such that $h^0(Q, \mathcal{I}_Z(a,b)) = z_i$ for all $Z \in \mathcal{G}_i$. Then we use the irreducible families $\mathcal{G}', \mathcal{G}_1, \ldots, \mathcal{G}_s$ of curvilinear schemes.

We will need only the case $a = 1$ of the next lemma, but its proof when $a \ge 2$ requires no modification.

Lemma 3.1. *Fix integers $a > 0$, $b > 0$ such that $a + b \ge 4$. Let D be a general element of $|\mathcal{O}_Q(a,b)|$. Then D has no flex and $T(D) = \emptyset$.*

Proof. We follow the classical approach outlined in Remark 3.3. The key step in the proof of the lemma is the computation of the integer $h^0(Q, \mathcal{I}_Z(a,b))$ for two types of zero-dimensional schemes Z.

With no loss of generality we may assume $b \geq a$ and hence $b \geq 2$. By Bertini's theorem D is smooth. Since $D \subset Q$, Bezout theorem implies that each flex tangent line of D is contained in Q and hence it is either an element of $|\mathcal{O}_Q(1,0)|$ or an element of $|\mathcal{O}_Q(0,1)|$.

(a) Take $L \in |\mathcal{O}_Q(1,0)|$ and any connected zero-dimensional scheme $F \subset L$ such that $\deg(F) = 3$. Since $\deg(\mathcal{O}_L(a,b)) = b \geq 2$, we have $h^1(L, \mathcal{I}_{F,L}(a,b)) = 0$. Since $h^1(\mathcal{O}_Q(0,b)) = 0$, the residual exact sequence of L gives $h^1(\mathcal{I}_F(a,b)) = 0$, i.e. $h^0(\mathcal{I}_F(a,b)) = h^0(\mathcal{O}_Q(a,b)) - 3$. Since $\dim |\mathcal{O}_Q(1,0)| = 1$ and each $L \in |\mathcal{O}_Q(1,0)|$ contains ∞^1 connected degree 3 subschemes, a general $D \in |\mathcal{O}_Q(a,b)|$ contains no F (for any L), i.e. no $L \in |\mathcal{O}_Q(1,0)|$ is a flex tangent of D.

(b) If $a \geq 2$ step (a) shows that no $R \in |\mathcal{O}_Q(0,1)|$ is a flex tangent of D. Now assume $a = 1$. Since $D \in |\mathcal{O}_Q(a,b)|$, we have $\deg(R \cap D) = 1$ for all $R \in |\mathcal{O}_Q(0,1)|$. Thus no element of $|\mathcal{O}_Q(0,1)|$ is a flex tangent line of D.

By steps (a) and (b) D has no flex. Thus it is sufficient to prove that each osculating plane of D has order of contact 4 with D at the osculating point. Fix a smooth element $A \in |\mathcal{O}_Q(1,1)|$ and $p \in A$. Let E be the connected zero-dimensional subscheme of A such that $E_{\mathrm{red}} = \{p\}$ and $\deg(E) = 5$.

Claim 1: We have $h^1(Q, \mathcal{I}_E(a,b)) = 0$.

Proof of Claim 1: We have $h^1(A, \mathcal{I}_{E,A}(a,b)) = 0$, because $A \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_A(1,b)) = b + 1 \geq 4$. Since $E \subset A$, $\mathrm{Res}_A(E) = \emptyset$. Thus it is sufficient to use the residual exact sequence of A in Q and that $h^1(\mathcal{O}_Q(0, b-1)) = 0$.

By Claim 1 we have $h^0(\mathcal{I}_E(a,b)) = (a+1)(b+1) - 5$ for all E. Since $\dim |\mathcal{O}_Q(1,1)| = 3$ and each smooth $A \in |\mathcal{O}_Q(1,1)|$ has ∞^1 points and hence ∞^1 schemes E's. Use Claim 1. □

Notation 3.2. *Let $D \subset Q$ be a reduced curve with no irreducible component of D being an element of $|\mathcal{O}_Q(1,0)|$ or $|\mathcal{O}_Q(1,0)|$ or $|\mathcal{O}_Q(1,1)|$. Let $F(D)$ be denote the set of all $C \in |\mathcal{O}_Q(1,1)|$ such that the scheme $C \cap D$ contains at least two connected components, both of them of degree at least 4.*

Lemma 3.2. *Fix integers $a > 0$ and $b > 0$ such that $a + b \geq 3$. Take a general $D \in |\mathcal{O}_Q(a,b)|$. Then $F(D) = \emptyset$.*

Proof. The curve D is smooth and for each line $L \subset Q$ every connected component of the scheme $L \cap D$ has connected components of degree 1 or 2, with at most one having degree 2. Thus it is sufficient to test the smooth $C \in |\mathcal{O}_Q(1,1)|$. Since $\deg(D \cap C) = a + b$, we may assume $a + b \geq 8$. Call $\mathcal{G}$ the set of all zero-dimensional schemes Z with 2 connected components, both of degree 4 and with Z contained in some smooth $C \in |\mathcal{O}_Q(1,1)|$. Each C contains ∞^2 elements of $\mathcal{G}$. Fix $Z \in \mathcal{G}$ and take C containing it. As in the proof of Lemma 3.1 it is sufficient to observe that $h^1(\mathcal{I}_Z(a,b)) = 0$, because $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(a,b)) = a + b \geq \deg(Z) - 1$ and hence

$h^1(C, \mathcal{O}_C(a,b)(-Z)) = 0$. Since $\mathrm{Res}_C(Z) = \emptyset$ and $h^1(Q, \mathcal{O}_Q(a-1,b-1)) = 0$, the residual exact sequence of C gives $h^1(Q, \mathcal{I}_{Z,Q}(a,b)) = 0$. □

Lemma 3.3. *Fix positive integers a, b, a', b' such that $(a,b) \neq (1,1)$ and $(a',b') \neq (1,1)$. Take a general $(D, D') \in |\mathcal{O}_Q(a,b)| \times |\mathcal{O}_Q(a',b')|$. Set $Y := D \cup D'$. Then there is no $C \in |\mathcal{O}_Q(1,1)|$ such that the scheme $C \cap Y$ has two connected components of degree at least 4.*

Proof. By Bertini's theorem D and D' are smooth and Y is nodal. For a general pair (D, D') for each line $L \subset Q$ the scheme $L \cap D$ has connected components of degree 1 or 2, with at most one being of degree 2. Thus it is sufficient to test all smooth $C \in |\mathcal{O}_Q(1,1)|$. Since (D, D') is general, each C contains at most 2 points of $D \cap D'$. Thus every smooth $C \in |\mathcal{O}_Q(1,1)|$ containing some $p \in D \cap D'$ satisfies the property that the connected component of $C \cap Y$ with p as its reduction has degree ≤ 3. Thus we only need to consider the schemes $C \cap (Y \setminus D \cap D')$ with C smooth. By Lemma 3.2 it is sufficient to exclude the smooth C such that $C \cap D$ has a connected component Z_1 of degree at least 4 and $C \cap (D' \setminus D \cap D')$ has a connected component Z_2 of degree at least 4. We may assume $a + b \geq 4$ and $a' + b' \geq 4$. As in the proof of Lemma 3.1 we find only finitely many smooth $C_i \in |\mathcal{O}_Q(1,1)|$, say C_i, $1 \leq i \leq t$, such that $C \cap D$ has a connected component of degree at least 4. For a general D', the curve D' is transversal to all C_i, $1 \leq i \leq t$. □

Lemma 3.4. *Fix positive integers $e \geq 2$, a_i, b_i, $1 \leq i \leq e$, such that for each $i \in \{1, \dots, e\}$ exactly one among a_i and b_i is 1. Let $D = D_1 \cup \dots \cup D_s \subset Q$ be a general union with each D_i general in $|\mathcal{O}_Q(a_i, b_i)|$. Then D is nodal, no two of the nodes of D are contained in the same line of Q, each line of Q passing through a singular point of D is transversal to each D_i, $T(D) = \emptyset$ and there is no line $J \subset Q$ such that $J \cap D$ has a connected component of degree at least 3.*

Proof. D is nodal by Bertini's theorem. Lemma 3.1 gives $T(D) \subseteq \mathrm{Sing}(D)$. Fix $p \in \mathrm{Sing}(D)$. Call D_i and D_j the irreducible components of D containing p. Since D is general, neither D_i nor D_j have a osculating plane at p with weight ≥ 2 and the tangent plane to one component, does not contain the tangent line to the other component. Thus $p \notin T(D)$.

For a general $(D_1, \dots, D_e)$ no two of the nodes of D are on the same line of Q, because $a_i b_i \neq 0$ for all i. We also see by induction on e that each line of Q passing through a singular point of D is transversal to each D_i.

Fix any line $J \subset \mathbb{P}^3$. Since $D \subset Q$, we have $\deg(D \cap J) \leq 2$ if $J \nsubseteq Q$. Now assume $L \in |\mathcal{O}_Q(1,0)|$ (resp. $R \in |\mathcal{O}_Q(1,0)|$). We have $\deg(L \cap D) = b$ (resp. $\deg(R \cap D) = a$). By Lemma 3.1 each connected component of the zero-dimensional schemes $L \cap D$ and $R \cap D$ has degree ≤ 2. □

Lemma 3.5. *Fix positive integers a, b and $q \in \mathbb{P}^3 \setminus Q$. Then $q \notin \tau(Y)$ for a general $Y \in |\mathcal{O}_Q(a,b)|$.*

Proof. The polar surface of Q with respect to Q is a plane, H, intersecting transversally Q and $q \in T_pQ$ if and only if $p \in H \cap Q$. Take Y intersecting transversally $H \cap Q$ and not containing the degree 2 subscheme of $\langle\{p, q\}\rangle$ with p as its reduction at all $p \in H \cap Q \cap Y$. $\square$

Lemma 3.6. *Fix positive integers $s \geq 4$, a_i, b_i, $1 \leq i \leq s$. Take a general $(D_1, \dots, D_s) \in \prod_{i=1}^{s} |\mathcal{O}_Q(a_i, b_i)|$. Then for every $q \in \mathbb{P}^3 \setminus Q$ there is $S_q \subset \{1, \dots, s\}$ such that $\sharp(S_q) \leq 3$ and $q \notin \tau(D_i)$ for all $i \in \{1, \dots, s\} \setminus S_q$.*

Proof. By Lemma 3.5 and the generality of $(D_1, \dots, D_s)$ we have $\dim((\mathbb{P}^3 \setminus Q) \cap \tau(D_1)) = 2$, $\dim((\mathbb{P}^3 \setminus Q) \cap \tau(D_1) \cap \tau(D_2)) \leq 1$, $\dim((\mathbb{P}^3 \setminus Q) \cap \tau(D_1) \cap \tau(D_2) \cap \tau(D_3)) \leq 0$ and $(\mathbb{P}^3 \setminus Q) \cap \tau(D_1) \cap \tau(D_2) \cap \tau(D_3) \cap \tau(D_4) = \emptyset$. Using all subsets of $\{1, \dots, s\}$ with cardinality 4 we get the lemma. $\square$

Lemma 3.7. *Fix positive integers a, b. Take a general $Y \in |\mathcal{O}_Q(a, b)|$. Then for every $q \in \mathbb{P}^3 \setminus Q$ there are at most 3 points $p \in Y$ such that $q \in T_pY$.*

Proof. With no loss of generality we may assume $b \geq a$. Y is smooth. If $a + b \leq 3$, then Y is a rational normal curve in its linear span and the lemma is trivial in this case. Thus we may assume $a + b \geq 4$. The lemma is also easy to check using the linear projection ℓ_q and the genus formula for plane curves if $(a, b) \in \{(2, 2), (1, 3), (2, 3)\}$ (all these cases are discussed in [9]).

For any $q \in \mathbb{P}^3 \setminus Q$ the polar plane H_q of Q with respect to q has the following properties. The curve $C_q := H_q \cap Q$ is a smooth conic and $q \in TpQ$, $p \in Q$, if and only if $p \in C_q$. For any $p \in C_q$ let z_p denote the degree 2 connected zero-dimensional subscheme of the line $\langle\{p, q\}\rangle$ with p as its reduction. For any curve $E \subset Q$ such that $p \in E_{reg}$ we have $q \in T_pE$ if and only if $z_p \subset E$. Let $\mathcal{U}$ denote the set of quadruples (Z_1, Z_2, Z_3, Z_4) with each Z_i a connected degree 2 zero-dimensional subscheme of Q such that there is $q \in \mathbb{P}^3 \setminus Q$ and $(p_1, p_2, p_3, p_4) \in C_q^4$ such that $p_i \neq p_j$ for all $i \neq j$ and $Z_i = z_{p_i}$. The lemma is equivalent to proving that a general Y contains no scheme $Z_1 \cup Z_2 \cup Z_3 \cup Z_4$ with $(Z_1, Z_2, Z_3, Z_4) \in \mathcal{U}$. For each smooth $C \in |\mathcal{O}_Q(1, 1)|$ there is a unique $q \in \mathbb{P}^3 \setminus Q$ such that $C = C_q$. Each smooth $C \in |\mathcal{O}_Q(1, 1)|$ has ∞^4 quadruples of distinct points. Since $\dim |\mathcal{O}_Q(1, 1)| = 3$, we get $\dim \mathcal{U} = 7$. Thus to prove the lemma it is sufficient to prove that $\dim |\mathcal{I}_{Z_1 \cup Z_2 \cup Z_3 \cup Z_4}(a, b)| = \dim |\mathcal{O}_Q(a, b)| - 8$. Fix $(Z_1, Z_2, Z_3, Z_4) \in \mathcal{U}$, say $Z_i = z_{p_i}$ with p_1, p_2, p_3, p_4 distinct points of a smooth $C \in |\mathcal{O}_Q(1, 1)|$. Set $Z := Z_1 \cup Z_2 \cup Z_3 \cup Z_4$. Since $\deg(Z) = 8$, it is sufficient to prove that $h^1(\mathcal{I}_Z(a, b)) = 0$. We have $C \cap Z = \{p_1, p_2, p_3, p_4\}$ (scheme-theoretically), because each tangent line of C is contained in the plane $\langle C \rangle$ and if $C = C_q$, then $q \notin \langle C \rangle$. Hence $\mathrm{Res}_C(Z) = \{p_1, p_2, p_3, p_4\}$. We have $h^1(C, \mathcal{I}_{Z \cap C}(a, b)) = 0$, because $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(a, b)) = a + b$. We have $h^1(C, \mathcal{I}_{\mathrm{Res}_C(Z)}(a - 1, b - 1)) = 0$, because $\deg(\mathcal{O}_C(a - 1, b - 1)) = a + b - 2 \geq 3$. We have $h^1(\mathcal{O}_Q(a - 2, b - 2)) = 0$. Use twice the residual exact sequence of C, first with $\mathcal{I}_Z(a, b)$ as its middle term and then with $\mathcal{I}_{\mathrm{Res}_C(Z)}(a - 1, b - 1)$ as its middle term. $\square$

Lemma 3.8. *Fix positive integers a, b such that $(a,b) \neq (1,1)$. Take a general $Y \in |\mathcal{O}_Q(a,b)|$. The set of all $q \in \mathbb{P}^3 \setminus Q$ such that there are 2 (resp. 3) points $p \in Y$ with $q \in T_pY$ has dimension ≤ 1 (resp. ≤ 0).*

Proof. Adapt the proof of Lemma 3.7 using $Z_1 \cup Z_2 \cup Z_3$ (resp. $Z_1 \cup Z_2$) instead of $Z_1 \cup Z_2 \cup Z_3 \cup Z_4$. □

Lemma 3.9. *Fix positive integers a_1, b_1, a_2, b_2. Take a general pair $(D_1, D_2) \in |\mathcal{O}_Q(a_1,b_1)| \times |\mathcal{O}_Q(a_2,b_2)|$. For each $q \in \mathbb{P}^3 \setminus Q$ the following properties are true:.*

(a) there is no $(p_1,p_2,p_3,p_4) \in D_1 \times D_1 \times D_2 \times D_2$ such that $p_1 \neq p_2$, $p_3 \neq p_4$ and $q \in T_{p_1}D_1 \cap T_{p_2}D_1 \cap T_{p_3}D_2 \cap T_{p_4}D_2$;

(b) there is no $(p_1,p_2,p_3,p_4) \in D_1 \times D_1 \times D_1 \times D_2$ such that $\sharp(\{p_1,p_2,p_3\}) = 3$ and $q \in T_{p_1}D_1 \cap T_{p_2}D_1 \cap T_{p_3}D_1 \cap T_{p_4}D_2$.

Proof. Part (b) follows from Lemmas 3.5 and 3.8.

Now we prove part (a). This is trivial if $(a_2,b_2) \in \{(2,1),(1,2)\}$, i.e. if D_2 is a rational normal curve. Thus we may assume $a_2 + b_2 \geq 4$. As in the proof of Lemma 3.7 let H_q be the polar hyperplane of Q with respect to q and $C_q := H_q \cap Q$. For any $p \in C_q$ let z_p denote the degree 2 connected zero-dimensional subscheme of the line $\langle\{p,q\}\rangle$ with p as its reduction. Let $\mathcal{U}$ denote the set of all quadruples Z_1, Z_2, Z_3, Z_4 such that there is a smooth $C \in |\mathcal{O}_Q(1,1)|$ and 4 distinct points $p_i \in C$, $1 \leq i \leq 4$, such that $z_{p_i} = Z_i$ for all i. For a fixed D_1 Lemma 3.8 shows that we have at most ∞^1 pairs (p_1,p_2) which may be prolonged to be the reduction of some (Z_1,Z_2,Z_3,Z_4). For a fixed p_1, p_2 we have $h^0(Q, \mathcal{I}_{p_1,p_2}(1,1)) = 2$ and hence there are only ∞^1 $C \in |\mathcal{O}_Q(1,1)|$ containing $\{p_1,p_2\}$. For a fixed C we have ∞^2 pairs $(p_3,p_4) \in C \times C$. We fix the general D_1. To prove that a general D_2 satisfies part (a) of the lemma it is sufficient to prove that $h^1(\mathcal{I}_{Z_1\cup Z_2\cup Z_3\cup Z_4}(a_2,b_2)) \leq 2$. We prove this inequality in the following way. Recall that $C \cap (Z_\cup Z_2 \cup Z_3 \cup Z_4) = \{p_1,p_2,p_3,p_4\}$ (scheme-theoretically), because each tangent line of C is contained in the plane $\langle C\rangle$ and if $C = C_q$, then $q \notin \langle C\rangle$. Thus $\mathrm{Res}_C(Z_1 \cup Z_2 \cup Z_3 \cup Z_4) = \{p_1,p_2,p_3,p_4\}$. Since $a_2 + b_2 \geq 4$, we have $h^1(C, \mathcal{I}_{\{p_1,p_2,p_3,p_4\}}(a_2,b_2)) = 0$ and $h^1(C, \mathcal{I}_{\{p_1,p_2,p_3,p_4\}}(a_2-1,b_2-1)) \leq 1$. Use twice the residual exact sequence of C, first with $\mathcal{I}_{Z_1\cup Z_2\cup Z_3\cup Z_4}(a_2,b_2)$ as its middle term and then with $\mathcal{I}_{\{p_1,p_2,p_3,p_4\}}(a_2-1,b_2-1)$ as its middle term. □

Lemma 3.10. *Fix positive integers $a_1, b_1, a_2, b_2, a_3, b_3$ such that $(a_i,b_i) \neq (1,1)$, $1 \leq i \leq 3$. Take a general $(D_1,D_2,D_3) \in |\mathcal{O}_Q(a_1,b_1)| \times |\mathcal{O}_Q(a_2,b_2)| \times |\mathcal{O}_Q(a_3,b_3)|$. Take any $q \in \mathbb{P}^3 \setminus Q$. There are no $(p_1,p_2,p_3,p_4) \in D_1 \times D_1 \times D_2 \times D_3$ such that $p_1 \neq p_2$ and $q \in T_{p_1}D_1 \cap T_{p_2}D_1 \cap T_{p_3}D_2 \cap T_{p_4}D_3$.*

Proof. The proof of part (a) of Lemma 3.9 shows that there are only finitely many triples $(p_1,p_2,p_3) \in D_1 \times D_1 \times D_2$ such that $p_1 \neq p_2$ and $T_{p_1}D_1 \cap T_{p_2}D_1 \cap T_{p_3}D_2$ is a point of $\mathbb{P}^3 \setminus Q$. Apply Lemma

3.5 to D_3. □

Proof of Theorem 1.1: Any $Y \in |\mathcal{O}_Q(a,b)|$ has arithmetic genus γ.

Claim 1: There are integers $e \ge 2$, a_i, b_i, $1 \le i \le e$, such that for each $i \in \{1,\dots,e\}$ exactly one among a_i and b_i is 1, $a_1+\cdots+a_e = a$ and $b_1+\cdots+b_e = b$.

Proof of Claim 1: If $d \equiv 0 \pmod 6$ we take $e = d/3$, $(a_i,b_i) = (1,2)$ for odd i and $(a_i,b_i) = (2,1)$ for even i. If $d \equiv i \pmod 6$, $1 \le i \le 5$, we take $e = (d-i)/3$, $(a_1,b_1) = (1,2+i)$, $(a_i,b_i) = (1,2)$ for odd $i \ge 3$ and $(a_i,b_i) = (2,1)$ for even i.

Take a nodal curve $D = D_1 \cup \cdots \cup D_e \subset Q$ satisfying the thesis of Lemma 3.4. Since each D_i is smooth and rational and $p_a(D) = \gamma$, we have $\sharp(\mathrm{Sing}(D)) = \gamma + e - 1$. Since $0 \le g \le \gamma$ and each D_i is irreducible, there is a set $A \subset \mathrm{Sing}(D)$ such that $\sharp(A) = \gamma - g$ and $D \setminus A$ is connected. We fix one such set A and call it the set of all *assigned nodes*. The set $\mathrm{Sing}(D)$ is called the set of all *unassigned nodes* (we are using the terminology of A. Tannenbaum ([10]) who extended to other rational surfaces the classical theory of nodal plane curves due to Severi). Since $D \setminus A$ is connected, [10, Lemma 2.2 and Theorem 2.13] gives the existence of a flat family $\{D_t\}_{t\in\Delta}$, Δ an integral affine curve, and $o \in \Delta$ such that $D_t \in |\mathcal{O}_Q(a,b)|$ for all $t \in \Delta$, $D_o = D$, each D_t, $t \in \Delta\setminus\{o\}$, is integral, nodal and with geometric genus g, and the nodes of D_t, $t \in \Delta\setminus\{o\}$, go to the set of assigned nodes. By Remark 3.1 we have $T(D_t) = \emptyset$ for a general $t \in \Delta$. Fix $c \in \Delta\setminus\{o\}$ such that $T(D_c) = \emptyset$ and set $X := D_c$. X is an integral and nodal curve with geometric genus g. To conclude the proof of the theorem it is sufficient to prove that $r_X(q) = 2$ for all $q \in \mathbb{P}^3 \setminus X$.

(a) Fix $q \in Q$. Let L be the element of $|\mathcal{O}_Q(1,0)|$ containing q. We have $\deg(L\cap X) = b$. By Lemma 3.1 each connected component of $L \cap X$ has degree ≤ 2. Thus $\sharp((L\cap X)_{\mathrm{red}}) \ge \lceil ba/2 \rceil$. Since $b \ge 3$, we get $r_X(q) = 2$.

(b) Fix $q \in \mathbb{P}^3 \setminus Q$. Assume $r_X(q) > 2$, i.e. assume $\ell_{q|X}$ is injective. Since $\ell_q(X)$ has degree $d = a+b$, it has arithmetic genus $(a+b-1)(a+b-2)/2$, while X has arithmetic genus $\gamma = ab - a - b + 1$. We silently use a small modification of Remark 3.1 to get $F(X) = \emptyset$ (for a general partial smoothing X) knowing that $F(D) = \emptyset$. We use Lemmas 3.1, 3.2, 3.3 to get $T(D) = \emptyset$ and hence (Remark 3.1) we get $T(X) = \emptyset$.

(b1) Assume for the moment that q is not in the tangent space of one of the nodes of X. Call o_i, $1 \le i \le s$, the points of X_{reg} such that $q \in T_{o_i}X$.

The following observation summarize lemmas 3.1, 3.2, 3.3, 3.4, 3.6, 3.7, 3.9, 3.10 first on D and then on X.

Observation 1: X has no flex, its osculating planes have weight 1 and each point of $\mathbb{P}^3 \setminus Q$ is contained in at most 3 tangent lines to smooth points of X.

A dimensional count similar to the one needed to prove Lemmas 3.2 and 3.7 gives the following observation.

Observation 2: At each $q \in \mathbb{P}^3 \setminus Q$ such that there are 3 different smooth points p_1, p_2, p_3 of X_{reg} with $q \in T_{p_i}X$, no $T_{p_i}(X)$ is a stall. At each point of X at which there are 2 different smooth points p_1, p_2 of X_{reg} with $q \in T_{p_i}X$ at most one among $T_{p_1}X$ and $T_{p_2}X$ is a stall.

By Observations 1 and 2 we have $p_a(\ell_q(X)) \le p_a(X) + 3$. Since $\ell_q(X)$ is a plane curve of degree $a+b$ and $p_a(X) = ab - a - b + 1$, we get $(a+b-1)(a+b-2)/2 \le ab - a - b + 4$, i.e. $a^2 + b^2 \le a + b + 6$, which is false if $a = 1$ and $b \ge 4$ or $a \ge 2$ and $b \ge 3$.

(b2) Assume $g < \gamma$ and that q is contained in at least one tangent plane at X at one of its points.

First assume that q is contained in the tangent cone at one of the nodes, o, of X. For a general D (and hence a general partial smoothing X) no line in the tangent cones of X at its singular points are stalls and tangent cones at different singular points are disjoints. At most another singular point o' of X has tangent plane containing q.

Now assume that q is not contained in any tangent cone at singular points. It is contained in at most 3 tangent spaces of X at its singular points and if at 3 it is not contained in any tangent line at a smooth point of X. We get a contradiction if $(a+b-1)(a+b-2)/2 \ge \gamma + 4$, i.e. if $a^2 + b^2 \ge a + b + 8$, which is true (for positive a, b) if and only if $a + b \ge 5$. $\square$

4 Curves in $\mathbb{P}^4$

Let $F_1 \subset \mathbb{P}^4$ be a smooth and non-degenerate surface such that $\deg(F_1) = 3$. All such surfaces are projectively equivalent. The smooth or nodal curves we use to prove Theorem 1.2 are contained in F_1. The surface F_1 is an embedding of the Hirzebruch surface with the same name ([7, §V.2]). We have $\mathrm{Pic}(F_1) \cong \mathbb{Z}^2$ and we take as free generators of it the class f, of a fiber of the ruling of F_1 and the section h of its ruling with negative self-intersection. We have $h^2 = -1$, $f^2 = 0$ and $h \cdot f = 1$. We have $\mathcal{O}_{F_1}(1) \cong \mathcal{O}_{F_1}(h+2f)$ and h and the elements of the ruling $|f|$ are the only lines contained in F_1. Each $\mathcal{O}_{F_1}(ah+bf)$, $b \ge a \ge 0$, is globally generated; it is ample (and very ample, too) if and only if $b > a > 0$. Fix $D \in |ah+bf|$, $b \ge a > 0$. Since $\omega_{F_1} \cong \mathcal{O}_{F_1}(-2h-3f)$, the adjunction formula gives $\omega_D \cong \mathcal{O}_D((a-2)h + (b-3)f)$. Thus $p_a(D) = 1 + ab - a(a+1)/2 - b$. For all $b \ge a-1$ we have $h^1(\mathcal{O}_{F_1}(ah+bf)) = 0$ and $h^0(\mathcal{O}_{F_1}(ah+bf)) = \sum_{i=0}^{a}(b+1-i) = (2b+2-a)(a+1)/2$.

Remark 4.1. *Take any curve $D \subset F_1$ and any line $L \subset \mathbb{P}^4$ such that* $\deg(D \cap L) \ge 3$. *Since F_1 is scheme-theoretically cut out by quadrics and $D \subset F_1$, Bezout theorem gives $L \subset F_1$.*

Lemma 4.1. *Fix an integer $q \in \mathbb{P}^4 \setminus F_1$. Then there is $C \in |h+f|$ such that $q \in \langle C \rangle$.*

Proof. Since 3 is a prime integer and $q \notin F_1$, $\ell_q(F_1)$ is an irreducible degree 3 ruled surface. This surface has a double line L meeting all lines of the ruling of $\ell_q(F_1)$ ([4, Theorem 9.2.1]). Thus there is a plane conic $C \subset F_1$ (a priori even a double line) mapped by ℓ_q onto L. All conics $C \subset F_1$ are elements of $|h+f|$. $\square$

Up to projective transformations there are exactly two degree 3 surfaces $\ell_q(F_1)$, $q \in \mathbb{P}^4 \setminus F_1$, distinguished by the fact that the unique conic $C \in |h+f|$ given by Lemma 4.1 is smooth or not ([4, Theorem 9.2.1]).

Proposition 4.1. *Let $X \subset F_1 \subset \mathbb{P}^4$ be a reduced and non-degenerate curve whose irreducible component have degrees at least 3. Assume the following conditions:*

(1) $\sharp((L \cap X)_{\mathrm{red}}) \geq 2$ *for all* $L \in |f|$*;*

(2) $\sharp((h \cap X)_{\mathrm{red}}) \geq 2$*;*

(3) $\sharp((C \cap X)_{\mathrm{red}}) \geq 3$ *for all smooth* $C \in |h+f|$*.*

Then $r_X(q) \leq 3$ *for all* $q \in \mathbb{P}^4$*.*

Proof. The assumptions on the irreducible components of X is equivalent to assuming that $X \cap C$ contains no curve for all $C \in |h+f|$. First assume $q \in F_1$. Let L be the only element of $|f|$ containing q. Since L is a line and $\sharp((L \cap X)_{\mathrm{red}}) \geq 2$, we have $r_X(q) \leq 2$.

Now assume $q \notin F_1$. Take $C \in |h+f|$ such that $q \in \langle C \rangle$. Note that $\langle C \rangle$ is a plane. If C is smooth (and hence it is a smooth conic), we have $r_X(q) \leq 3$, because $\sharp((C \cap X)_{\mathrm{red}}) \geq 3$ and hence $(C \cap X)_{\mathrm{red}}$ spans $\langle C \rangle$. Now assume that C is singular, i.e. $C = h + L$ for some $L \in |f|$. Both h and L are lines and $h \cap L$ is a single point. By assumption there are $p_1, p_2 \in (L \cap X)_{\mathrm{red}}$ with $p_1 \neq p_2$ and hence $L = \langle \{p_1, p_2\} \rangle$. Since $\sharp((h \cap X)_{\mathrm{red}}) \geq 2$, there is $p_3 \in (h \cap X)_{\mathrm{red}}$ such that $p_3 \neq h \cap L$. Since $h = \langle \{p_3, h \cap L\} \rangle$, we have $\langle C \rangle = \langle \{p_1, p_2, p_3\} \rangle$ and hence $r_X(q) \leq 3$. $\square$

Lemma 4.2. *Let Δ a quasi-projective variety and $\mathcal{X} \subset F_1 \times \Delta$ a closed algebraic set such that the restriction $u : \mathcal{X} \longrightarrow \Delta$ to $\mathcal{X}$ of the projection $F_1 \times \Delta \longrightarrow \Delta$ is proper and flat. For each $t \in \Delta$ set $X_t := u^{-1}(t)$. Assume that all fibers of u are reduced curves with no irreducible component of degree ≤ 2. Fix $o \in \Delta$ and assume $\sharp((C \cap X_o)_{\mathrm{red}}) \geq 3$ for all $C \in |h+f|$. Then for a general $t \in \Delta$ we have $\sharp((C \cap X_t)_{\mathrm{red}}) \geq 3$ for all $C \in |h+f|$.*

Proof. Assume that the lemma is false. Taking a neighborhood Ω of o in Δ and then a branch covering of Ω we may assume that for each $t \in \Omega \setminus \{o\}$ there is $C_t \in |h+f|$ with $\sharp((C_t \cap X_t)_{\mathrm{red}}) \leq 2$. Since $|h+f|$ is a projective set, the family $\{C_t\}_{t \in \Omega \setminus \{o\}}$ has at least one limit point, C', and $\sharp((C' \cap X_o)_{\mathrm{red}}) \leq 2$. $\square$

Lemma 4.3. *Fix integers a, b such that either $a = 1$ and $b \geq 5$ or $a \geq 2$ and $b \geq \max\{4, a\}$. Let X be a general element of $|ah + bf|$. Then $\sharp((C \cap X)_{\mathrm{red}}) \geq 3$ for all smooth $C \in |h + f|$.*

Proof. For each $C \in |h + f|$ we have $\deg(X \cap C) = b$. For $e \in \{1, 2\}$ let $\mathcal{U}(e)$ denote the set of all degree zero-dimensional schemes $Z \subset F_1$ such that $\deg(Z) = b$, Z has exactly e connected components and there is a smooth $C \in |h+f|$ containing Z. Since each smooth $C \in |h+f|$ has ∞^e elements of $\mathcal{U}(e)$, we have $\dim \mathcal{U}(e) = 2 + e$. Thus (since $e \leq 2$) to prove the lemma it is sufficient to prove that $\dim |\mathcal{I}_Z(ah + bf)| = \dim |ah + bf| - 5$ for all $Z \in \mathcal{U}(e)$, $i = 1, 2$. Fix $Z \in \mathcal{U}(e)$ and take a smooth $C \in |h + f|$ containing it. Since $\deg(Z) = b \geq 5$, it is sufficient to prove that $h^1(\mathcal{I}_Z(ah + bf)) = 0$. Since $h^1(\mathcal{O}_{F_1}((a - 1)h + (b - 1)f)) = 0$, the residual exact sequence of C shows that it is sufficient to prove that $h^1(C, \mathcal{I}_{Z,C}(ah + bf)) = 0$. This is true, because $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(ah + bf)) = b$. □

Lemma 4.4. *Fix $q \in F_1$. There is a smooth $C \in |h + f|$ such that $q \in C$ if and only if $q \in F_1 \setminus h$.*

Proof. Since $h \cdot (h + f) = 0$, no irreducible $C \in |h + f|$ (i.e. no smooth $C \in |h + f|$) meets h. Now assume $q \in |h + f|$. Since $\dim |\mathcal{I}_q(h + f)| = \dim |h + f| - 1 = 1$ and there is a unique singular element of $|h + f|$ containing q, there is a smooth $C \in |h + f|$ such that $q \in C$. □

Proposition 4.2. *Fix integer a, b such that $a \geq 1$ and $b \geq 2a + 3$.*

(1) There is a nodal $D \in |ah+bf|$ with exactly a smooth irreducible components, all of them rational and neither lines nor conics, such that $\sharp((D \cap C)_{\mathrm{red}}) \geq 3$ for all $C \in |h + f|$.

(2) If $a \geq 2$ we have $r_D(q) \leq 3$ for all $q \in \mathbb{P}^4$.

Proof. Set $b_i := 2$ for $2 \leq i \leq a$ and $b_1 := b - 2a + 2$. Take a general $(D_1, \dots, D_a) \in \prod_{i=1}^{a} |h + b_i f|$ and set $D := D_1 \cup \cdots \cup D_a$. By Bertini's theorem each D_i is smooth and connected and D is nodal. Set $S := \mathrm{Sing}(D)$. Each D_i is rational and $p_a(D) = 1 + ab - a(a + 1)/2 - b$. Thus $\sharp(S) = p_a(D) + a - 1 = ab - a(a - 1)/2 - b$. In the case $a = 1$ we have $D = D_1$ with D_1 a general element of $|h + bf|$. For a general $(D_1, \dots, D_a)$ the nodal curve D is transversal to h and hence $\sharp((h \cap D)_{\mathrm{red}}) = b - a \geq 4$. Hence part (1) is true for all singular $C \in |h + f|$.

Now we check part (1) for all smooth $C \in |h + f|$.

If $a = 1$ it is sufficient to quote Lemma 4.3.

Now assume $a \geq 2$. Since $\sharp((D_1 \cap C)_{\mathrm{red}}) \geq 3$ by Lemma 4.3, we get part (1) for all smooth $C \in |h + f|$.

Now we prove part (2). By Lemma 4.4 we have $r_D(q) \leq 3$ for all $q \in F_1 \setminus h$. Since $\sharp((h \cap D)_{\mathrm{red}}) = b - a \geq 2$, we have $r_D(q) \leq 2$ for all $q \in h$.

Take $q \in \mathbb{P}^4 \setminus F_1$. Take $C \in |h + f|$ such that $q \in \langle C \rangle$ (Lemma 4.1). If C is smooth we get $r_D(q) \le 3$ by Proposition 4.1. Now assume C singular, say $C = h \cup L$ with $L \in |f|$. Since D contains $b - a$ points of D, it is sufficient to prove that L contains a point of $D \setminus D \cap h$. This is true, because $a \ge 2$ and D is transversal to h. □

Proof of Theorem 1.2: Take the curve D given by Proposition 4.2. Use Remark 2.1 to get X as in the proof of Theorem 1.1. Apply part (1) of Proposition 4.2 and Lemma 4.2. □

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