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Caputo fractional Iyengar type Inequalities

George A. Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

ABSTRACT

Here we present Caputo fractional Iyengar type inequalities with respect to L_p norms, with $1 \leq p \leq \infty$. The method is based on the right and left Caputo fractional Taylor's formulae.

RESUMEN

Aquí presentamos desigualdades de tipo Caputo fraccional Iyengar con respecto a las normas L_p, con $1 \leq p \leq \infty$. El método se basa en las fórmulas de Taylor fraccionales de Caputo derecha e izquierda.

Keywords and Phrases: Iyengar inequality, right and left Caputo fractional, Taylor formulae, Caputo fractional derivative.

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1 Introduction

We are motivated by the following famous Iyengar inequality (1938), [4].

Theorem 1. *Let* f *be a differentiable function on* $[a,b]$ *and* $|f'(x)| \leq M$. *Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \quad (1)$$

We need

Definition 2. *([1], p. 394) Let* $\nu > 0$, $n = \lceil \nu \rceil$ *(*$\lceil \cdot \rceil$ *the ceiling of the number),* $f \in AC^n([a,b])$ *(i.e.* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$*). The left Caputo fractional derivative of order* ν *is defined as*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \quad (2)$$

$\forall\ x \in [a,b]$, *and it exists almost everywhere over* $[a,b]$.

We need

Definition 3. *([2], p. 336-337) Let* $\nu > 0$, $n = \lceil \nu \rceil$, $f \in AC^n([a,b])$. *The right Caputo fractional derivative of order* ν *is defined as*

$$D_{b-}^{\nu} f(x) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (z-x)^{n-\nu-1} f^{(n)}(z)\,dz, \quad (3)$$

$\forall\ x \in [a,b]$, *and exists almost everywhere over* $[a,b]$.

2 Main Results

We present the following Caputo fractional Iyengar type inequalities:

Theorem 4. *Let* $\nu > 0$, $n = \lceil \nu \rceil$ *(*$\lceil \cdot \rceil$ *is the ceiling of the number), and* $f \in AC^n([a,b])$ *(i.e.* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$*). We assume that* $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_{\infty}([a,b])$. *Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_{\infty}([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_{\infty}([a,b])} \right\}}{\Gamma(\nu+2)} \left[(t-a)^{\nu+1} + (b-t)^{\nu+1} \right], \quad (4)$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$*, the right hand side of (4) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{5}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$*, for all* $k = 0, 1, ..., n-1$*, we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{6}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$*, it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right|$$

$$\leq \frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{7}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$*,* $k = 1, ..., n-1$*, from (7) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \left(\frac{b-a}{N} \right)^{\nu+1} \left[j^{\nu+1} + (N-j)^{\nu+1} \right], \tag{8}$$

$j = 0, 1, 2, ..., N,$

vi) when $N = 2$ *and* $j = 1$*, (8) turns to*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\max\left\{ \|D_{*a}^{\nu} f\|_{L_\infty([a,b])}, \|D_{b-}^{\nu} f\|_{L_\infty([a,b])} \right\}}{\Gamma(\nu+2)} \frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{9}$$

vii) when $0 < \nu \leq 1$*, inequality (9) is again valid without any boundary conditions.*

Proof. Let $\nu > 0$, $n = \lceil \nu \rceil$, and $f \in AC^n([a,b])$. Then by ([3], p. 54) left Caputo fractional Taylor's formula we have

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{*a}^{\nu} f(t)\, dt, \tag{10}$$

$\forall\, x \in [a,b]$.

Also by ([2], p. 341) right Caputo fractional Taylor's formula we get:

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k = \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} D_{b-}^{\nu} f(z)\, dz, \tag{11}$$

$\forall\, x \in [a,b]$.

By (10) we derive

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| \leq \frac{\left\| D_{*a}^{\nu} f \right\|_{L_\infty([a,b])}}{\Gamma(\nu+1)} (x-a)^{\nu}, \tag{12}$$

and by (11) we obtain

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k \right| \leq \frac{\left\| D_{b-}^{\nu} f \right\|_{L_\infty([a,b])}}{\Gamma(\nu+1)} (b-x)^{\nu}, \tag{13}$$

$\forall\, x \in [a,b]$.

Call

$$\gamma_1 := \frac{\left\| D_{*a}^{\nu} f \right\|_{L_\infty([a,b])}}{\Gamma(\nu+1)}, \tag{14}$$

and

$$\gamma_2 := \frac{\left\| D_{b-}^{\nu} f \right\|_{L_\infty([a,b])}}{\Gamma(\nu+1)}. \tag{15}$$

Set

$$\gamma := \max(\gamma_1, \gamma_2). \tag{16}$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| \leq \gamma (x-a)^{\nu}, \tag{17}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k \right| \leq \gamma (b-x)^{\nu}, \tag{18}$$

$\forall\, x \in [a,b]$.

Hence it holds

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^{k}-\gamma(x-a)^{\nu}\leq f(x)\leq\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^{k}+\gamma(x-a)^{\nu}\qquad(19)$$

and

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^{k}-\gamma(b-x)^{\nu}\leq f(x)\leq\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^{k}+\gamma(b-x)^{\nu},\qquad(20)$$

$\forall\ x\in[a,b]$.

Let any $t\in[a,b]$, then by integration over $[a,t]$ and $[t,b]$, respectively, we obtain

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{(k+1)!}(t-a)^{k+1}-\frac{\gamma}{(\nu+1)}(t-a)^{\nu+1}\leq\int_a^t f(x)\,dx\leq$$

$$\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{(k+1)!}(t-a)^{k+1}+\frac{\gamma}{(\nu+1)}(t-a)^{\nu+1},\qquad(21)$$

and

$$-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{(k+1)!}(t-b)^{k+1}-\frac{\gamma}{(\nu+1)}(b-t)^{\nu+1}\leq\int_t^b f(x)\,dx\leq$$

$$-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{(k+1)!}(t-b)^{k+1}+\frac{\gamma}{(\nu+1)}(b-t)^{\nu+1}.\qquad(22)$$

Adding (21) and (22), we obtain

$$\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}-f^{(k)}(b)(t-b)^{k+1}\right]\right\}-$$

$$\frac{\gamma}{(\nu+1)}\left[(t-a)^{\nu+1}+(b-t)^{\nu+1}\right]\leq\int_a^b f(x)\,dx\leq$$

$$\left\{\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}-f^{(k)}(b)(t-b)^{k+1}\right]\right\}+$$

$$\frac{\gamma}{(\nu+1)}\left[(t-a)^{\nu+1}+(b-t)^{\nu+1}\right],\qquad(23)$$

$\forall\ t\in[a,b]$.

Consequently we derive:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\gamma}{(\nu+1)}\left[(t-a)^{\nu+1}+(b-t)^{\nu+1}\right],\qquad(24)$$

$\forall\ t \in [a,b]$.

Let us consider

$$g(t) := (t-a)^{\nu+1} + (b-t)^{\nu+1},\ \forall\ t \in [a,b].$$

Hence

$$g'(t) = (\nu+1)\left[(t-a)^{\nu} - (b-t)^{\nu}\right] = 0,$$

giving $(t-a)^{\nu} = (b-t)^{\nu}$ and $t-a = b-t$, that is $t = \frac{a+b}{2}$ the only critical number here.

We have $g(a) = g(b) = (b-a)^{\nu+1}$, and $g\left(\frac{a+b}{2}\right) = \frac{(b-a)^{\nu+1}}{2^{\nu}}$, which the minimum of g over $[a,b]$.

Consequently the right hand side of (24) is minimized when $t = \frac{a+b}{2}$, with value $\frac{\gamma}{(\nu+1)}\frac{(b-a)^{\nu+1}}{2^{\nu}}$.

Assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, for $k = 0,1,...,n-1$, then we obtain that

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\gamma}{(\nu+1)}\frac{(b-a)^{\nu+1}}{2^{\nu}}, \tag{25}$$

which is a sharp inequality.

When $t = \frac{a+b}{2}$, then (24) becomes

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\right| \leq$$

$$\frac{\gamma}{(\nu+1)}\frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{26}$$

Next let $N \in \mathbb{N}$, $j = 0,1,2,...,N$ and $t_j = a + j\left(\frac{b-a}{N}\right)$, that is $t_0 = a$, $t_1 = a + \frac{b-a}{N}, ..., t_N = b$.

Hence it holds

$$t_j - a = j\left(\frac{b-a}{N}\right),\quad (b-t_j) = (N-j)\left(\frac{b-a}{N}\right),\quad j = 0,1,2,...,N. \tag{27}$$

We notice that

$$(t_j-a)^{\nu+1} + (b-t_j)^{\nu+1} = \left(\frac{b-a}{N}\right)^{\nu+1}\left[j^{\nu+1} + (N-j)^{\nu+1}\right], \tag{28}$$

$j = 0,1,2,...,N$,

and (for $k = 0,1,...,n-1$)

$$\left[f^{(k)}(a)(t_j-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t_j)^{k+1}\right] =$$

$$\left[f^{(k)}(a)\,j^{k+1}\left(\frac{b-a}{N}\right)^{k+1} + (-1)^k f^{(k)}(b)(N-j)^{k+1}\left(\frac{b-a}{N}\right)^{k+1}\right] =$$

$$\left(\frac{b-a}{N}\right)^{k+1}\left[f^{(k)}(a)\,j^{k+1}+(-1)^k f^{(k)}(b)\,(N-j)^{k+1}\right], \tag{29}$$

$j = 0, 1, 2, ..., N$.

By (24) we get

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[f^{(k)}(a)\,j^{k+1}+(-1)^k f^{(k)}(b)\,(N-j)^{k+1}\right]\right|$$

$$\leq \frac{\gamma}{(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{30}$$

$j = 0, 1, 2, ..., N$.

If $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, then (30) becomes

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)[jf(a)+(N-j)\,f(b)]\right| \leq$$

$$\frac{\gamma}{(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu+1}\left[j^{\nu+1}+(N-j)^{\nu+1}\right], \tag{31}$$

$j = 0, 1, 2, ..., N$.

When $N = 2$ and $j = 1$, then (31) becomes

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \leq$$

$$\frac{\gamma}{(\nu+1)}2\left(\frac{b-a}{2}\right)^{\nu+1}=\frac{\gamma}{(\nu+1)}\frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{32}$$

Let $0 < \nu \leq 1$, then $n = \lceil \nu \rceil = 1$. In that case, without any boundary conditions, we derive from (32) again that

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \leq \frac{\gamma}{(\nu+1)}\frac{(b-a)^{\nu+1}}{2^{\nu}}. \tag{33}$$

The theorem is proved in all cases. □

We give

Theorem 5. *Let* $\nu \geq 1$, $n = \lceil \nu \rceil$, *and* $f \in AC^n([a,b])$. *We assume that* $D_{*a}^{\nu}f, D_{b-}^{\nu}f \in L_1([a,b])$. *Then*

i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)\,(t-a)^{k+1}+(-1)^k f^{(k)}(b)\,(b-t)^{k+1}\right]\right| \leq$$

$$\frac{\max\left\{\left\|D^{\nu}_{*a}f\right\|_{L_1([a,b])}, \left\|D^{\nu}_{b-}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left[(t-a)^{\nu}+(b-t)^{\nu}\right], \tag{34}$$

$\forall\ t \in [a,b]$,

ii) when $\nu = 1$*, from (34), we have*

$$\left|\int_a^b f(x)\,dx - [f(a)(t-a)+f(b)(b-t)]\right| \leq$$

$$\|f'\|_{L_1([a,b])}(b-a), \quad \forall\ t\in[a,b], \tag{35}$$

iii) from (35), we obtain ($\nu = 1$ *case)*

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \leq \|f'\|_{L_1([a,b])}(b-a), \tag{36}$$

iv) at $t = \frac{a+b}{2}$*,* $\nu > 1$*, the right hand side of (34) is minimized, and we get:*

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right| \leq$$

$$\frac{\max\left\{\left\|D^{\nu}_{*a}f\right\|_{L_1([a,b])}, \left\|D^{\nu}_{b-}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{37}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$*, for all* $k = 0,1,...,n-1$*;* $\nu > 1$*, from (37), we obtain*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\max\left\{\left\|D^{\nu}_{*a}f\right\|_{L_1([a,b])}, \left\|D^{\nu}_{b-}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}, \tag{38}$$

which is a sharp inequality,

vi) more generally, for $j = 0,1,2,...,\ N \in \mathbb{N}$*, it holds*

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq \frac{\max\left\{\left\|D^{\nu}_{*a}f\right\|_{L_1([a,b])}, \left\|D^{\nu}_{b-}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{39}$$

vii) if $f^{(k)}(a) = f^{(k)}(b) = 0$*,* $k = 1,...,n-1$*, from (39) we get:*

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)[jf(a)+(N-j)f(b)]\right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\left(\frac{b-a}{N}\right)^{\nu}\left[j^{\nu}+(N-j)^{\nu}\right], \tag{40}$$

$j = 0, 1, 2, ..., N,$

viii) when $N = 2$ *and* $j = 1$*, (40) turns to*

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{2}(f(a)+f(b))\right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}\right\}}{\Gamma(\nu+1)}\frac{(b-a)^{\nu}}{2^{\nu-1}}. \tag{41}$$

Proof. Here $\nu \geq 1$ and $D_{*a}^{\nu}f, D_{b-}^{\nu}f \in L_1([a,b])$. By (10) we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \frac{1}{\Gamma(\nu)}(x-a)^{\nu-1}\int_a^x\left|D_{*a}^{\nu}f(t)\right|dt$$

$$\leq \frac{(x-a)^{\nu-1}}{\Gamma(\nu)}\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])}, \tag{42}$$

$\forall\, x \in [a,b]$.

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \frac{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])}}{\Gamma(\nu)}(x-a)^{\nu-1}, \tag{43}$$

$\forall\, x \in [a,b]$.

By (11) we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \frac{1}{\Gamma(\nu)}(b-x)^{\nu-1}\int_x^b\left|D_{b-}^{\nu}f(z)\right|dz$$

$$\leq \frac{(b-x)^{\nu-1}}{\Gamma(\nu)}\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}, \tag{44}$$

$\forall\, x \in [a,b]$.

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \frac{\left\|D_{b-}^{\nu}f\right\|_{L_1([a,b])}}{\Gamma(\nu)}(b-x)^{\nu-1}, \tag{45}$$

$\forall\, x \in [a,b]$.

Call

$$\delta_1 := \frac{\left\|D_{*a}^{\nu}f\right\|_{L_1([a,b])}}{\Gamma(\nu)}, \tag{46}$$

and

$$\delta_2 := \frac{\left\| D_{b-}^{\nu} f \right\|_{L_1([a,b])}}{\Gamma(\nu)}. \tag{47}$$

Set

$$\delta := \max(\delta_1, \delta_2). \tag{48}$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \delta (x-a)^{\nu-1}, \tag{49}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \delta (b-x)^{\nu-1}, \tag{50}$$

$\forall\, x \in [a,b]$.

As in the proof of Theorem 4, we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\delta}{\nu} \left[(t-a)^{\nu} + (b-t)^{\nu} \right], \tag{51}$$

$\forall\, t \in [a,b]$.

The rest of the proof is similar to the proof of Theorem 4. □

We continue with

Theorem 6. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$; $f \in AC^n([a,b])$, *with* $D_{*a}^{\nu} f, D_{b-}^{\nu} f \in L_q([a,b])$. *Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)\left(\nu + \frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}} \left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}} \right], \tag{52}$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, *the right hand side of (52) is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{53}$$

iii) if $f^{(k)}(a)=f^{(k)}(b)=0$*, for all* $k=0,1,...,n-1$*, we obtain*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{54}$$

which is a sharp inequality,

iv) more generally, for $j=0,1,2,...,$ $N\in\mathbb{N}$*, it holds*

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right|$$

$$\leq \frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{55}$$

v) if $f^{(k)}(a)=f^{(k)}(b)=0$*,* $k=1,...,n-1$*, from (55) we get:*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)[jf(a)+(N-j)f(b)]\right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\left(\frac{b-a}{N}\right)^{\nu+\frac{1}{p}}\left[j^{\nu+\frac{1}{p}}+(N-j)^{\nu+\frac{1}{p}}\right], \tag{56}$$

for $j=0,1,2,...,N$*,*

vi) when $N=2$ *and* $j=1$*, (56) turns to*

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \leq$$

$$\frac{\max\left\{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])},\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}\right\}}{\Gamma(\nu)\left(\nu+\frac{1}{p}\right)(p(\nu-1)+1)^{\frac{1}{p}}}\frac{(b-a)^{\nu+\frac{1}{p}}}{2^{\nu-\frac{1}{q}}}, \tag{57}$$

vii) when $1/q<\nu\leq 1$*, inequality (57) is again valid but without any boundary conditions.*

Proof. Here $\nu>0$, $n=\lceil\nu\rceil$, $f\in AC^n([a,b])$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, with $D_{*a}^{\nu}f, D_{b-}^{\nu}f\in L_q([a,b])$. By (10) we have

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\left|D_{*a}^{\nu}f(t)\right|dt \leq$$

$$\frac{1}{\Gamma(\nu)}\left(\int_a^x (x-t)^{p(\nu-1)}\,dt\right)^{\frac{1}{p}}\left(\int_a^x \left|D_{*a}^{\nu}f(t)\right|^q dt\right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\nu)}\frac{(x-a)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])}. \quad (58)$$

Here we assume that $\nu > \frac{1}{q} \Leftrightarrow p(\nu-1)+1>0$. So, we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \frac{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}(x-a)^{\nu-\frac{1}{q}}, \quad (59)$$

$\forall\, x \in [a,b]$.

By (11) we have

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \frac{1}{\Gamma(\nu)}\int_x^b (z-x)^{\nu-1}\left|D_{b-}^{\nu}f(z)\right| dz \leq$$

$$\frac{1}{\Gamma(\nu)}\left(\int_x^b (z-x)^{p(\nu-1)}\,dz\right)^{\frac{1}{p}}\left(\int_x^b \left|D_{b-}^{\nu}f(z)\right|^q dz\right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\nu)}\frac{(b-x)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}. \quad (60)$$

So, we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \frac{\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}(b-x)^{\nu-\frac{1}{q}}, \quad (61)$$

$\forall\, x \in [a,b]$.

Call

$$\rho_1 := \frac{\left\|D_{*a}^{\nu}f\right\|_{L_q([a,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}, \quad (62)$$

and

$$\rho_2 := \frac{\left\|D_{b-}^{\nu}f\right\|_{L_q([a,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}. \quad (63)$$

Set

$$\rho := \max(\rho_1,\rho_2), \quad m := \nu - \frac{1}{q} > 0. \quad (64)$$

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right| \leq \rho(x-a)^m, \quad (65)$$

and

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right| \leq \rho(b-x)^m, \quad (66)$$

$\forall\ x \in [a, b]$.

As in the proof of Theorem 4, we obtain:

$$\left| \int_a^b f(x)\, dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\rho}{(m+1)} \left[(t-a)^{m+1} + (b-t)^{m+1} \right] =$$

$$\frac{\max\left\{ \left\| D_{*a}^{\nu} f \right\|_{L_q([a,b])}, \left\| D_{b-}^{\nu} f \right\|_{L_q([a,b])} \right\}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p} \right)} \left[(t-a)^{\nu+\frac{1}{p}} + (b-t)^{\nu+\frac{1}{p}} \right], \tag{67}$$

$\forall\ t \in [a, b]$.

The rest of the proof is similar to the proof of Theorem 4. □

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