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Odd Vertex Equitable Even Labeling of Cycle Related Graphs

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ABSTRACT

Let G be a graph with p vertices and q edges and $A = \{1, 3, ..., q\}$ if q is odd or $A = \{1, 3, ..., q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, ..., 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph. Here, we prove that the subdivision of double triangular snake $(S(D(T_n)))$, subdivision of double quadrilateral snake $(S(D(Q_n)))$, $DA(Q_m) \odot nK_1$ and $DA(T_m) \odot nK_1$ are odd vertex equitable even graphs.

RESUMEN

Sea G un grafo con p vértices y q aristas, y $A = \{1, 3, ..., q\}$ si q es impar o $A = \{1, 3, ..., q+1\}$ si q es par. Se dice que un grafo G admite un etiquetado par equitativo de vértices impares si existe un etiquetado de vértices $f : V(G) \rightarrow A$ que induce un etiquetado de ejes f^* definido por $f^*(uv) = f(u) + f(v)$ para todos los ejes uv tales que para todo a y b en A, $|v_f(a) - v_f(b)| \leq 1$ y las etiquetas de ejes inducidas son $2, 4, ..., 2q$ donde $v_f(a)$ es el número de vértices v con $f(v) = a$ para $a \in A$. Un grafo que admite un etiquetado par equitativo de vértices impares se dice grafo par equitativo de vértices impares. Aquí demostramos que la subdivisión de serpientes triangulares dobles $(S(D(T_n)))$, la subdivisión de serpientes cuadriláteras dobles $(S(D(Q_n)))$, $DA(Q_m) \odot nK_1$ y $DA(T_m) \odot nK_1$ son grafos pares equitativos de vértices impares.

Keywords and Phrases: Odd vertex equitable even labeling, odd vertex equitable even graph, double triangular snake, subdivision of double quadrilateral snake, double alternate triangular snake, double alternate quadrilateral snake, subdivision graph.

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1 Introduction:

All graphs considered here are simple, finite, connected and undirected. Let $G(V,E)$ be a graph with p vertices and q edges. We follow the basic notations and terminology of graph theory as in [2]. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions and a detailed survey of graph labeling can be found in [1]. The concept of vertex equitable labeling was due to Lourdusamy and Seenivasan [6]. Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, ..., \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, ..., q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. The vertex labeling f is known as vertex equitable labeling. A graph G is said to be a vertex equitable if it admits vertex equitable labeling. Motivated by the concept of vertex equitable labeling [6], Jeyanthi, Maheswari and Vijayalakshmi extend this concept and introduced a new labeling namely odd vertex equitable even (OVEE) labeling in [3]. A graph G with p vertices and q edges and $A = \{1, 3, ..., q\}$ if q is odd or $A = \{1, 3, ..., q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $v_f(a) - v_f(b) \leq 1$ and the induced edge labels are $2, 4, ..., 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even (OVEE) labeling then G is called an odd vertex equitable even (OVEE) graph. In [3], [4] and [5] the same authors proved that nC_4-snake, $CS(n_1, n_2, ..., n_k$, $n_i \equiv 0(mod 4)$, $n_i \geq 4$, be a generalized kC_n -snake, $T\widehat{O}QS_n$ and $T\widetilde{O}QS_n$ are odd vertex equitable even graphs. They also proved that the graphs path, $P_n \odot P_m (n, m \geq 1)$, $K_{1,n} \cup K_{1,n-2} (n \geq 3)$, $K_{2,n}$, T_p-tree, cycle C_n ($n \equiv$ 0 or 1 (mod4)), quadrilateral snake Q_n, ladder L_n, $L_n \odot K_1$, arbitrary super subdivision of any path P_n, $S(L_n)$, $L_m \widehat{O} P_n$, $L_n \odot \overline{K_m}$ and $\left\langle L_n \widehat{O} K_{1,m} \right\rangle$ are odd vertex equitable even graphs. Also they proved that the graphs $K_{1,n}$ is an odd vertex equitable even graph iff $n \leq 2$ and the graph $G = K_{1,n+k} \cup K_{1,n}$ is an odd vertex equitable even graph if and only if $k = 1, 2$ and cycle C_n is an odd vertex equitable even graph if and only if $n \equiv 0$ or 1(mod4). Let G be a graph with p vertices and q edges and $p \leq \lceil \frac{q}{2} \rceil + 1$, then G is not an odd vertex equitable even graph. In addition they proved that if every edge of a graph G is an edge of a triangle, then G is not an odd vertex equitable even graph.
We use the following definitions in the subsequent section.

Definition 1.1. *The double triangular snake* $D(T_n)$ *is a graph obtained from a path* P_n *with vertices* $v_1, v_2, ..., v_n$ *by joining* v_i *and* v_{i+1} *to the new vertices* w_i *and* u_i *for* $i = 1, 2, ..., n - 1$.

Definition 1.2. *The double quadrilateral snake* $D(Q_n)$ *is a graph obtained from a path* P_n *with vertices* $u_1, u_2, ..., u_n$ *by joining* u_i *and* u_{i+1} *to the new vertices* v_i, x_i *and* w_i, y_i *respectively and then joining* v_i, w_i *and* x_i, y_i *for* $i = 1, 2, ..., n - 1$.

Definition 1.3. *A double alternate triangular snake* $DA(T_n)$ *consists of two alternate triangular snakes that have a common path. That is, a double alternate triangular snake is obtained from* *a path* $u_1, u_2, ..., u_n$ *by joining* u_i *and* u_{i+1} *(alternatively) to the two new vertices* v_i *and* w_i *for* $i = 1, 2, ..., n-1$.

Definition 1.4. *A double alternate quadrilateral snake* $DA(Q_n)$ *consists of two alternate quadrilateral snakes that have a common path. That is, a double alternate quadrilateral snake is obtained from a path* $u_1, u_2, ..., u_n$ *by joining* u_i *and* u_{i+1} *(alternatively) to the two new vertices* v_i, x_i *and* w_i, y_i *respectively and adding the edges* $v_i w_i$ *and* $x_i y_i$ *for* $i = 1, 2, ..., n-1$.

Definition 1.5. *Let* G *be a graph. The subdivision graph* $S(G)$ *is obtained from* G *by subdividing each edge of* G *with a vertex.*

Definition 1.6. *The corona* $G_1 \odot G_2$ *of the graphs* G_1 *and* G_2 *is defined as the graph obtained by taking one copy of* G_1 *(with* p *vertices) and* p *copies of* G_2 *and then joining the* i^{th} *vertex of* G_1 *to every vertex of the* i^{th} *copy of* G_2.

2 Main Results

In this section, we prove that $S(D(T_n))$, $S(D(Q_n))$, $DA(Q_m) \odot nK_1$ and $DA(T_m) \odot nK_1$ are odd vertex equitable even graphs.

Theorem 2.1. *Let* $G_1(p_1, q_1)$, $G_2(p_2, q_2)$,...,$G_m(p_m, q_m)$ *be an odd vertex equitable even graphs with each* q_i *is even for* $i = 1, 2, ..., m-1$, q_m *is even or odd and let* u_i, v_i *be the vertices of* $G_i (1 \leq i \leq m)$ *labeled by 1,* q_i *if* q_i *is odd or* $q_i + 1$ *if* q_i *is even. Then the graph* G *obtained by identifying* v_1 *with* u_2 *and* v_2 *with* u_3 *and* v_3 *with* u_4 *and so on until we identify* v_{m-1} *with* u_m *is also an odd vertex equitable even graph.*

Proof. The graph G has $p_1 + p_2 + ... + p_m - (m-1)$ vertices and $\sum_{i=1}^{m} q_i$ edges and f_i be an odd vertex equitable even labeling of $G_i (1 \leq i \leq m)$.

Let $A = \left\{ \begin{array}{ll} 1, 3, 5, ..., \sum_{i=1}^{m} q_i, & \text{if } \sum_{i=1}^{m} q_i \text{ is odd} \\ 1, 3, 5, ..., \sum_{i=1}^{m} q_i + 1, & \text{if } \sum_{i=1}^{m} q_i \text{ is even} \end{array} \right\}$.

Define a vertex labeling $f : V(G) \rightarrow A$ as follows: $f(x) = f_1(x)$ if $x \in V(G_1)$, $f(x) = f_i(x) + \sum_{k=1}^{i-1} q_k$ if $x \in V(G_i)$ for $2 \leq i \leq m$. The edge labels of the graph G_1 will remain fixed, the edge labels of the graph $G_i (2 \leq i \leq m)$ are $2q_1 + 2, 2q_1 + 4, ..., 2(q_1 + q_2)$;$2(q_1 + q_2) + 2, 2(q_1 + q_2) + 4, ..., 2(q_1 + q_2 + q_3)$;...,$2\sum_{i=1}^{m-1} q_i + 2, 2\sum_{i=1}^{m-1} q_i + 4, ..., 2\sum_{i=1}^{m} q_i$. Hence the edge labels of G are distinct and is $\{2, 4, 6, ..., 2\sum_{i=1}^{m} q_i\}$. Also $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$. Hence G is an odd vertex equitable even graph. □

Theorem 2.2. *The graph* $S(D(T_n))$ *is an odd vertex equitable even graph.*

Proof. Let $G_i = S(D(T_2))$ $1 \leq i \leq n-1$ and u_i, v_i be the vertices with labels 1 and $q + 1$ respectively. By Theorem 2.1, $S(D(T_2))$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = S(D(T_2))$ is given in Figure 1.

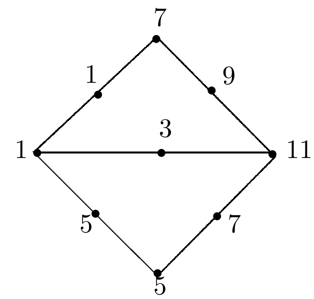


Figure 1. □

Theorem 2.3. *The graph* $S(D(Q_n))$ *is an odd vertex equitable even graph.*

Proof. Let $G_i = S(D(Q_2))$ $1 \leq i \leq n-1$ and u_i, v_i be the vertices with labels 1 and $q+1$ respectively. By Theorem 2.1, $S(D(Q_2))$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = S(D(Q_2))$ is given in Figure 2.

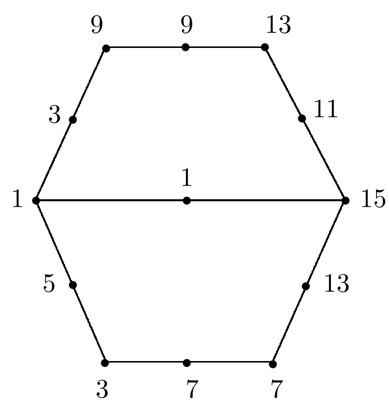


Figure 2. □

Theorem 2.4. *The double quadrilateral graph* $D(Q_{2n})$ *is an odd vertex equitable even graph.*

Proof. Let $G_i = D(Q_4)$ $1 \leq i \leq n-1$ and u_i, v_i be the vertices with labels 1 and $q+1$ respectively. By Theorem 2.1, $D(Q_4)$ admits an odd vertex equitable even labeling. An odd vertex equitable even labeling of $G_i = D(Q_4)$ is given in Figure 3.

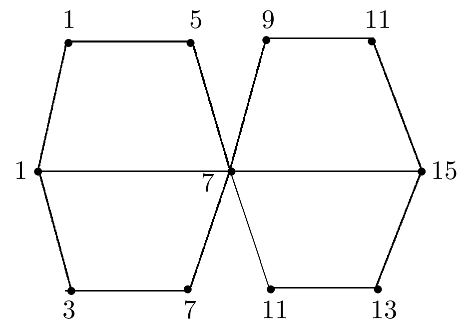


Figure 3. ☐

Theorem 2.5. *Let $G_1(p_1, q), G_2(p_2, q), ..., G_m(p_m, q)$ be an odd vertex equitable even graphs with q odd and u_i,v_i be vertices of $G_i(1 \leq i \leq m)$ labeled by 1 and q. Then the graph G obtained by joining v_1 with u_2 and v_2 with u_3 and v_3 with u_4 and so on until joining v_{m-1} with u_m by an edge is also an odd vertex equitable even graph.*

Proof. The graph G has $p_1 + p_2 + ... + p_m$ vertices and $mq + (m - 1)$ edges.
Let f_i be the odd vertex equitable even labeling of $G_i(1 \leq i \leq m)$ and
let $A = \{1, 3, ..., mq + (m - 1)\}$.
Define a vertex labeling $f : V(G) \to A$ as
$f(x) = f_i(x) + (i - 1)(q + 1)$ if $x \in G_i$ for $1 \leq i \leq m$.
The edge labels of G_i are incresed by $2(i - 1)(q + 1)$ for $i = 1, 2, ..., m$ under the new labeling f.
The bridge between the two graphs G_i, G_{i+1} will get the label $2i(q + 1)$, $1 \leq i \leq m - 1$.
Hence the edge labels of G are distinct and is $\{2, 4, ..., 2(mq + m - 1)\}$.
Also $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.
Then the graph G is an odd vertex equitable even graph. ☐

Theorem 2.6. *The graph $DA(T_2) \odot nK_1$ is an odd vertex equitable even graph for $n \geq 1$.*

Proof. Let $G = DA(T_2) \odot nK_1$. Let $V(G) = \{u_1, u_2, u, w\} \cup \{u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and
$E(G) = \{u_1u_2, u_1v, vu_2, u_1w, wu_2\} \cup \{u_iu_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\} \cup \{vv_i, ww_i : 1 \leq i \leq n\}$.
Here $|V(G)| = 4(n + 1)$ and $|E(G)| = 4n + 5$.
Let $A = \{1, 3, ..., 4n + 5\}$.
Define a vertex labeling $f : V(G) \to A$ as follows:
For $1 \leq i \leq n$ $f(u_1) = 1$, $f(u_2) = 4n + 5$, $f(v) = 2n + 1$, $f(w) = 2n + 5$, $f(u_{1i}) = 2i - 1$, $f(u_{2i}) = 4n + 5 - 2(i - 1)$,

$$f(v_i) = \begin{cases} 3 & \text{if } i=1 \\ 2i + 3 & \text{if } 2 \leq i \leq n, \end{cases}$$

$$f(w_i) = \begin{cases} 2(n + i) + 1 & \text{if } 1 \leq i \leq n - 1 \\ 4n + 3 & \text{if } i=n. \end{cases}$$

It can be verified that the induced edge labels of $DA(T_2) \odot nK_1$ are $2, 4, ..., 8n+10$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.
Hence f is an odd vertex equitable even labeling $DA(T_2) \odot nK_1$.
An odd vertex equitable even labeling of $DA(T_2) \odot 3K_1$ is shown in Figure 4.

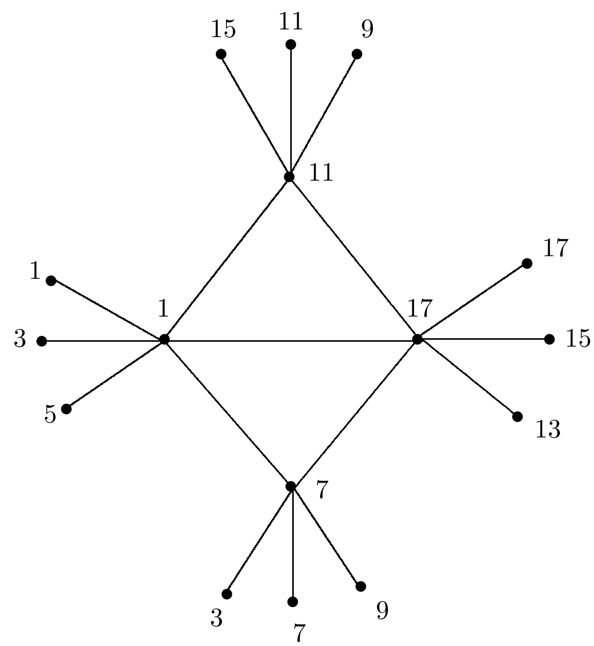


Figure 4.

□

Theorem 2.7. *The graph* $DA(Q_2) \odot nK_1$ *is an odd vertex equitable even graph for* $n \geq 1$.

Proof. Let $G = DA(Q_2) \odot nK_1$. Let $V(G) = \{u_1, u_2, v, w, x, y\} \cup \{v_i, w_i, x_i, y_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$ and
$E(G) = \{u_1u_2, u_1v, vw, wu_2, u_1x, xy, yu_2\} \cup \{vv_i, ww_i, xx_i, yy_i : 1 \leq i \leq n\} \cup \{u_iu_{ij} : 1 \leq i \leq 2, 1 \leq j \leq n\}$.
Here $|V(G)| = 6(n+1)$ and $|E(G)| = 6n+7$.
Let $A = \{1, 3, ..., 6n+7\}$.
Define a vertex labeling $f : V(G) \to A$ as follows:
For $1 \leq i \leq n$ $f(u_1) = 1$, $f(u_2) = 6n+7$, $f(u_{1i}) = 2i-1$, $f(u_{2i}) = 6n-2i+9$, $f(v) = 2n+1$, $f(w) = 2n+3$, $f(x) = 4n+5$, $f(y) = 4n+7$, $f(v_i) = 2i+1$, $f(w_i) = f(x_i) = 2n+2i+3$, $f(y_i) = 4n+2i+5$.
It can be verified that the induced edge labels of $DA(Q_2) \odot nK_1$ are $2, 4, ..., 12n+14$ and $|v_f(a) - v_f(b)| \leq 1$ for all $a, b \in A$.
Hence f is an odd vertex equitable even labeling of $DA(Q_2) \odot nK_1$.

An odd vertex equitable even labeling of $DA(Q_2) \odot 4K_1$ is shown in Figure 5.

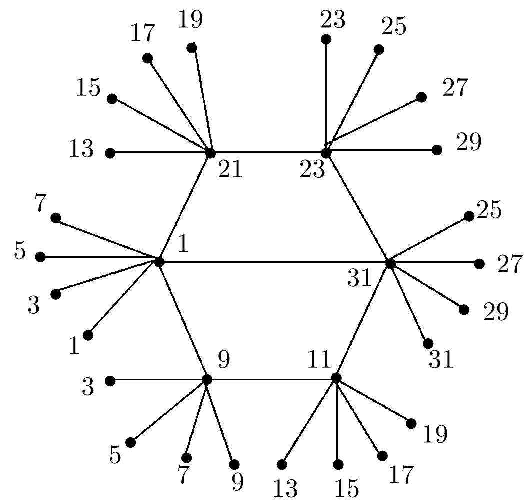
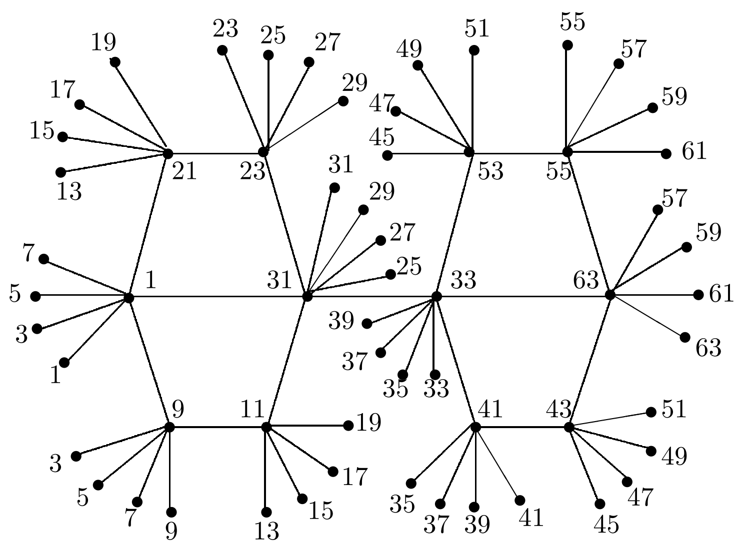


Figure 5. □

Theorem 2.8. *The graph* $DA(Q_m) \odot nK_1$ *is an odd vertex equitable even graph for* $m, n \geq 1$.

Proof. By Theorem 2.7, $DA(Q_2) \odot nK_1$ is an odd vertex equitable even graph. Let $G_i = DA(Q_2) \odot nK_1$ for $1 \leq i \leq m-1$. Since each G_i has 6n+7 edges, by Theorem 2.5, $DA(Q_m) \odot nK_1$ admits odd vertex equitable even labeling.
An odd vertex equitable even labeling of $DA(Q_4) \odot 4K_1$ is shown in Figure 6.



□

Figure 6.

Theorem 2.9. *The graph* $DA(T_m) \odot nK_1$ *is an odd vertex equitable even graph for* $m, n \geq 1$.

Proof. By Theorem 2.6, $DA(T_2) \odot nK_1$ is an odd vertex equitable even graph. Let $G_i = DA(T_2) \odot nK_1$ for $1 \leq i \leq m-1$. Since each G_i has 4n+5 edges, by Theorem 2.5, $DA(T_m) \odot nK_1$ admits odd vertex equitable even labeling.
An odd vertex equitable even labeling of $DA(T_4) \odot 3K_1$ is shown in Figure 7.

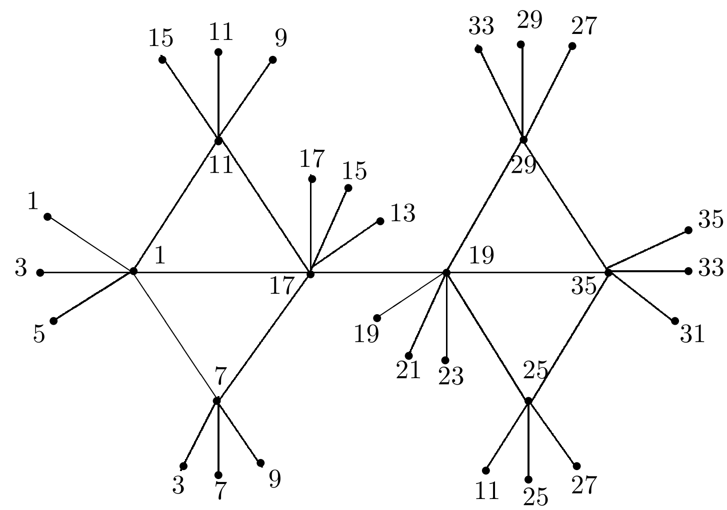


Figure 7.

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