CUBO A Mathematical Journal
Vol.20, N^{o}01, (79–94). March 2018
http://dx.doi.org/10.4067/S0719-06462018000100079

Anti-invariant $\xi^{\perp}$-Riemannian Submersions From Hyperbolic β-Kenmotsu Manifolds

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ABSTRACT

In this paper, we introduce anti-invariant $\xi^{\perp}$-Riemannian submersions from Hyperbolic β-Kenmotsu Manifolds onto Riemannian manifolds. Necessary and sufficient conditions for a special anti-invariant $\xi^{\perp}$-Riemannian submersion to be totally geodesic are studied. Moreover, we obtain decomposition theorems for the total manifold of such submersions.

RESUMEN

En este artículo se introducen las submersiones $\xi^{\perp}$-Riemannianas anti-invariantes desde variedades hiperbólicas β-Kenmotsu sobre variedades Riemannianas. Se estudian condiciones necesarias y suficientes para que ciertas submersiones $\xi^{\perp}$-Riemannianas anti-invariantes especiales sean totalmente geodésicas. Más aún, se obtienen teoremas de descomposión para la variedad total de dichas submersiones.

Keywords and Phrases: Riemannian submersion Anti-invariant $\xi^{\perp}$-Riemannian submersions, Hyperbolic β-Kenmotsu Manifolds, Integrability Conditions. geometry.

2010 AMS Mathematics Subject Classification: 53C25, 53C20, 53C50, 53C40.

1 Introduction

The geometry of Riemannian submersions between Riemannian manifolds has been intensively studied and sevral results has been pulished (see O'Neill [7] and Gray [4]). In [11] Waston defined almost Hermitian submersion between almost Hermitian manifolds and in most cases he show that the base manifold and each fiber has the same kind of structure as the total space. He also show that the vertical and horizontal distributions are invariant. On the other hand, the geometry of anti-invariant Riemannian submersions is different from the geometry of almost Hermitian submersions. For example, since every holomorphic map between Kahler manifolds is harmonic [2], it follows that any holomorphic submersion between Kahler manifolds is harmonic. However, this result is not valid for anti-invariant Riemannian submersions, which was first studied by Sahin in [8]. Similarly, Ianus and Pastore [5] shows ϕ-holomorphic maps between contact manifolds are harmonic. This implies that any contact submersion is harmonic. However, this result is not valid for anti-invariant Riemannian submersions. In [1], Chinea defined almost contact Riemannian submersion between almost contact metric manifolds. In [6], Lee studied the vertical and horizontal distribution are ϕ-invariant. Moreover, the characteristic vector field ξ is horizontal. We note that only ϕ-holomorphic submersions have been consider on an almost contact manifolds [3]. It was 1976, Upadhyay and Dube [10] introduced the notion of almost hyperbolic contact (f, g, η, ξ)-structure. Some properties of CR-submanifolds of trans hyperbolic Sasakian manifold were studied in [9]. In this paper, we consider a Riemannian submersion from a Hyperbolic β-Kenmotsu Manifolds under the assumption that the fibers are anti-invariant with respect to the tensor field of type $(1, 1)$ of almost hyperbolic contact manifold. This assumption implies that the horizontal distribution is not invariant under the action of tensor field of the total manifold of such submersions. In other words, almost hyperbolic contact are useful for describing the geometry of base manifolds, anti-invariant submersion are however served to determine the geometry of total manifold.

The paper is organized as follows: In Section 2, we present the basic information needed for this paper. In Section 3, we give the definition of anti-invariant $\xi^{\perp}$-Riemannian submersions. We also introduce a special anti-invariant $\xi^{\perp}$-Riemannian submersions and obtain necessary and sufficient conditions for such submersions to be totally geodesic or harmonic. In Section 4, we give decomposition theorems by using the existence of anti-invariant $\xi^{\perp}$-Riemannian submersions and observe that such submersions put some restrictions on the geometry of the total manifold.

2 Preliminaries

In this section, we define almost hyperbolic contact manifolds, recall the notion of Riemannian submersion between Riemannian manifolds and give a brife review of basic facts if Riemannian submersion.

Let M be an almost hyperbolic contact metric manifold with an almost hyperbolic contact metric structure (ϕ, ξ, η, g_M), where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and

g_M is a compatible Riemannian metric on M such that

$$\phi^2 = I - \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g_M(\phi X, \phi Y) = -g_M(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g_M(X, \phi Y) = -g_M(\phi X, Y), \quad g_M(X, \xi) = \eta(X) \tag{2.3}$$

An almost hyperbolic contact metric structure (ϕ, ξ, η, g_M) on M is called trans-hyperbolic Sasakian [9] if and only if

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)\phi X) + \beta(g(\phi X, Y) - \eta(Y)\phi X) \tag{2.4}$$

for all X, Y tangent to M, α and β are smooth functions on M and we say that the trans-hyperbolic Sasakian structure of type (α, β). From the above condition it follows that

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \tag{2.6}$$

where ∇ is the Riemannian connection of Levi-Civita covariant differentiation.

More generally one has the notion of a hyperbolic β-Kenmotsu structure which be defined by

$$(\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.7}$$

where β is non-zero smooth function. Also we have

$$\nabla_X \xi = \beta[X - \eta(X)\xi]. \tag{2.8}$$

Thus $\alpha = 0$ and therefore a trans-hyperbolic Sasakian structure of type $(0, \beta)$ with a non-zero constant is always hyperbolic β-Kenmotsu manifold.

Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds, where $dim M = m$, $dim N = N$ and $m > n$. A Riemannian submersion $F : M \to N$ is a map from M onto N satisfying the following axioms:

(1) (S1) F has maximal rank

(2) (S2) The differential F_* preserves the lengths of horizontal vectors.

For each $q \in N$, $F^{-1}(q)$ is an $(m - n)$-dimensional submanifold of M. The submanifold $F^{-1}(q)$ are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and F-related to a vector field X_* on N, i.e., $F_* X_p = X_* F(p)$ for all $p \in M$. Note that we denote the projection morphisms on the distributions $ker F_*$ and $(ker F_*)$ by V and H, respectively.

We recall the following lemma from O'Neill [7].

Lemma 2.1. *Let* $F : M \to N$ *be a Riemannian submersion between Riemannian manifolds and* X, Y *be basic vector fields of* M. *Then*

(1) *(1)* $g_M(X, Y) = g_N(X_*, Y_*) \circ F.$

(2) *(2)* the horizontal part $[X, Y]^H$ *of* $[X, Y]$ *is a basic vector field and corresponds to* $[X_*, Y_*]$, *i.e.,* $F_*([X, Y]) = [X_*, Y_*]$.

(3) *(3)* $[V, X]$ *is vertical for any vector field* V *of* $\ker F_*$.

(4) *(4)* $((\nabla)^M_X Y)^H$ *is the basic vector field corresponding to* $\nabla^N_{X_*} Y_*$.

The geometry of Riemannian submersion is characterized by O'Neill's tensor T and A defined for vector fields E, F on M by

$$A_E F = H\nabla_{HE} VF + V\nabla_{HE} HF \tag{2.9}$$

$$T_E F = H\nabla_{VE} VF + V\nabla_{VE} HF \tag{2.10}$$

where ∇ is the Levi-Civita connection of g_M. It is easy to see that a Riemannian submersion $F : M \to N$ has totally geodesic fibers if and only if T vanishes identically. For any $E \in (TM)$, $T_C = T_{VC}$ and A is horizontal, $A = A_{HE}$. We note that the tensor T and A satisfy

$$T_U W = T_W U, \qquad U, W \in (\ker F_*) \tag{2.11}$$

$$A_X Y = -A_Y X = \frac{1}{2} V[X, Y], \qquad X, Y \in (\ker F_*)^{\perp} \tag{2.12}$$

On the other hand, from (2.6) and (2,7), we have

$$\nabla_V W = T_V W + \bar{\nabla}_V W \tag{2.13}$$

$$\nabla_V X = H\nabla_V X + T_V X \tag{2.14}$$

$$\nabla_X V = A_X V + V\nabla_X V \tag{2.15}$$

$$\nabla_X Y = H\nabla_X Y + A_X V \tag{2.16}$$

for $X, Y \in (\ker F_*)^{\perp}$ and $V, W \in (\ker F_*)$, where $\bar{\nabla}_V W = V\nabla_V W$. If X is basic then $H\nabla_V X = A_X V$.

Finally, we recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and supposed that $\phi : M \to N$ is a smooth map. Then the differential ϕ_* of ϕ can be viewed a section of the bundle $\mathrm{Hom}(TM, \phi^{-1}TN) \to M$, where $\phi^{-1}TN$ is the pullback bundle which has fibers $(\phi^{-1}TN)_p = T_{\phi(p)}N$, $p \in M$. $\mathrm{Hom}(TM, \phi^{-1}TN)$

has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection ∇^{ϕ}. Then the second fundamental form of ϕ is given by

$$(\nabla\phi*)(X,Y) = \nabla^{\phi}_X \phi * (Y) - \phi * (\nabla^M_X Y) \tag{2.17}$$

for $X, Y \in TM$. It is known that the second fundamental form is symmetric. A smooth map $\phi : (M, g_M) \to (N, g_N)$ is said to be harmonic if $trace(\nabla\phi*) = 0$. On the other hand, the tensor field of ϕ is the section $\tau(\phi)$ of $(\phi^{-1}TN)$ defined by

$$\tau(\phi) = div\phi* = \sum_{i=1}^{m} (\nabla\phi*)(e_i, e_i), \tag{2.18}$$

where $\{e_1, ...e_m\}$ is the orthogonal frame on M. Then it follows that ϕ is harmonic if and only if $\tau(\phi) = 0$ (see [7]).

3 Anti-invariant $\xi^{\perp}$- Riemannian Submersions

In this section, we define anti-invariant $\xi^{\perp}$- Riemannian submersion from hyperbolic β-Kenmotsu manifold onto a Riemannian manifold and investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map. We also investigate the harmonicity of a special Riemannian submersion.

Definition 3.1. *Let* $(M, g_M, \phi, \xi, \eta)$ *be a hyperbolic* β*-Kenmotsu manifold and* (N, g_N) *a Riemannian manifold. Suppose that there exists a Riemannian submersion* $F : M \to N$ *such that* ξ *is normal to* $kerF_*$ *and* $kerF_*$ *is anti-invariant with respect to* ϕ*, ie.,* $\phi(kerF_*) \subset (kerF_*)^{\perp}$*. Then we say that* F *is an anti-invariant* $\xi^{\perp}$*-Riemannian submersion.*

Now, we assume that $F : (M, g_M, \phi, \xi, \eta) \to (N, g_N)$ is an anti-invariant $\xi^{\perp}$-Riemannian submersion. First of all, from Definition 3.1, we have $(kerF_*)^{\perp} \cap (kerF_*) \neq 0$. We denote the complementary orthogonal distribution to $\phi(kerF_*)$ in $(kerF_*)^{\perp}$ by μ. Then we have

$$(kerF_*)^{\perp} = \phi(kerF_*) \oplus \mu, \tag{3.1}$$

where $\phi(\mu) \subset \mu$. Hence μ contains ξ. Thus, for $X \in (kerF_*)^{\perp}$, we have

$$\phi X = BX + CX, \tag{3.2}$$

where $BX \in (kerF_*)$ and $CX \in (\mu)$. On the other hand, since $F_*(kerF_*)^{\perp} = TN$ and F is a Riemannian submersion, using (3.2), we have

$$g_N(F_*\phi V, F_*\phi CX) = 0$$

for any $X \in (kerF_*)^{\perp}$ and $V \in (kerF_*)$, which implies

$$TN = F_*(\phi((kerF_*)) \oplus F_*(\mu).$$

Example 3.2. *Let us consider a 5-dimensional manifold* $\bar{M} = \{(x_1, x_2, x_3, x_4, z) \in \mathbb{R}^5 : z \neq 0\}$, *where* (x_1, x_2, x_3, x_4, z) *are standard coordinates in* $\mathbb{R}^5$.

We choose the vector fields
$E_1 = e^{-z}\frac{\partial}{\partial x_1}$, $E_2 = e^{-z}\frac{\partial}{\partial x_2}$, $E_3 = e^{-z}\frac{\partial}{\partial x_3}$, $E_4 = e^{-z}\frac{\partial}{\partial x_4}$, $E_5 = e^{-z}\frac{\partial}{\partial x_1}$,
which are linearly independent at each point of $\bar{M}$. We define g by

$$g = e^{2z}G,$$

where G is the Euclidean metric on R^5. Hence $\{E_1, E_2, E_3, E_4, E_5\}$ is an orthonormal basis of $\bar{M}$.

We consider an 1-form η defined by

$$\eta = e^z dz, \quad \eta(X) = g(X, E_5), \qquad \forall X \in T\bar{M}.$$

We defined the $(1,1)$ tensor field ϕ by

$$\phi\left\{\sum_{i=2}^{2}\left(x_i\frac{\partial}{\partial x_i} + x_{i+2}\frac{\partial}{\partial x_{i+2}} + z\frac{\partial}{\partial z}\right)\right\} = \sum_{i=2}^{2}\left(x_i\frac{\partial}{\partial x_{i+2}} - x_{i+2}\frac{\partial}{\partial x_i}\right).$$

Thus, we have

$$\phi(E_1) = E_3, \ \phi(E_2) = E_4, \ \phi(E_3) = -E_1, \ \phi(E_4) = -E_2, \ \phi(E_5) = 0.$$

The linear property of g and ϕ yields that

$$\eta(E_5) = -1, \quad \phi^2(X) = X - \eta(X)E_5$$

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on $\bar{M}$. Thus, $\bar{M}$ (ϕ, ξ, η, g) defines an almost hyperbolic contact metric manifold with $\xi = E_5$. Moreover, let $\bar{\nabla}$ be the Levi-Civita connection with respect to metric g. Then we have $[E_1, E_2] = 0$. Similarly $[E_1, \xi] = e^{-z}E_1$, $[E_2, \xi] = e^{-z}E_2$, $[E_3, \xi] = e^{-z}E_3$, $[E_4, \xi] = e^{-z}E_4$, $[E_i, E_j] = 0$, $1 \leq i \neq \leq 4$.
The Riemannian connection $\bar{\nabla}$ of the metric g is given by

$$2g(\bar{\nabla}_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

By Koszul's formula, we obtain the following equations

$$\bar{\nabla}_{E_1}E_1 = -e^{-z}\xi, \ \bar{\nabla}_{E_2}E_2 = -e^{-z}\xi, \ \bar{\nabla}_{E_3}E_3 = -e^{-z}\xi, \ \bar{\nabla}_{E_4}E_4 = -e^{-z}\xi,$$

$$\bar{\nabla}_\xi \xi = 0, \ \bar{\nabla}_\xi E_i = 0, \ \bar{\nabla}_{E_i}\xi = e^{-z}E_i, \ 1 \leq i \leq 4$$

and $\bar{\nabla}_{E_i}E_i = 0$ for all $1 \leq i, j \leq 4$. Thus, we see that M is a trans-hyperbolic Sasakian manifold of type $(0, e^{-z})$, which is hyperbolic β-Kenmotsu manifold. Here $\alpha = 0$ and $\beta = e^{-z}$.

Now, we define $(1,1)$ tensor field as follows

$$\phi(x_1, x_2, x_3, x_4, z) = (-x_3, -x_4, x_1, x_3, z).$$

Now, we can give the following example.

Example 3.3. *Let* $(M_1, g_1 = e^{2z}G, \phi, \xi, \eta)$ *be an almost Hyperbolic contact manifolds and* M_2 *be* $\mathbb{R}^3$*. The Riemannian metric tensor field* g_2 *is defined by* $g_2 = e^{2z}(dy_1 \otimes dy_1 + dy_2 \otimes dy_2 + dy_3 \otimes dy_3)$ *on* M_2*.*

Let ϕ be a submersion defined by

$$\begin{array}{cccc} \phi: & \mathbb{R}^5 & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3, x_4, z) & & (\frac{x_1 + x_3}{\sqrt{2}}, z, \frac{x_1 + x_2}{\sqrt{2}}) \end{array}$$

Then it follows that

$$\ker\phi_* = \mathrm{span}\{V_1 = \partial x_1 - \partial x_3, \quad V_2 = \partial x_2 - \partial x_2\}$$

and

$$(\ker\phi_*)^{\perp} = \mathrm{span}\{X_1 = \partial x_1 + \partial x_3, \quad X_2 = \partial x_2 + \partial x_2, \quad X_3 = z = \xi\}$$

Hence we have $\phi V_1 = X_1$ and $\phi V_2 = X_2$. It means that $\phi(\ker\phi) \subset (\ker\phi)^{\perp}$. A straight computations, we get $\phi_* X_1 = \partial y_1$, $\phi_* X_2 = \partial y_3$ and $\phi_* X_3 = \partial y_2$. Hence, we have

$$g_1(X_i, X_i) = g_2(\phi_* X_i, \phi_* X_i), \quad \text{for} \quad i = 1, 2, 3.$$

Thus ϕ is a anti-invariant $\xi^{\perp}$ Riemannian submersion.

Lemma 3.4. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N)*. Then we have*

$$g_M(CY, \phi V) = 0, \tag{3.3}$$
$$g_M(\nabla_X CY, \phi V) = -g_M(CY, \phi A_X V) \tag{3.4}$$

for $X, Y \in ((\ker F_*)^{\perp})$ and $V \in (\ker F_*)$.

Proof. For $Y \in ((\ker F_*)^{\perp})$ and $V \in (\ker F_*)$, using (2.2), we have

$$g_M(CY, \phi V) = g_M(\phi Y - BY, \phi V) = g_M(\phi Y, \phi V) = -g_M(Y, V) - \eta(Y)\eta(V) = -g_M(Y, V) = 0$$

since $BY \in (\ker F_*)$ and $\phi V, \xi \in ((\ker F_*)^{\perp})$. Differentiating (3.3) with respect to X, we get

$$\begin{aligned} g_M(\nabla_X CY, \phi V) =& - g_M(CY, \nabla_X \phi V) \\ =& g_M(CY, (\nabla_X \phi)V) - g_M(CY, \phi(\nabla_X V)) \\ =& - g_M(CY, \phi(\nabla_X V)) \\ =& - g_M(CY, \phi A_X V) - g_M(CY, \phi \nu \nabla_X V) \\ =& - g_M(CY, \phi A_X V) \end{aligned}$$

due to $\phi\nu\nabla_X V \in (\ker F_*))$. Our assertion is complete. $\square$

We study the integrability of the distribution $(\ker F_*)^\perp$ and then we investigate the geometry of leaves of $\ker F_*$ and $(\ker F_*)^\perp$. We note it is known that the distribution $(\ker F_*)$ is integrable.

Theorem 3.5. *Let F be an anti-invaraint $\xi^\perp$-Riemannian submersion from a hyperbolic β-Kenmotsu manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N). The followings are equivalent.*

(1) $(\ker F_*)^\perp$ is integrable,

(2)
$$\begin{aligned} g_N((\nabla F_*)(Y, BX), F_*\phi V) = g_N((\nabla F_*)(X, BY), F_*\phi V) \\ +g_M(CY, \phi A_X V) - g_M(CX, \phi A_Y V) \\ +\beta\eta(Y)g_M(X, V) - \beta\eta(X)g_M(Y, V), \end{aligned}$$

(3)
$$\begin{aligned} g_M(A_X BY - A_Y BY, \phi V) = g_M(CY, \phi A_X V) - g_M(CX, \phi A_Y V) \\ +\beta\eta(Y)g_M(X, V) - \beta\eta(X)g_M(Y, V). \end{aligned}$$

for $X, Y \in (\ker F_*)^\perp$ and $V \in (\ker F_*)$.

Proof. For $Y \in (\ker F_*)^\perp$ and $V \in (\ker F_*)$, from Definition 3.1, $\phi V \in (\ker F_*)^\perp$ and $\phi Y \in (\ker F_*) \oplus \mu$. Using (2.2) and (2.4), we note that for $X \in (\ker F_*)^\perp$,

$$\begin{aligned} g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V) - \beta\eta(Y)g_M(X, V) \\ -(\alpha + \beta)\eta(X)\eta(Y)\eta(V). \end{aligned} \tag{3.5}$$

Therefore, from (3.5), we get

$$\begin{aligned} g_M([X, Y], V) = g_M(\nabla_X \phi Y, \phi V) - g_M(\nabla_Y \phi X, \phi V) \\ = \beta\eta(X)g_M(Y, V) - \beta\eta(Y)g_M(X, V) \\ = g_M(\nabla_X BY, \phi V) + g_M(\nabla_X CY, \phi V) \\ -g_M(\nabla_Y BX, \phi V) - g_M(\nabla_Y CX, \phi V) \\ -\beta\eta(Y)g_M(X, V) + \beta\eta(X)g_M(Y, V). \end{aligned}$$

Since F is a Riemannian submersion, we obtain

$$\begin{aligned} g_M([X, Y], V) = g_N(F_*\nabla_X BY, F_*\phi V) + g_M(\nabla_X CY, \phi V) \\ -g_N(F_*\nabla_Y BX, F_*\phi V) - g_M(\nabla_Y CX, \phi V) \\ -\beta\eta(Y)g_M(X, V) + \beta\eta(X)g_M(Y, V). \end{aligned}$$

Thus, from (2.15) and (3.4), we have

$$g_M([X,Y],V) = g_N(-(\nabla F_*(X,BY) + (\nabla F_*)(Y,BX), F_*\phi V)$$
$$-g_M(CY, \phi A_X V + g_M(CX, \phi A_Y V)$$
$$-\beta\eta(Y)g_M(X,V) + \beta\eta(X)g_M(Y,V).$$

which proves $(1) \iff (2)$.
On the other hand, using (2.14), we obtain

$$(\nabla F_*)(Y,BX) - (\nabla F_*)(X,BY) = -F_*(\nabla_Y BX - \nabla_X BY) = -F_*(A_Y BX - A_X BY),$$

which shows that $(2) \iff (3)$ □

Corollary 3.6. *Let* F *be an anti-invaraint* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N) *with* $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus <\xi>$*. Then the following are equivalent:*

(1) $(\ker F_*)^{\perp}$ is integrable

(2) $(\nabla F_*)(X,\phi Y) + \beta\eta(X)F_*Y = (\nabla F_*)(Y,\phi X) + \beta\eta(Y)F_*X$

(3) $A_X\phi Y + \beta\eta(X)Y = A_Y\phi X + \beta\eta(Y)X$, for $X, Y \in (\ker F_*)^{\perp}$.

Theorem 3.7. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N)*. The following are equivalent:*

(1) $(\ker F_*)^{\perp}$ defines a totally geodesic foliation on M.

(2) $g_M(A_X BY, \phi V) = g_M(CY, \phi A_X Y) - \beta\eta(X)g_M(X,V) - \beta\eta(X)g_M(Y,V)$,

(3) $g_N((\nabla F_*)(Y,\phi X), F_*\phi V) = g_M(CY, \phi A_X V) - \beta\eta(X)g_M(X,V) - \beta\eta(X)g_M(Y,V)$, for $X, Y \in (\ker F_*)^{\perp}$ and $V \in (\ker F_*)$.

Proof. For $X, Y \in (\ker F_*)^{\perp}$ and $V \in (\ker F_*)$, from (3.5), we have

$$g_M(\nabla_X Y, V) = g_M(A_X BY, \phi V) + g_M(\nabla_X CY, \phi V) - \beta\eta(Y)g_M(X,V) - \beta\eta(X)\eta(Y)\eta(V)$$

Then from (3.4), we have

$$g_M(\nabla_X Y, V) = g_M(A_X BY, \phi V) + g_M(CY, \phi A_X V) - \beta\eta(Y)g_M(X,V) - \beta\eta(X)\eta(Y)\eta(V)$$

which shows $(1) \iff (2)$. On the other hand, from (2.12) and (2.14), we have

$$g_M(A_X BY, \phi V) = g_N(-(\nabla F_*)(X,BY), F_*\phi V),$$

which proves $(2) \iff (3)$. □

Corollary 3.8. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N) *with* $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus <\xi>$*. Then the following are equivalent:*

(1) $(\ker F_*)^{\perp}$ defines a totally geodesic folition on M

(2) $A_X \phi Y = \beta\eta(Y)X - (\alpha + \beta)\eta(X)Y$

(3) $(\nabla F_*)(Y, \phi X) = \beta\eta(Y)F_*X - \beta)\eta(X)F_*Y$

for $X, Y \in (\ker F_*)^{\perp}$.

Theorem 3.9. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N)*. The following are equivalent:*

(1) $\ker F_*$ defines a totally geodesic folition on M

(2) $-g_N(\nabla F_*)(V, \phi X, F_*\phi W) = 0$

(3) $T_V BX + A_{CX} V \in (\mu)$,

for $X, \in (\ker F_*)^{\perp}$ and $V, W \in (\ker F_*)$

Proof. For $X, \in (\ker F_*)^{\perp}$ and $V, W \in (\ker F_*)$, $g_M(W, \xi) = 0$ implies that from (2.4)

$$g_M(\nabla_V W, \xi) = -g_M(W, \nabla_V \xi) = g_M(W, \beta(V - \eta(V)\xi)) = 0.$$

Thus we have

$$\begin{aligned} g_M(\nabla_V W, X) &= -g_M(\phi\nabla_V W, \phi X) - \eta((\nabla_V W)\eta(X) \\ &= -g_M(\phi\nabla_V W, \phi X) \\ &= -g_M(\nabla_V \phi W, \phi X) + g_M((\nabla_V \phi)W, \phi X) \\ &= g_M(\phi W, \nabla_V \phi X). \end{aligned}$$

Since F is Riemannian submersion, we have

$$g_M(\nabla_V W, X) = g_N(F_*\phi W, F_*\nabla_V \phi X) = -g_N(F_*\phi W, (\nabla F_*)(V\phi X)),$$

which proves $(1) \iff (2)$.
By direct calculation, we derive

$$-g_N(F_*\phi W, (\nabla F_*)(V\phi X)) = g_M(\phi W, \nabla_V \phi X)$$

$$= g_M(\phi W, \nabla_V BX + \nabla_V CX)$$

$$= g_M(\phi W, \nabla_V BX + [V, CX] + \nabla_{CX} V).$$

$\square$

Since $[V, CX] \in (\ker F_*)$, from (2.10) and (2.12), we obtain

$$-g_N(F_*\phi W, (\nabla F_*)(V\phi X)) = g_M(\phi W, T_V BX + A_{CX}V),$$

which proves (2) $\Longleftrightarrow$ (3).
As an analouge of a Lagrangian Riemannian submersion in [11], we have a similar result;

Corollary 3.10. *Let* F *be an anti-invaraint* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N) *with* $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus <\xi, >$*. Then the following are equivalent:*

(1) $(\ker F_*)^{\perp}$ defines a totally geodesic folition on M

(2) $-(\nabla F_*)(V, \phi X) = 0$

(3) $T_V \phi W = 0$,

$X, \in (\ker F_*)^{\perp}$ and $V, W \in (\ker F_*)$.

Proof. From Theorem 3.6, it is enough to show (2) $\Longleftrightarrow$ (3). Using (2.14) and (2.11), we have

$$\begin{aligned} -g_N(F_*\phi W, (\nabla F_*)(V\phi X)) &= g_M(\nabla_V \phi W, \phi X) \\ &= g_M(T_V \phi W, \phi X). \end{aligned}$$

Since $T_V \phi W \in (\ker F_*)$, the proof is complete. □

We note that a differentiable map F between two Riemannian manifolds is called totally geodesic if $\nabla F_* = 0$. For the special Riemannian submersion, we have the following characterization.

Theorem 3.11. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Riemannian manifold* (N, g_N) *with* $(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus <\xi, >$*. Then* F *is a totally geodesic map if and only if*

$$T_V \phi W = 0, \qquad V, W \in (\ker F_*) \tag{3.6}$$

and

$$A_X \phi W = 0, \qquad X \in (\ker F_*^{\perp}). \tag{3.7}$$

Proof. First of all, we recall that the second fundamental form of a Riemannian submersion satisfies

$$(\nabla F_*)(X, Y) = 0 \qquad \forall \ X, Y \in (\ker F_*^{\perp}). \tag{3.8}$$

For $V, W \in (\ker F_*)$, we get

$$(\nabla F_*)(X,Y) = F_*(\phi T_V \phi W). \tag{3.9}$$

On the other hand, from (2.1), (2.2) and (2.14), we get

$$(\nabla F_*)(X,W) = F_*(\phi A_X \phi W), X \in (\ker F_*^\perp). \tag{3.10}$$

Therefore, F is totally geodesic if and only if

$$\phi(T_V \phi W) = 0 \quad \forall \ V, W \in (\ker F_*^\perp). \tag{3.11}$$

and

$$\phi(A_X \phi W) = 0 \quad \forall \quad X \in (\ker F_*^\perp). \tag{3.12}$$

From (2.2), (2.6) and (2.7), we have

$$T_V \phi W = 0 \quad \forall \quad V, W \in (\ker F_*). \tag{3.13}$$

and

$$A_X \phi W = 0 \quad \forall \quad X \in (\ker F_*^\perp).$$

From (2.4), F is totally geodesic if and only the equation (3.6) and (3.7) hold □

Finally, in this section, we give a necessary and sufficient condition for a special Riemannian submersion to be harmonic as an analouge of Lagrangian Riemannian submersion in [11].

Theorem 3.12. *Let F be an anti-invaraint $\xi^\perp$-Riemannian submersion from a hyperbolic β-Kenmotsu manifold $(M, g_M, \phi, \xi, \eta)$ onto a Riemannian manifold (N, g_N) with $(\ker F_*)^\perp = \phi(\ker F_*) \oplus <\xi>$. Then F is harmonic if and only if $\mathrm{Trace}(\phi T_V) = 0$ for $V \in (\ker F_*)$.*

Proof. From [5], we know that F is harmonic if and only if F has minimal fibers. Thus F is harmonic if and only if $\sum_{i=1}^{m_1} T_{e_i} e_i = 0$. On the other hand, from (2.4), (2.11) and (2.10), we have

$$T_V \phi W = \phi T_V W \tag{3.14}$$

due to $\xi \in (\ker F_*^\perp)$ for any $V, W \in (\ker F_*)$. Using (3.14), we get

$$\sum_{i=1}^{m_1} g_M(T_{e_i} \phi e_i, V) = \sum_{i=1}^{m_1} g_M(\phi T_{e_i} \phi e_i, V) = -\sum_{i=1}^{m_1} g_M(T_{e_i} e_i, \phi V)$$

for any $V \in (\ker F_*)$. Thus skew-symmetric T implies that

$$\sum_{i=1}^{m_1} g_M(\phi T_{e_i} \phi e_i, V) = -\sum_{i=1}^{m_1} g_M(T_{e_i} e_i, \phi V).$$

Using (2.8) and (2.2), we have

$$\sum_{i=1}^{m_1} g_M(e_i, \phi T_V e_i) = -\sum_{i=1}^{m_1} g_M(\phi e_i, T_V e_i) = -\sum_{i=1}^{m_1} g_M(T_{e_i} e_i, \phi V)$$

which shows our assertion. □

4 Decomposition theorems

In this section, we obtain decomposition theorems by using the existence of anti-invariant $\xi^{\perp}$-Riemannian submersions. First, we recall the following.

Theorem 4.1. [10] *Let* g *be a Riemannian metric on the manifold* $B = M \times N$ *and assume that the canonical foliations* D_M *and* D_N *intersect perpendicular every where. Then* g *is the metric tensor of*

(1) (i) a twisted product $M \times_f N$ if and only if D_M is totally geodesic foliation and D_N is a totally umbilical foliation.

(2) (ii) a warped product $M \times_f N$ if and only if D_M is totally geodesic foliation and D_N is a spheric foliation, i.e., it is umbilical and its mean curvature vector field is parallel.

(3) (iii) a usual product of Riemannian manifold if and only if D_M and D_N are totally geodesic foliations.

Our first decomposition theorem for anti-invariant $\xi^{\perp}$-Riemannian submersion comes from Theorem 3.4 and 3.6 in terms of the second fundamental forms of such submersions.

Theorem 4.2. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *on to a Riemannian manifold* (N, g_N)*. Then* M *is locally product manifold if and only if*

$$-g_N((\nabla F_*)(Y, \phi X), F_*\phi V) = g_M(CY, \phi A_X V) - \beta\eta(Y)g_M(X, V)$$

and

$$-g_N((\nabla F_*)(V, \phi X), F_*\phi W) = 0$$

for $X, Y \in (\ker F_*^{\perp})$ and $V, W \in (\ker F_*)$.
From Corollary 3.5 and 3.7, we have the following decomposition theorem:

Theorem 4.3. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *on to a Riemannian manifold* (N, g_N) *with* $(\ker F_*^{\perp}) \oplus <\xi>$*. Then* M *is a locally product manifold if and only if* $A_X\phi Y = (\alpha + \beta)\eta(Y)X$ *and* $T_V\phi W = 0$*, for* $X, Y \in (\ker F_*^{\perp})$ *and* $V, W \in (\ker F_*)$.

Next we obtain a decomposition theorem which is related to the notion of a twisted product manifold.

Theorem 4.4. *Let* F *be an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from a hyperbolic* β*-Kenmotsu manifold* $(M, g_M, \phi, \xi, \eta)$ *on to a Riemannian manifold* (N, g_N) *with* $(\ker F_*^{\perp}) \oplus <\xi>$*. Then* M *is locally twisted product manifold of the form* $M_{\ker F_*^{\perp}} \times_f M_{\ker F_*}$ *if and only if*

$$T_V\phi X = -g_M(X, T_V V)\,\|V\|^{-2} - \beta\eta(Y)g_M(\phi X, \phi V).$$

and

$$A_X\phi Y = \beta\eta(Y)X$$

for $X, Y \in (\ker F_*^{\perp})$ and $V \in (\ker F_*)$, where $M_{(\ker F_*^{\perp})}$ and $M_{(\ker F_*)}$ are integrable manifolds of the distributions $(\ker F_*^{\perp})$ and $(\ker F_*)$.

Proof. For $X \in (\ker F_*^{\perp})$ and $V \in (\ker F_*)$, from (2.4) and (2.11), we obtain

$$g_M(\nabla_V W, X) = g_M(T_V\phi W, \phi X) = -g_M(\phi W, T_V\phi X)$$

Since T_V is skew-symmetric. This implies that $\ker F_*$ is totally umbilical if and only if

$$T_V\phi X - \beta\eta(V)g_M(\phi X, \phi V) = -X(\lambda)\phi V,$$

where λ is a function on M. By direct computation,

$$T_V\phi X = -g_M(X, T_V V)\,\|V\|^{-2} - \beta\eta(Y)g_M(\phi X, \phi V).$$

□

Then the proof follows from Corollary 3.5
However, in the sequel, we show that the notion of anti-invariant $\xi^{\perp}$-Riemannian submersion puts some restrictions on the source manifold.

Theorem 4.5. *Let* $(M, g_M, \phi, \xi, \eta)$ *be a hyperbolic* β*-Kenmotsu manifold and* (N, g_N) *be a Riemannian manifold . Then there does not exist an anti-invariant* $\xi^{\perp}$*-Riemannian submersion from* M *to* N *with* $(\ker F_*)^{\perp} = \phi(\ker F_*)^{\perp} \oplus < \xi >$ *such that* M *is a locally proper twisted product manifold of the form* $M_{\ker F_*} \times_f M_{(\ker F_*)^{\perp}}$.

Proof. Suppose that $F : (M, g_M, \phi, \xi, \eta) \longrightarrow (N, g_N)$ is an anti-invaraiant $\xi^{\perp}$-Riemannian submersion with $(\ker F_*)^{\perp} = \phi(\ker F_*)^{\perp} \oplus < \xi >$ and M is a locally twisted product of the form $M_{\ker F_*} \times_f M_{(\ker F_*)^{\perp}}$ .Then $M_{\ker F_*}$ is a totally geodesic foliation and $M_{(\ker F_*^{\perp})}$ is a totally umbilical foliation. We denote the second fundamental form of $M_{(\ker F_*^{\perp})}$ by h. Then we have

$$g_M(\nabla_X Y, V) = g_M(h(X, Y), V) \qquad X, Y \in ((\ker F_*)^{\perp}, V \in (\ker F_*). \tag{4.1}$$

Since $M(^{\perp}_{\ker F_*})$ is a totally umbilical foliation, we have

$$g_M(\nabla_X Y, V) = g_M(H, V)g_M(X, Y),$$

where H is the mean curvature vector field of $M_{(\ker F_*)^{\perp}}$. On the other hand, from (3.5), we derive

$$g_M(\nabla_X Y, V) = -g_M(\phi Y, \nabla_X\phi V) - \beta\eta(Y)g(X, V) - \beta\eta(X)\eta(Y)\eta(V). \tag{4.2}$$

Using (2.13), we obtain

$$g_M(\nabla_X Y, V) = g_M(\phi Y, A_X \phi V) - \beta\eta(Y)g(X, V) - \beta\eta(X)\eta(Y)\eta(V) \tag{4.3}$$
$$= g_M(Y, A_X \phi V) - \beta g(X, V) - \beta\eta(X)\eta(V)\xi)$$

Therefore, from (4.1), (4.3) and (2.2), we have

$$A_X \phi V = g_M(H, V)\phi X + \eta(A_X \phi V)\xi.$$

Since $A_X \phi V \in (\ker F_*)$,

$$\eta(A_X \phi V) = g_M(A_X \phi V, \xi) = 0.$$

Thus, we have

$$A_X \phi V = g_M(H, V)\phi X.$$

Hence, we derive

$$g_M(A_X \phi V, \phi X) - \beta\eta(X)\eta(V)g(Y, \phi X) = -g_M(H, V)\left\{\|X\|^2 - \eta^2(X)\right\}$$

$$g_M(\nabla_X \phi V, \phi X) = -g_M(H, V)\left\{\|X\|^2 - \eta^2(X)\right\} + \beta\eta(X)\eta(V)g(Y, \phi X)$$
$$g_M(\nabla_X Y, V) + \beta\eta(Y)g(X, V) - \beta\eta(X)\eta(Y)\eta(V)$$
$$= -g_M(H, V)\left\{\|X\|^2 - \eta^2(X)\right\} + \beta\eta(X)\eta(V)g(Y, \phi X).$$

Thus using (2.9), we have $A_X X = 0$, which implies

$$\beta\eta(X)g_M(X, V) = -g_M(H, V)\left\{\|X\|^2 - \eta^2(X)\right\} + \beta\eta(X)\eta(Y)[\eta(V) - g_M(Y, \phi X)]$$

for every $X \in ((\ker F_*^{\perp})$, $V \in (\ker F_*)$. Choosing X which is orthogonal to ξ, $g_M(H, V)\|X\|^2 = 0$. Since g_M is the Riemannian metric and $H \in (\ker F_*)$, we conclude that $H = 0$, which shows $\ker F_*^{\perp}$ is totally geodesic, so M is usual product of Riemannian manifolds. □

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