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W_2-Curvature Tensor on Generalized Sasakian Space Forms

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ABSTRACT

In this paper, we study W_2-pseudosymmetric, W_2-locally symmetric, W_2-locally ϕ-symmetric and W_2-ϕ-recurrent generalized Sasakian space form. Further, illustrative examples are given.

RESUMEN

En este artículo, estudiamos formas espaciales Sasakianas generalizadas W_2-seudosimétricas, W_2-localmente ϕ-simétricas y W_2-ϕ-recurrentes. Ejemplos ilustrativos son dados.

Keywords and Phrases: Generalized Sasakian space form, W_2-curvature tensor, pseudosymmetric, ϕ-recurrent, Einstein manifold.

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1 Introduction

The nature of a Riemannian manifold depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine its curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as a real space form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

Representation for these spaces are hyperbolic spaces ($c < 0$), spheres ($c > 0$) and Euclidean spaces ($c = 0$).

The ϕ-sectional curvature of a Sasakian space form is defined by Sasakian manifold and it has a specific form of its curvature tensor. Same notion also holds for Kenmotsu and cosymplectic space forms. In order to generalize such space forms in a common frame Alegre, Blair and Carriazo [1] introduced and studied generalized Sasakian space forms.

A generalized Sasakian space form is an almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, whose curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y \\ &- g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.1}$$

The Riemanian curvature tensor of a generalized Sasakian space form $M^{2n+1}(f_1, f_2, f_3)$ is simply given by

$$R = f_1R_1 + f_2R_2 + f_3R_3,$$

where f_1, f_2, f_3 are differential functions on $M^{2n+1}(f_1, f_2, f_3)$ and

$$\begin{aligned} R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \quad and \\ R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi, \end{aligned}$$

where $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$. Here c denotes the constant ϕ-sectional curvature. The properties of generalized Sasakian space form was studied by many geometers such has [2, 9, 10, 14, 17, 18, 19, 21, 26]. The concept of local symmetry of a Riemanian manifold has been studied by many authors in several ways to a different extent. The locally ϕ-symmetry of Sasakian manifold was introduce by Takahashi in [28]. De and et al generalize this to the notion of ϕ-symmetry and then introduced the notion of ϕ-recurrent Sasakian manifold in [11]. Further ϕ-recurrent condition was studied on Kenmotsu manifold [8], LP-Sasakian manifold [29] and $(LCS)_n$-manifold [20].

In[16], Pokhariyal and Mishra have defined the W_2-curvature tensor, given by

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2n}\{g(X,Z)QY - g(Y,Z)QX\}, \tag{1.2}$$

here R and Q are the Riemanian curvature tensor and Ricci operator of Riemanian manifold respectively.

In a generalized Sasakian space forms, the W_2-curvature tensor satisfies the condition

$$\eta(W_2(X,Y)Z) = 0. \tag{1.3}$$

Many Geometers studied the W_2 curvature tensor studied on different manifolds such has generalized Sasakian space forms [13], Lorentzian para Sasakian manifolds [30] and Kenmotsu manifolds [25]

Motivated by these ideas, we made an attempt to study the properties of generalized Sasakian space form. The present paper is organized as follows: In section 2, we review some preliminary results. In section 3, we study W_2-pseudosymmetric generalized Sasakian space form. Section 4, deals with the W_2-locally symmetric generalized Sasakian space forms and it is shown that a generalized Sasakian space form of dimension greater than three is W_2-locally symmetric if and only if it is conformally flat. Section 5, is devoted to the study of W_2-locally ϕ-symmetric generalized Sasakian space forms. Finally in last section, we discus the W_2-ϕ-recurrent generalized Sasakian space form and found to be Einstein manifold.

2 Generalized Sasakian space-forms

The Riemannian manifold M^{2n+1} is called an almost contact metric manifold if the following result holds [5, 6]:

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0 \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \qquad \forall\ X, Y \in (T_p M). \tag{2.5}$$

A almost contact metric manifold is said to be Sasakian if and only if [5, 23]

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.6}$$

$$\nabla_X \xi = -\phi X. \tag{2.7}$$

Again we know that [1] in $(2n+1)$-dimensional generalized Sasakian space form:

$$\begin{aligned} S(X,Y) &= (2nf_1+3f_2-f_3)g(X,Y) \\ &- (3f_2+(2n-1)f_3)\eta(X)\eta(Y), \end{aligned} \tag{2.8}$$

$$S(\phi X,\phi Y) = S(X,Y)+2n(f_1-f_3)\eta(X)\eta(Y), \tag{2.9}$$

$$\begin{aligned} QX &= (2nf_1+3f_2-f_3)X \\ &- (3f_2+(2n-1)f_3)\eta(X)\xi, \end{aligned} \tag{2.10}$$

$$r = 2n(2n+1)f_1+6nf_2-4nf_3, \tag{2.11}$$

$$R(X,Y)\xi = (f_1-f_3)\{\eta(Y)X-\eta(X)Y\}, \tag{2.12}$$

$$R(\xi,X)Y = (f_1-f_3)\{g(X,Y)\xi-\eta(Y)X\}, \tag{2.13}$$

$$\eta(R(X,Y)Z) = (f_1-f_3)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}, \tag{2.14}$$

$$S(X,\xi) = 2n(f_1-f_3)\eta(X). \tag{2.15}$$

Here R, S, Q and r are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature tensor of generalized Sasakian space forms in that order.

3 W_2-pseudosymmetric generalized Sasakian space forms

The concept of a pseudosymmetric manifold was introduced by Chaki [7] and Deszcz [12]. In this article we shall study properties of pseudosymmetric manifold according to Deszcz. Semisymmetric manifolds satisfies the condition $R\cdot R=0$ and were categorized by Szabo in [27]. Every pseudosymmetric manifold is semisymmetric but semisymmetric manifold need not be pseudosymmetric.

An $(2n+1)$-dimensional Riemannian manifold M^{2n+1} is said to be pseudosymmetric, if

$$(R(X,Y)\cdot R)(U,V)W = L_R\{((X\wedge Y)\cdot R)(U,V)W)\}. \tag{3.1}$$

where L_R is some smooth function on $U_R=\{x\in M^{2n+1}|R-\frac{r}{n(n-1)}G\neq 0 \text{ at } x\}$, where G is the $(0,4)$-tensor defined by $G(X_1,X_2,X_3,X_4)=g((X_1\wedge X_2)X_3,X_4)$ and $(X\wedge Y)Z$ is the endomorphism and it is defined as,

$$(X\wedge Y)Z = g(Y,Z)X-g(X,Z)Y \tag{3.2}$$

An $(2n+1)$-dimensional generalized Sasakian space form M^{2n+1} is said to be W_2-pseudosymmetric, if

$$(R(X,Y)\cdot W_2)(U,V)Z = L_{W_2}\{(X\wedge Y)\cdot W_2)(U,V)Z\}, \tag{3.3}$$

holds on the set $U_{W_2}=\{x\in M^{2n+1}|W_2\neq 0 \text{ at } x\}$, where L_{W_2} is some function on U_{W_2}.
Suppose that generalized Sasakian space form is W_2-pseudosymmetric.
Now the left- hand side of (3.3) is

$$\begin{aligned} &R(\xi,Y)W_2(U,V)Z-W_2(R(\xi,Y)U,V)Z \\ -\; &W_2(U,R(\xi,Y)V)Z-W_2(U,V)R(\xi,Y)Z=0. \end{aligned} \tag{3.4}$$

In the view of (2.12) the above expression becomes

$$\begin{aligned} & (f_1 - f_3)\{g(Y, W_2(U,V)Z)\xi - \eta(W_2(U,V)Z)Y \\ - & g(Y,U)W_2(\xi,V)Z + \eta(U)W_2(Y,V)Z \\ - & g(Y,V)W_2(U,\xi)Z + \eta(V)W_2(U,Y)Z \\ - & g(Y,Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)Y\} = 0. \end{aligned} \tag{3.5}$$

Next the right hand side of (3.3) is

$$\begin{aligned} & L_{W_2}\{(\xi \wedge Y)W_2(U,V)Z - W_2((\xi \wedge Y)U,V)Z \\ - & W_2(U,(\xi \wedge Y)V)Z - W_2(U,V)(\xi \wedge Y)Z\} = 0. \end{aligned} \tag{3.6}$$

By virtue of (3.2), (3.6) becomes

$$\begin{aligned} & L_{W_2}\{g(Y, W_2(U,V)Z)\xi - \eta(W_2(U,V)Z)Y \\ - & g(Y,U)W_2(\xi,V)Z + \eta(U)W_2(Y,V)Z \\ - & g(Y,V)W_2(U,\xi)Z + \eta(V)W_2(U,Y)Z \\ - & g(Y,Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)Y\} = 0. \end{aligned} \tag{3.7}$$

Using the expressions (3.5) and (3.7) in (3.3) and taking inner product with ξ, we obtain

$$\begin{aligned} & \{L_{W_2} - (f_1 - f_3)\}\{W_2(U,V,Z,Y) - \eta(W_2(U,V)Z)\eta(Y) \\ - & g(Y,U)\eta(W_2(\xi,V)Z) + \eta(U)\eta(W_2(Y,V)Z) \\ - & g(Y,V)\eta(W_2(U,\xi)Z) + \eta(V)\eta(W_2(U,V)Z) \\ - & g(Y,Z)\eta(W_2(U,V)\xi) + \eta(Z)\eta(W_2(U,V)Z)\} = 0, \end{aligned} \tag{3.8}$$

where $W_2(U,V,Z,Y) = g(Y, W_2(U,V)Z)$ and using(1.3) we get either

$$L_{W_2} = (f_1 - f_3) \text{ or } W_2(U,V,Z,Y) = 0. \tag{3.9}$$

Thus we have following:

Theorem 3.1. *If $M^{2n+1}(f_1, f_2, f_3)$ is W_2-pseudosymmetric generalized Sasakian space form, then $M^{2n+1}(f_1, f_2, f_3)$ is either W_2-flat, or $L_{W_2} = (f_1 - f_3)$ if $(f_1 \neq f_3)$.*

Also in a generalized Sasakian space form, Singh and Pandey [24] proved the following,

Theorem 3.2. *A (2n+1)-dimensional $(n > 1)$ generalized Sasakian space form satisfying $W_2 = 0$ is an η-Einstein manifolds.*

In view of theorem (3.1) and theorem (3.2) we can state the following corollary.

Corolary 1. *If $M^{2n+1}(f_1, f_2, f_3)$ is a W_2-pseudosymmetric generalized Sasakian space forms then M^{2n+1} is either η-Einstein manifold or $L_{W_2} = (f_1 - f_3)$ if $(f_1 \neq f_3)$.*

4 W_2-locally symmetric generalized Sasakian space forms

Definition 1. *A* $(2n+1)$ *dimensional* $(n>1)$ *generalized Sasakian space form is called projectively locally symmetric if it satisfies [18].*

$$(\nabla_W P)(X,Y)Z = 0.$$

for all vector fields X, Y, Z *orthogonal to* ξ, *and an arbitrary vector field* W.

Analogous to this definition, we define a $(2n+1)$ dimensional $(n>1)$ W_2-locally symmetric generalized Sasakian space form if

$$(\nabla_W W_2)(X,Y)Z = 0,$$

for all vector fields X, Y, Z orthogonal to ξ and an arbitrary vector field W.
From (1.1) and (1.2), we have

$$\begin{aligned} W_2(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\} + \frac{1}{2n}\{g(X,Z)QY - g(Y,Z)QX\}. \end{aligned} \tag{4.1}$$

Taking covariant differentiation of (4.1) with respect to an arbitrary vector field W, we get

$$\begin{aligned} (\nabla_W W_2)(X,Y)Z &= df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\ &+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_2\{g(X,\phi Z)(\nabla_W\phi)Y \\ &+ g(X,(\nabla_W\phi)Z)\phi Y - g(Y,\phi Z)(\nabla_W\phi)X \\ &- g(Y,(\nabla_W\phi)Z)\phi X + 2g(X,\phi Y)(\nabla_W\phi)Z \\ &+ 2g(X,(\nabla_W\phi)Y)\phi Z\} + df_3(W)\{\eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\ &+ f_3\{(\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y \\ &- (\nabla_W\eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W\eta)\eta(Z)X \\ &+ g(X,Z)(\nabla_W\eta)(Y)\xi + g(X,Z)\eta(Y)\nabla_W\xi \\ &- g(Y,Z)(\nabla_W\eta)(X)\xi - g(Y,Z)\eta(X)\nabla_W\xi\} \\ &+ \frac{1}{2n}\{g(X,Z)(\nabla_W Q)(Y) - g(Y,Z)(\nabla_W Q)(X)\}. \end{aligned} \tag{4.2}$$

where ∇ denotes the Riemannian connection on the manifold.
Differentiating (2.10) covariantly with respect to a W, one can get

$$\begin{aligned} (\nabla_W Q)(Y) &= d(2nf_1 + 3f_2 - f_3)(W)Y - d(3f_2 + (2n-1)f_3)(W)\eta(Y)\xi \\ &- (3f_2 + (2n-1)f_3)[(\nabla_W\eta)(Y)\xi + \eta(Y)(\nabla_W\xi)]. \end{aligned} \tag{4.3}$$

In view of (4.3) and (4.2), it follows that

$$\begin{aligned}
(\nabla_W W_2)(X,Y)Z &= df_1(W)\{g(Y,Z)X - g(X,Z)Y\}\\
&+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X\\
&+ 2g(X,\phi Y)\phi Z\} + f_2\{g(X,\phi Z)(\nabla_W\phi)Y\\
&+ g(X,(\nabla_W\phi)Z)\phi Y - g(Y,\phi Z)(\nabla_W\phi)X\\
&- g(Y,(\nabla_W\phi)Z)\phi X + 2g(X,\phi Y)(\nabla_W\phi)Z\\
&+ 2g(X,(\nabla_W\phi)Y)\phi Z\} + df_3(W)\{\eta(X)\eta(Z)Y\\
&- \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\
&+ f_3\{(\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y\\
&- (\nabla_W\eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W\eta)\eta(Z)X\\
&+ g(X,Z)(\nabla_W\eta)(Y)\xi + g(X,Z)\eta(Y)\nabla_W\xi\\
&- g(Y,Z)(\nabla_W\eta)(X)\xi - g(Y,Z)\eta(X)\nabla_W\xi\}\\
&+ \frac{1}{2n}[g(X,Z)\{d(2nf_1 + 3f_2 - f_3)(W)Y - d(3f_2\\
&+ (2n-1)f_3)(W)\eta(Y)\xi - (3f_2 + (2n-1)f_3)[(\nabla_W\eta)(Y)\xi\\
&+ \eta(Y)(\nabla_W\xi)]\} - g(Y,Z)\{d(2nf_1 + 3f_2 - f_3)(W)X\\
&- d(3f_2 + (2n-1)f_3)(W)\eta(X)\xi\\
&- (3f_2 + (2n-1)f_3)[(\nabla_W\eta)(X)\xi + \eta(X)(\nabla_W\xi)]\}]. \qquad (4.4)
\end{aligned}$$

Taking X, Y, Z orthogonal to ξ in (4.4) and then taking the inner product of the resultant equation with V, followed by setting $V = Z = e_i$ in the above equation, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, i = 1, 2, ..., 2n+1$, we get

$$\begin{aligned}
& f_2\{-g(\phi X,(\nabla_W\phi)Y) + \sum_{i=1}^{n} g(X,(\nabla_W\phi)e_i)g(\phi Y, e_i)\\
&+ g(\phi Y,(\nabla_W\phi)X) - \sum_{i=1}^{n} g(Y,(\nabla_W\phi)e_i)g(\phi X, e_i)\\
&+ 2\sum_{i=1}^{n} g(X,\phi Y)g((\nabla_W\phi)e_i, e_i)\} = 0. \qquad (4.5)
\end{aligned}$$

For Levi Civita connection ∇,

$$(\nabla_W g)(X,Y) = 0,$$

which gives

$$(\nabla_W g)(X,Y) - g(\nabla_W X, Y) - g(X, \nabla_W Y) = 0.$$

Putting $X = e_i$ and $Y = \phi e_i$ in the above equation, we obtain

$$-g(\nabla_W e_i, \phi e_i) - g(e_i, (\nabla_W\phi)e_i) = 0,$$

which can be written as

$$g(e_i, \phi(\nabla_W e_i)) - g(e_i, (\nabla_W \phi)e_i) = 0.$$

Thus we have

$$g(e_i, (\nabla_W \phi)e_i) = 0. \tag{4.6}$$

By the virtue of (4.5) and (4.6) takes the form

$$\begin{aligned} & f_2\{-g(\phi X, (\nabla_W \phi)Y) + \sum_{i=1} g(X, (\nabla_W \phi)e_i)g(\phi Y, e_i) \\ + \quad & g(\phi Y, (\nabla_W \phi)X) - \sum_{i=1} g(Y, (\nabla_W \phi)e_i)g(\phi X, e_i)\} = 0. \end{aligned} \tag{4.7}$$

The above equation yields $f_2 = 0$. It is known that a generalized Sasakian space form of dimension greater than three is conformally flat if and only if $f_2 = 0$ [14]. Hence the manifold under consideration is conformally flat. Conversely, suppose that the manifold is conformally flat. Then $f_2 = 0$. In addition, if we consider X, Y, Z orthogonal to ξ, then (1.1) yields

$$R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\}.$$

The above equation gives,

$$r = 2n(2n + 1)f_1. \tag{4.8}$$

In view of (2.11) and (4.8), we obtain $f_3 = 0$. Hence from (4.4), we get

$$(\nabla_W W_2)(X, Y)Z = 0.$$

Therefore, the manifold is W_2-locally symmetric.
Thus we have the following assertion.

Theorem 4.1. *A* $(2n + 1)$ *dimensional* $(n > 1)$ *generalized Sasakian space form is* W_2*-locally symmetric if and only if it is conformally flat.*

or

Theorem 4.2. *A* $(2n + 1)$ *dimensional* $(n > 1)$ *generalized Sasakian space form is* W_2*-locally symmetric if and only if* f_1 *is constant.*

5 W_2-Locally ϕ-symmetric generalized Sasakian space forms

Definition 2. *A generalized Sasakian space form* $M^{2n+1}(f_1, f_2, f_3)$ *of dimension greater than three is called* W_2*-locally* ϕ*-symmetric if it satisfies*

$$\phi^2((\nabla_W W_2)(X, Y)Z) = 0, \tag{5.1}$$

for all vector fields X, Y, Z *orthogonal to* ξ*, on* M^{2n+1}*. Let us consider a* W_2*-locally* ϕ*-symmetric generalized Sasakian space form of dimension greater than three. Then from the definition and* (2.1)*, we have*

$$-((\nabla_W W_2)(X,Y)Z) + \eta(\nabla_W W_2)(X,Y)Z)\xi = 0, \tag{5.2}$$

Taking the inner product g in both sides of the above equation with respect to W, we get

$$-g((\nabla_W W_2)(X,Y)Z, W) + \eta(\nabla_W W_2)(X,Y)Z)\eta(W) = 0, \tag{5.3}$$

If we take orthogonal to W, then the above equation yields,

$$g((\nabla_W W_2)(X,Y)Z, W) = 0, \tag{5.4}$$

The above equation is true for all W orthogonal to ξ. If we choose $W \neq 0$ and not orthogonal to $(\nabla_W W_2)(X,Y)Z$, then it follows that

$$(\nabla_W W_2)(X,Y)Z = 0 \tag{5.5}$$

Hence, the manifold is W_2-locally symmetric and hence by theorem 4.3, it is conformally flat. Conversely, let the manifold is conformally flat and hence $f_2 \neq 0$. Again, for X, Y, Z orthogonal to ξ, we have applying ϕ^2 on both side to equation (4.4), one can get

$$\begin{aligned} \phi^2(\nabla_W W_2)(X,Y)Z &= -df_2(W)\{g(X,\phi Z)\phi X - g(Y,\phi Z) + 2g(X,\phi Y)\phi Z\} \\ &- \frac{1}{2n}\{d(3f_2 - f_3)(W)[g(X,Z)Y - g(Y,Z)X]\}. \end{aligned} \tag{5.6}$$

if $f_2 = f_3 = 0$, the above equation yields

$$\phi^2(\nabla_W W_2)(X,Y)Z = 0$$

for all X, Y, Z are orthogonal to ξ, therefore the manifold is W_2-locally ϕ-symmetric.
Now we are in a position to state the following statement,

Theorem 5.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form M^{2n+1} is W_2-locally ϕ-symmetric if and only if it is conformally flat.*

6 W_2-ϕ-recurrent generalized Sasakian Space form

Definition 3. *A generalized Sasakian space form is said to be ϕ-recurrent if there exists a non-zero 1-form A such that,(see[11])*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z,$$

for arbitrary vector fields X, Y, Z, W. If the 1-form A vanishes, then the manifold reduces to a ϕ-symmetric manifold.

According to the definition of ϕ-recurrent generalized Sasakian space form, we define W_2-ϕ-recurrent generalized sasakian space form by

$$\phi^2((\nabla_W W_2)(X,Y)Z) = A(W)W_2(X,Y)Z. \tag{6.1}$$

Then by (2.1) and (6.1), we have

$$-(\nabla_W W_2)(X,Y)Z + \eta((\nabla_W W_2)(X,Y)Z)\xi = A(W)W_2(X,Y)Z, \tag{6.2}$$

for arbitrary vector fields X, Y, Z, W. From the above equation it follows that

$$\begin{aligned} &- \quad g((\nabla_W W_2)(X,Y)Z,U) + \eta((\nabla_W W_2)(X,Y)Z)\eta(U) \\ &= \quad A(W)g(W_2(X,Y)Z,U). \end{aligned} \tag{6.3}$$

Let $\{e_i\}, i = 1,2,......2n+1$, be an orthogonal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (6.3) and taking summation over $i, 1 \leq i \leq 2n+1$, we get

$$\begin{aligned} &- \quad (\nabla_W S)(Y,Z) - \frac{1}{2n}[(\nabla_W S(Y,Z)) - g(Y,Z)dr(W)] \\ &+ \quad \sum_{i=1}^{2n+1} \eta((\nabla_W W_2)(e_i,Y)Z)\eta(e_i) = A(W)\{(\nabla_W S)(Y,Z) \\ &- \quad \frac{1}{2n}[(\nabla_W S)(Y,Z) - g(Y,Z)dr(W)]\}. \end{aligned} \tag{6.4}$$

Setting $Z = \xi$ in (6.4) then using (2.5), (2.13) and (2.7) and then replace Y by ϕY in (6.4), we get

$$S(Y,W) = 2n(f_1 - f_3)g(Y,W). \tag{6.5}$$

Hence we can state following theorem:

Theorem 6.1. *Let generalized Sasakian space forms M^{2n+1} is W_2-ϕ-recurrent, then it is an Einstein manifold, provided $(f_1 - f_3) \neq 0$.*

7 Example

In [1], generalized complex space-form of dimension two is $N(a,b)$ and the warped product $M = R \times N$ endowed with the almost contact metric structure is a three dimensional generalized Sasakian-space-form whose smooth functions $f_1 = \frac{a-(f')^2}{f^2}$, $f_2 = \frac{b}{f^2}$ and $f_3 = \frac{a-(f')^2}{f^2} + \frac{f''}{f}$. Here $f = f(t)$, $t \in R$ and f' indicates the derivative of f with respect to t. Suppose we set $a = 2$, $b = 0$ and $f(t) = t$ with $t \neq 0$, then $f_1 = \frac{1}{t^2}$, $f_2 = 0$ and $f_3 = \frac{1}{t^2}$, we have from (1.2)

$$\begin{aligned} W_2(X,Y)Z &= \frac{1}{t^2}\{g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} + \frac{1}{2t^2}\{g(X,Z)Y - g(Y,Z)X \\ &- g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{7.1}$$

Now differentiating covariantly with respect to W and taking X, Y, Z are orthogonal to ξ, and then apply ϕ^2 on both side of the above equation

$$\phi^2(\nabla_W W_2(X,Y)Z) = -\frac{3}{2}d(\frac{1}{t^2})\{g(X,Z)Y - g(Y,Z)X\}. \tag{7.2}$$

By the virtue of (7.2) we can easily say generalized Sasakian space forms is W_2-locally ϕ-symmetric if and only if $\frac{1}{t^2}$ is constant or both f_1 and f_2 are constants.

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