ON THE CONSTRUCTION OF JACOBI MATRICES FROM SPECTRAL DATA

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ABSTRACT

The problem of constructing an n by n Jacobi matrix J with prescribed spectrum $\{\lambda_i\}^n_1$, such that the submatrix $J(\rho)$, obtained from J by deleting its ρ^{th} row and column, also has a prescribed spectrum $\{\mu_i\}^{n-1}_1$ is studied. The cases $\rho=1$ and $\rho=n$ are well Known. For the case $2 \le \rho \le n-1$ it is shown that the problem has a unique solution under the condition $\lambda_i<\mu_i<\lambda_{i+1}$, $i=1, 2, \ldots n-1$.

1.- INTRODUCTION

A Jacobi matrix is any real, symmetric, tridiagonal matrix of the form

$$\begin{bmatrix} a_1 & b_1 & 0 & \cdot & \cdot & 0 \\ b_1 & a_2 & b_2 & & & 0 \\ & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot & b_{n-1} \\ 0 & & & & b_{n-1} & a_n \end{bmatrix}$$

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where $b_i > 0$, $1 \le i \le n-1$. For any square matrix X we denote by X(ρ) the truncated matrix obtained from X by deleting its ρ^{th} row and column, and by σ(X) we denote the set of eigenvalues $\{\lambda_i(X)\}$ of X.
This paper deals with the following Inverse Eigenvalue problem : Given the sequences of real numbers $\{\lambda_i\}^n{}_1$ and $\{\mu_i\}^{n-1}{}_1$, which satisfy the interlacing condition $\lambda_i < \mu_i < \lambda_{i+1}$, $1 \le i \le n-1$, find an n by n Jacobi matrix J such that $\sigma(J) = \{\lambda_i\}$ and $\sigma(J(\rho)) = \{\mu_i\}$.

Most of the research about this problem has taken as initial spectral data the set of eigenvalues of J and the set of eigenvalues of J(\1) (or J(\n)). The reason for this is that J(\1)(and J(\n))is also a Jacobi matrix and therefore its eigenvalues are distinct and strictly separate those of J, that is, $\lambda_i(J) < \lambda_i(J(\backslash 1)) < \lambda_{i+n}(J)$, $1 \le i \le n-1$, (see Wilkinson [11]).
The case ρ = 1 (ρ=n is the analogous case) has been studied by Hochstadt [9], Gray and Wilson [7], Hald [8], de Boor and Golub [2] Gragg and Harrow [6].

The situation is completely different if we consider as initial data the spectra of J and J(ρ) for $2 \le \rho \le n-1$. In this case, J(ρ) is not a Jacobi matrix and hence its eigenvalues need not to be distinct nor strictly to separate those of J.

The following example illustrate this situation:

$$J = \begin{bmatrix} 0 & \sqrt{1.5} & 0 \\ \sqrt{1.5} & 0 & \sqrt{2.5} \\ 0 & \sqrt{2.5} & 0 \end{bmatrix}$$

is a Jacobi matrix with eigenvalues{-2,0,2}.J(\1) and J(\3) have eigenvalues which strictly interlace the eigenvalues of J. However, J(\2) = 0 does not satisfy that property. The case $2 \le \rho \le n-1$ has been considered in [3], [5], [10]. A numerical algorithm to compute the entries of J is given in [10].

The paper is self-contained and is organized as follows: in section 2 we discuss the cases ρ=1 and ρ=n; in section 3 we show how we may construct a Jacobi matrix J with prescribed spectrum such that J(ρ), $2 \le \rho \le n-1$, also has a prescribed spectrum.
We also show section 4 that if J is a persymmetrix Jacobi matrix, that is , symmetric with respect to its second diagonal, we may uniquely reconstruct J from only one spectrum and one single additional piece of information.

2.- The case $\rho=1$ and $\rho=n$.

Let Q be the orthogonal matrix of eigenvectors of J. Then $Q^TJQ=\Lambda=\mathrm{diag}\{\lambda_1,\ldots,\lambda_n\}$. Since $Q^T(\lambda I-J)Q=\lambda I-\Lambda$, then

(1) $(\lambda I-J)^{-1}=Q(\lambda I-\Lambda)^{-1}Q^T$.

As

$$(\lambda I-J)^{-1}=\frac{1}{\det(\lambda I-J)}\,\mathrm{adj}(\lambda I-J)=\frac{1}{p(\lambda)}\begin{bmatrix} p(1;\lambda) & x & x & x \\ x & \cdot & x & x \\ x & x & \cdot & x \\ x & x & x & p(n;\lambda)\end{bmatrix}$$

where $p(\rho;\lambda)$ denotes the characteristic polynomial of $J(\rho)$,the diagonal entries of $(\lambda I-J)^{-1}$ are given by

$$<(\lambda I-J)^{-1}\rho_\rho,\ \rho_\rho>=\frac{p(\rho;\lambda)}{p(\lambda)},$$

where e_ρ is the ρ^{th} unit vector. The right-hand side of (1) is

$$Q(\lambda I-\Lambda)^{-1}Q^T=Q(\bar{Q}_1\,|\,\bar{Q}_2\,|\ldots|\,\bar{Q}_n),$$

where

$$\bar{Q}_j=\left(\frac{q_{j1}}{\lambda-\lambda_1},\ldots\ldots,\frac{q_{jn}}{\lambda-\lambda_n}\right)^T;\ j=1,2,\ldots,n.$$

Hence, by comparing the entries in position (1,1) in both sides of (1) we find

$$\frac{p(1,\lambda)}{p(\lambda)}=\sum_{K=1}^{n}\frac{z_{1K}^2}{\lambda-\lambda_K}$$

Taking the limit when λ tends to λ_j we obtain

(2) $$q_{1j}^2=\frac{p(1;\lambda_j)}{p'(\lambda_j)}$$

We remark that right-hand side of (2) is positive because of the interlacing condition $\lambda_i(J) < \lambda_i(J(\backslash 1)) < \lambda_{i+1}(J)$. Since $J=Q\Lambda_J Q^T$, we have $J^K=Q\Lambda^K{}_J Q^T$. Then

$$\langle J^K e_1, e_1\rangle = \langle Q\Lambda^K{}_J Q^T e_1, e_1\rangle = \sum_{i=1}^{n} \lambda_i^K q_{1i}^2 = \sum_{i=1}^{n} \lambda_i^K \frac{p(1;\lambda_i)}{p'(\lambda_i)}$$

That is,

$$(3)\quad \langle J^K e_1, e_1\rangle = \sum_{i=1}^{n} \lambda_i^K \frac{p(1;\lambda)}{p'(\lambda_i)}$$

We hasten to point out that the identity (3) was given by Hochstadt [9]. Hence, if we are given the spectra of J and $J(\backslash 1)$, Hochstadt showed that we way construct the matrix J uniquely. In fact, for K = 1,2 in (3), we have $\langle Je_1, e_1\rangle = a_1$ and $\langle J^2 e_1, e_1\rangle = a^2{}_1 + b^2{}_1$, so that we compute a_1 and b_1, that is, the first row of J from

$$a_1 = \sum_{i=1}^{n} \lambda_i \frac{p(1;\lambda_i)}{p'(\lambda_i)}, \quad \text{and} \quad a_1^2 + b_1^2 = \sum_{i=1}^{n} \lambda_i^2 \frac{p(1;\lambda_i)}{p'(\lambda_i)}$$

Next, K=3 and K=4 will give a_2 and b_2, and so forth. Following this procedure we determine all the entries of J uniquely.
We note that by comparing the entries in position (n,n) in both sides of (1) we obtain

$$(4)\quad q_{nj}^2 = \frac{p(n;\lambda_j)}{p'(\lambda_j)}$$

and

$$(5)\quad \langle J^K e_n, e_n\rangle = \sum_{i=1}^{n} \lambda_i^K \frac{p(n;\lambda_i)}{p'(\lambda_i)}$$

Thus, in the case that we are given the spectra of J and $J(\backslash n)$ we may apply the Hochstadt's technique to compute the entries of J backwards in the order $a_n, b_{n-1}, a_{n-1}, \ldots, a_2, b_1, a_1$.
The Hochstadt uniqueness result was complemented by Gray and Wilson [7], and Hald [8], who proved that if we are given the

sequences of real numbers $\{\lambda_i\}^n_1$ and $\{\mu_i\}^{n-1}_1$, which strictly interlace, then there exists a unique n by n Jacobi matrix J such that $\sigma(J)=\{\lambda_i\}$ and $\sigma(J(\backslash 1))=\{\mu_i\}$.

3. The case $2 \le \rho \le n-1$

Let be ρ an integer, $2 \le \rho \le n-1$. The matrix $J(\rho)$ has the form $J(\rho) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, where A and B are Jacobi matrices of order $(\rho-1)$ and $(n-\rho)$, respectively. Let $\sigma(A)=\{\alpha_i\}$ and $\sigma(B)=\{\beta_i\}$. Then $\sigma(J(\rho)=\sigma(A) \cup \sigma(B)$. Let X and Y be the orthogonal matrices of eigenvectors of A and B, respectively. Then, by(4) we find that

$$(6) \qquad x^2_{\rho-1,J} = \frac{\bar{q}(\alpha_j)}{q'(\alpha_j)}; \quad j=1, 2, \ldots, \rho-1,$$

where $q(\lambda)=\det(\lambda I-A)$ and $\bar{q}(\lambda)=\det(\lambda I-\bar{A})$, with $\bar{A}=A(\rho-1)$ as the left principal submatrix of A. Concerning the matrix B we have by (2)

$$(7) \qquad y^2_{1j} = \frac{\bar{r}(\beta_j)}{r'(\beta_j)}; \quad j= 1, 2, \ldots, n-\rho$$

where $r(\lambda)=\det(\lambda I-B)$ and $\bar{r}(\lambda)=\det(\lambda I-\bar{B})$, with $\bar{B}=B(\backslash 1)$ the right principal submatrix of B.

We note that by using Hochstadt's technique we may construct the matrices A and B uniquely provided that we know their spectra $\{\alpha_i\}$ and $\{\beta_k\}$ and the values $\bar{q}(\alpha_i)$ and $\bar{r}(\beta_k)$.

By expanding $\det(\lambda I-J)$ along the ρ^{th} row we find that the characteristic polynomial of J is given by

$$(8) \qquad p(\lambda)=(\lambda-a_\rho)q(\lambda)r(\lambda)-b^2_{\rho 1}\bar{q}(\lambda)r(\lambda)-b^2_\rho \bar{q}(\lambda)r(\lambda).$$

Hence,

$$(9) \qquad \bar{q}(\alpha_i)=\frac{-1}{b^2_{\rho-1}} \frac{p(\alpha_i)}{r(\alpha_i)}, \quad i=1, 2, \ldots, \rho-1 \text{ and}$$

$$(10)\quad \bar{r}(\beta_k) = \frac{-1}{b_\rho^2}\,\frac{p(\beta_k)}{q(\beta_k)},\quad k=1,2,\ldots,n-\rho$$

THEOREM 1.

Let $\{\lambda_i\}_{i=1}^{n}$ and $\{\alpha_i\}_{i=1}^{\rho-1} \cup \{\beta_k\}_{k=1}^{n-\rho} = \{\mu_j\}_{j=1}^{n-1}$ be sequences of real numbers, which satisfy the interlacing condition

$$(11)\quad \lambda_i < \mu_i < \lambda_{i+1},\quad 1 \le i \le n-1.$$

Let ρ be an integer, $2 \le \rho \le n-1$. Then, there exists a unique Jacobi matrix J such that $\sigma(J)=\{\lambda_i\}$ and $\sigma(J(\rho))=\{\mu_i\}$.

PROOF:

We form the polynomials $p(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i)$ and $\varepsilon(\lambda)=\prod_{j=1}^{n-1}(\lambda-\mu_j)=q(\lambda)r(\lambda)$, where $q(\lambda)=\prod_{i\in I}(\lambda-\mu_i)$ and $r(\lambda)=\prod_{k\in K}(\lambda-\mu_k)$, with I and K being the sets of indices for which $\mu_i=\alpha_i$ and $\mu_k=\beta_k$, respectively.

If the matrix J exists, its characteristic polynomial $p(\lambda)$ must satisfy (8). Consider the quotient

$$\frac{p(\lambda)}{s(\lambda)} = (\lambda-a_\rho) - b_{\rho-1}^2\frac{\bar{q}(\lambda)}{q(\lambda)} - b_\rho^2\frac{\bar{r}(\lambda)}{r(\lambda)}$$

$$= (\lambda-a_\rho - \frac{b_{\rho-1}^2\bar{q}(\lambda)r(\lambda) + b_\rho^2 q(\lambda)\bar{r}(\lambda)}{s(\lambda)}$$

$$= (\lambda-a_\rho) - \sum_{i=1}^{n-1}\frac{c_i}{\lambda-\mu_i},$$

where the constants c_i are determined as the residues

$$c_i=\operatorname*{Res}_{\lambda=\mu_i}\frac{b_{\rho-1}^2\bar{q}(\lambda)r(\lambda)+b_\rho^2 q(\lambda)\bar{r}(\lambda)}{s(\lambda)}=-\frac{p(\mu_i)}{s'(\mu_i)}$$

by integrating around a circle C_i sufficiently small so that its interior contains only the pole μ_i. Hence

$$\frac{p(\lambda)}{s(\lambda)}=(\lambda-a_\rho)+\sum_{i=1}^{n-1}\frac{p(\mu_i)}{s'(\mu_i)(\lambda-\mu_i)},$$

whence,

$$(12)\quad b_{\rho-1}^2\frac{\bar{q}(\lambda)}{q(\lambda)}=-\sum_{i\in I}\frac{p(\mu_i)}{s'(\mu_i)(\lambda-\mu_i)}$$

$$(13)\quad b_\rho^2\frac{\bar{r}(\lambda)}{r(\lambda)}=-\sum_{k\in K}\frac{p(\mu_k)}{s'(\mu_k)(\lambda-\mu_k)}$$

Taking the limit as λ goes to ∞ yields

$$(14)\quad b_{\rho-1}^2=-\sum_{i\in I}\frac{p(\mu_i)}{s'(\mu_i)}$$

$$(15)\quad b_\rho^2=-\sum_{k\in K}\frac{p(\mu_k)}{s'(\mu_k)}$$

Since the zeros of $p(\lambda)$ and $s(\lambda)$ satisfy (11), $b_{\rho-1}^2>0$ and $b_\rho^2>0$.

Now, by (9) and (10) we have

$$(16)\quad \frac{\bar{q}(\mu_i)}{q'(\mu_i)}=\frac{-1}{b_{\rho-1}^2}\frac{p(\mu_i)}{s'(\mu_i)},\quad i\in I$$

$$(17)\quad \frac{\bar{r}(\mu_k)}{r'(\mu_k)}=\frac{-1}{b_\rho^2}\frac{p(\mu_k)}{s'(\mu_k)},\quad k\in K$$

Since $b_{\rho-1}$ and b_ρ are required to be positive, (14) and (15) give $b_{\rho-1}$ and b_ρ uniquely and consequently $\frac{q(\mu_i)}{q'(\mu_i)}$, $i \in I$ and $\frac{r(\mu_k)}{r'(\mu_k)}$, $k \in K$ are also obtained uniquely.
We now use (5) (with J=A and n=ρ-1) and (3) (with J=B and n as n-ρ) to compute uniquely all the entries of A and B, respectively. It only remains to compute the entry a_ρ, which is uniquely obtained from

$$(18)\ a_\rho = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu_i .$$

Thus, the proof is completed.

4. <u>The case of a persymmetric Jacobi matrix</u>

If J is a persymmetric Jacobi matrix, that is,

$$(19)\ a_i = a_{n-i+1} \text{ and } b_i = b_{n-i},$$

then there are particular cases in which we can uniquely reconstruct J from only one spectrum, the spectrum of J(ρ), and one single additional piece of information. We show this in the following corollaries:

COROLLARY 2.

Let $\sigma(A)=\{\alpha_i\}_{i=1}^{\rho-1}$, $\sigma(B)=\{\beta_i\}_{i=1}^{n-\rho}$ and $\prod_{i=1}^{n-1} b_i$ be given, where $J(\rho)=\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\sigma(A) \cap \sigma(B) = \emptyset$ and the entries of J satisfy (19). Let $n \geq 4$ be an even number. Then J can be uniquely reconstructed if $\rho=\frac{n+2}{2}$ (or $\rho=\frac{n}{2}$).

PROOF:

Observe that the number of unknowns equal the number of data. If A denotes the left principal submatrix of A, then from the symmetric condition (19), B is similar to A and we have $q(\lambda) = \det(\lambda I-A) = \det(\lambda I-B) = r(\lambda)$. Then $\sigma(A) = \{\beta_i\}$, and the β_i's strictly interlace the α_i's. Hence, we can determine the matrix A uniquely. Once we have found A we have

already computed B since all the entries of B are also of A. It only remains to compute the ρ^{th} row of J, that is, the entries $b_{\rho-1}$, a_ρ, b_ρ. However, from (19) $b_\rho = b_{\rho-2} \in A$ and $a_\rho = a_{\rho-1} \in A$. Finally, $b_{\rho-1}$ is obtained from $\pi\, b_i$.
For $\rho = n/2$, A become similar to the right principal submatrix of B and we construct the matrix B.

COROLLARY 3.

Let $\sigma(A)$ and $\sigma(B)$ be given and disjoint. Assume that the trace of J is known and its entries satisfy (19). Let $n \geq 4$ be an odd number. Then J can be uniquely reconstructed if $\rho = \frac{n+3}{2}$ (or $\rho = \frac{n-1}{2}$).

PROOF:

Here B is similar to the left principal submatrix of $\bar{A}$.
Then,

(20) $$q(\lambda) = (\lambda - a_{\rho-1})\bar{q}(\lambda) - b_{\rho-2}^2\, r(\lambda),$$

whence,

$$\bar{q}(\alpha_j) = \frac{b_{\rho-2}^2\, r(\alpha_j)}{\alpha_j - a_{\rho-1}} \text{ and}$$

(21) $$\frac{\bar{q}(\alpha_j)}{q'(\alpha_j)} = b_{\rho-2}^2 \frac{r(\alpha_j)}{(\alpha_j - a_{\rho-1})q'(\alpha_j)}$$

Note that α_j cannot equal a_{e-1} because if that is the case, from (20) we would have $r(\alpha_j)=0$ and $\alpha_j \in \sigma(B)$, contradicting the hypothesis of the corollary. Taking the sum from $j=1$ to $j=\rho-1$ in (21) we have by (6)

(22) $$1 = b_{\rho-2}^2 \sum_{j=1}^{\rho-1} \frac{r(\alpha_j)}{(\alpha_j - a_{\rho-1})q'(\alpha_j)}.$$

Next, we compute a_ρ and $a_{\rho-1}$ from $a_\rho = tr(J)-tr(J(\rho))$ and $a_{\rho-1} = tr(A)-tr(B)-a_\rho$. Hence, $b_{\rho-2}$ can be uniquely determined from (22) and consequently we also compute $\frac{q(\alpha_j)}{q'(\alpha_j)}$ uniquely from (21). Now, we are in position to

determine uniquely all the entries of A by the use of the identity (5). Because of the symmetric condition (19), all the entries of B as well as the entries of the ρ^{th} row of J,are computed as elements of A. For $\rho = \frac{n-1}{2}$ the role of the submatrices A and B is interchanged.

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