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CONDITIONALLY INTEGRABLE PERTURBATIONS OF LINEAR DIFFERENTIAL SYSTEMS

by

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Abstract. We determine the asymptotic behavior of the solutions of differential systems with conditionally integrable coefficients:

$$X'=(\Delta(t)+V(t))X, \quad X'=(A+V(t))X,$$

where $\Delta(t)=\mathrm{diag}\{\lambda_1(t),\lambda_2(t),\dots\lambda_n(t)\}$ satisfies the dichotomic conditions of Levinson's Theorem [5], A is a constant n×n matrix and V is a conditionally integrable n×n matrix.

1. Introduction. We call a function f defined on $[t_0,\infty)$, t_0 a given real number, conditionally integrable if $\int_t^\infty f(s)ds$ exists for all $t\ge t_0$. In that case, we will write $f\in L_C(t\ge t_0)$. We wish to study the differential system:

$$y'=(\Delta(t) + V(t))y, \qquad (*)$$

where the matrix $V\in L_C(t\ge t_0)$. This kind of systems are of interest in physics (Adiabatic oscillators Theory, [1],[8]). Throughout this article $\Delta(t)$ will denote a diagonal matrix $\mathrm{diag}\{\lambda_1(t),\lambda_2(t),\dots,\lambda_n(t)\}=\Delta(t)$, where the eigenvalues satisfy the dichotomic conditions of Levinson's Theorem [5] on asymptotic integration, namely " For each index $i\ne j$ either

$$D_1)\ \int_a^t \mathrm{Re}(\lambda_i(s)-\lambda_j(s))ds\to\infty \text{ as } t\to\infty \text{ and } \int_a^t \mathrm{Re}(\lambda_i(s)-\lambda_j(s))ds>-K \text{ for } a\le\tau\le t$$

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or

D_2) $\int_\tau^t \mathrm{Re}(\lambda_1(s)-\lambda_j(s))ds<K$ for all $a\le\tau\le t$".

We will write this last condition: $\Delta\in\mathrm{Dic(Lev.)}$.

This fundamental theorem of Levinson cannot be applied to that kind of system (*). In this paper, we study the validation of Levinson's Theorem for systems as (*).

In section two we obtain similar results to Levinson's theorem which determine the behavior of the solutions of system (*). In section 3 and 4 we study the perturbation of a system with constant coefficients

$$y'=(A+V(t))y,$$

where A is a constant n×n matrix with simple eigenvalues (section 3) and non-simple eigenvalues (section 4). Using the results obtained on section 3, we obtain an analog L_c of GHIZZETTI'S Theorem [3]. However, the general solution is not the analogous to the L_1-case. We remark that on account of Pinto [7], the results obtained are immediately extended to systems whose unperturbed systems

$$x'=\Delta(t)x$$

has an exponential dichotomy (hence Δ diagonal is not necessary). Finally, in section 5 several examples are shown.

2. General results. We begin with a corrected version of a Harris-Lutz's Theorem [4].

Theorem 2.1. Let $\Delta(t),V(t)$ and $R(t)$ be n×n continuous matrices for $t\ge t_0$ such that:

i) $\Delta(t)=\mathrm{diag}\{\lambda_1(t),\dots,\lambda_n(t)\}\in\mathrm{Dic(Lev)}$

ii) $Q(t)=-\int_t^\infty V(s)ds$ exists for $t\ge t_0$; and

iii) VQ, ΔQ, $Q\Delta$ and $R\in L_1(t\ge t_0)$.

Then

$$y'=(\Delta(t)+V(t)+(R(t))y \tag{1}$$

has a fundamental matrix Y(t)such that

$$Y(t) = [I+o(1)] \exp\left(\int_{t_0}^t \Delta(s)ds\right) \text{ for } t\ \infty \tag{2}$$

Proof. If we put $y=(I+Q)z$, then replacing y in (1) we get

$$z'=\{(I+Q)^{-1}[\Delta+V+R](I+Q)-(I+Q)^{-1}V\}z \tag{3}$$

Since $[I+Q(t)]^{-1}$ exists for t large enough, let us say for $t\ge t_1\ge t_0$.

Then using the identity $(I+Q)^{-1}=I-(I+Q)^{-1}Q$, (3) becomes

$$z'=(\Delta(t)+\tilde{R}(t))z, \tag{4}$$

where

$$\tilde{R}(t)=\Delta Q+VQ-(I+Q^{-1})Q(\Delta+\Delta Q+VQ)+(I+Q)^{-1}R(I+Q).$$

Since $\Delta\in Dic(Lev)$ and by iii), $\tilde{R}\in L_1(t\geq t_0)$ the system (4) satisfies Levinson's theorem [5]. Thus system (1) has a fundamental matrix Y(t) which verifies (2).

Remark. Harris-Lutz [4] has proved a similar theorem (Th. 3.1) which demands incorrect conditions of integrability and the extra assumption that diag V=0.

Theorem 2.2. (Non resonant case). Let V(t) be a continuous n×n matrix for $t\geq t_0$ and ϕ be a fundamental matrix of

$$x'=A(t)x, \tag{5}$$

such that $\phi^{-1}V\phi\in L_c(t\geq t_0)$ and $(\phi^{-1}V\phi)\int_t^\infty\phi^{-1}(s)V(s)\phi(s)ds$, $\phi^{-1}R\phi\in L_1(t\geq t_0)$. Then there exists a fundamental matrix Y of

$$y'=(A(t)+V(t)+R(t))y, \tag{6}$$

such that

$$Y(t)=\phi[I+o(1)] \text{ for } t\to\infty. \tag{7}$$

Proof. Using the change of variable $y=\phi z$, (6) becomes $z'=\phi^{-1}(V+R)\phi\, z$

Let $z=(I+Q)w$, where $Q(t)=\int_t^\infty \phi^{-1}(s)V(s)\phi(s)ds$. Then

$$w'=(I+Q)^{-1}\{(\phi^{-1}V\phi)Q+\phi^{-1}R\phi(I+Q)\}w \tag{8}$$

Since $(I+Q)^{-1}$ exists and is bounded for $t\geq t_1\geq t_0$ large enough and $(\phi^{-1}V\phi)Q\in L_1(t\geq t_0)$, then by theorem 2.1, system (5) has a fundamental matrix W(t) such that $W(t)=I+o(1)$ for $t\to\infty$.

Therefore (6) has a fundamental matrix Y(t) which satisfies (7).

The conditional integrability of $\phi^{-1}V\phi$ differentiates the non-resonant from the resonant case:

Theorem 2.3 (Resonant case). Let ϕ be a fundamental matrix of (5). Assume

i) $\Delta=\text{diag}(\phi^{-1}V\phi)\in Dic(Lev.)$,

ii) $Q(t)=\int_t^\infty(\phi^{-1}(s)V(s)\phi(s)-\Delta(s))ds$ exists for $t\geq t_0$ and

iii) $\phi^{-1}V\phi Q, \phi^{-1}R\phi, \Delta Q$ and $Q\Delta\in L_1(t\geq t_0)$.

Then (6) possesses a fundamental matrix Y(t) such that

$$Y(t)=[I+o(1)]\exp\left(\int_{t_0}^t \text{diag}(\phi^{-1}V\phi)\right) \text{ for } t\to\infty. \tag{9}$$

Proof. We perform $y=\phi z$ to transform (6) in

$$z'=(\Delta(t)+\tilde{V}(t)+\tilde{R}(t))z, \qquad (10)$$

where $\Delta=\mathrm{diag}(\phi^{-1}V\phi)$, $\phi V=\phi^{-1}V\phi-\Delta$ and $R=\phi^{-1}R\phi$. Using i) and ii) we have that $\Delta \in \mathrm{Dic(Lev.)}$ and $Q(t)$ exist for $t\geq t_0$. Furthermore from iii) $(\phi^{-1}V\phi-\Delta)Q, \Delta Q, Q\Delta$ and $\phi^{-1}R\phi \in L_1(t\geq t_0)$. Then, by Theorem 2.1, system (10) has a fundamental matrix $Z(t)$ such that $z=[I+o(1)]\exp(\int_{t_0}^{t}\Delta)$ for t ∞. Hence (6) has a fundamental matrix Y which satisfies (9).

The difference between the resonant and non-resonant case is represented by the apparition of the factor $[\int_{t_0}^{t}\mathrm{diag}(\phi^{-1}V\phi)ds]$ which gives an exponentially stable part and a non-exponentially stable part.

As application, by using only Theorem 2.2. we can obtain the two first theorems in Harris-Lutz [4] for the equation:

$$y''+(1+g)y = 0.$$

In fact, his theorem 2.1 [4] follows from our Theorem 2.2, taking the fundamental matrix corresponding to the fundamental system $\phi_1(t)=\cos t$, $\phi_2(t)=\sin t$. Theorem 2.2 [4] is obtained taking $\phi_1(t)=e^{it}$ and $\phi_2(t)=ie^{-it}$.

Remark. If we haven that (4) that $\tilde{R} \notin L_1(t\geq t_0)$, but its integrability is better than that of $V(t)$, we can iterate our method a finite number of times. The two following Theorems represent the type of results which can be obtained by iterating the method.

Theorem 2.4. Assume

i) $\Delta\in\mathrm{Dic(Lev.)}$,

ii) $Q(t)=\int_t^\infty V(s)ds$ and $\tilde{Q}(t)=\int_t^\infty V(s)Q(s)+[\Delta,Q](s)ds$ exist for $t\geq t_0$, where $[\Delta,Q]=\Delta Q-Q\Delta$ and

iii) $\Delta\tilde{Q}$, $\tilde{Q}\Delta$, $V\circ Q\circ\tilde{Q}$, $\Delta Q\tilde{Q}$, $Q^2V, Q^2\Delta$ and $R\in L_1$.

Then (1) has fundamental matrix $Y(t)$ which satisfies (2).

Proof. Since $(I+Q)^{-1}$exists and is bounded for $t\geq t_1\geq t_0$ large enough, by using the identity

$$(I+Q)^{-1}=I-Q+(I+Q)^{-1}Q^2, \qquad (11)$$

the vector $z=(I+Q)^{-1}y$ satisfies

$$z'=(\Delta+\tilde{V}+\tilde{R})z, \qquad (12)$$

where $\tilde{V}= VQ+[\Delta,Q]$ and $\tilde{R}= (I+Q)^{-1}[Q^2\Delta(I+Q)+Q^2VQ+QR(I+Q)]$. Now, since $\tilde{V}\in L_c(t \geq t_0)$ and $\tilde{R} \in L_1(t \geq t_0)$, applying Theorem 2.1 to system (12) gives that (1) has a fundamental matrix Y which satisfies (2).

Theorem 2.5. (Iterated version of Theorem 2.1). Assume

i) $\Delta=\mathrm{diag}(A+VQ)\in Dic(Lev.)$,

ii) $Q(t)=\int_t^\infty V$ and $\tilde{Q}(t)=\int_t^\infty(A+VQ-\Delta)+[A,Q]$ exist for $t\geq t_0$ and

iii) $A\tilde{Q}$, $\tilde{Q}\Delta$, $\Delta\tilde{Q}$, $VQ\tilde{Q}$, $AQ\tilde{Q}$, $Q(A+V)Q$, $Q^2(A+V)$ and $R\in L_1$.

Then (6) possesses a fundamental matrix Y such that

$$Y(t)=[I+o(1)]\ \exp(\int_{t_0}^t \mathrm{diag}(A(s)+V(s)Q(s))ds \text{ for } t\to\infty.$$

Proof. Let $y=(I+Q)z$. Then using $C=(I+Q)^{-1}$ and (11) we have

$$z'=\{\Delta+\tilde{V}+\tilde{R}\}z \qquad (13)$$

where $\tilde{V}=(A+VQ-\Delta)+[A,Q]$ and $\tilde{R}=Q(A+V)Q+CQ^2(A+V)(I+Q)-CQR(I+Q)$.

We have that by hypotheses $\tilde{V}\in L_c(t\geq t_0)$ and $\tilde{R}\in L_1(t\geq t_0)$. Then by Theorem 2.1, system (13) has a fundamental matrix $Z(t)$ such that $Z(t)=[I+o(1)]\ \exp\int_{t_0}^t \Delta$ for $t\to\infty$, from where the conclusion follows.

Remark. This Theorem allows the possibility that the entries which are not on the diagonal of A+VQ could be weak if the terms of diagonal belong to Dic(Lev.). The iterated version in **Pinto** [6] is an example of its application.

3. Perturbation of Constant System. Simple Eigenvalues. The asymptotic behavior of the solutions of the linear system with constant coefficients:

$$x'=Ax \qquad (14)$$

are determined by the spectrum of the constant matrix A. If B(t) is a "small" perturbation, then the asymptotic behavior of the solutions of the system

$$x'=(A+B(t))x \qquad (15)$$

are again determined by system (14). In this section, we study the validation of the fundamental results of Levinson [5] and Coppel [2] for a class of system (15), where A has simple eigenvalues and the perturbation B(t) is conditionally integrable. If A has only one eigenvalue, Theorem 4, Chap IV of Coppel [2] gives the asymptotic behavior of system (15) when $B\in L_1(t\geq t_0)$ but we cannot apply that theorem in the weaker situation $B\in L_c(t\geq t_0)$.

By Theorem 2.1.we have the following corollary:

Corollary 3.1. Let A be a constant matrix with simple eigenvalues. Let $B\in L_c(t\geq t_0)$, Q and $QB\in L_1(t\geq t_0)$, $(Q(t)=\int_t^\infty B(s)ds)$. Then (15) has a fundamental matrix Y(t) such that

$$Y(t)=[I+o(1)]\exp(tA) \text{ for } t\to\infty$$

We remark (see example 1, section 5) that Theorem 4 Cap. IV of Coppel [2] an Corollary 3.1. solve different problems.

Now, suppose that A is in the canonical Jordan form consisting of the block with λ_0 in the main diagonal and let $B(t) = \left(b_{ij}(t)\right)_{\substack{1<i<n\\1<j<n}}$ a continuous matrix for $t\ge t_0$.

Theorem 3.2.. Assume

1) $v_{ij}(t)=t^{j-i}b_{ij}(t) \in L_c(t\ge t_0)$, $i,j\in\{1,2,\dots,n\}$ and

2) $t^{-1}q_{ij}$, $q_{ij}v_{jk}\in L_1(t\ge t_0)$, $i,j,k\in\{1,2,\dots,n\}$, $(q_{ij}(t)=\int_t^\infty v_{ij}(s)ds)$.

Then

$$y'=(J(\lambda_0)+B(t))Y. \qquad J(\lambda_0)=\lambda_0 I+J, \tag{16}$$

where

$$J=\begin{vmatrix} 0 & \dots & 0 & 0\\ 1 & \dots & 0 & 0\\ 0 & \dots & 0 & 0\\ \vdots & & \vdots & \vdots\\ 0 & \dots & 1 & 0\end{vmatrix} \tag{17}$$

has a fundamental matrix Y(t) such that $Y(t)=[I+o(1)]\exp(tJ(\lambda_0))$ for $t\to\infty$.

Proof. Let $D_n(t)=\mathrm{diag}(1,t,\dots,t^{n-1})$. We have

$$e^{tJ(\lambda_0)}D_n(t)=e^{\lambda_0 t}e^{tJ_0}D_n(t)=e^{\lambda_0 t}D_n(t)e^{J_0}.$$

Henceusingthechangeofvariables $y=e^{\lambda_0 t}e^{tJ_0}D_n(t)z$, system (16) becomes

$$z'=(\Delta(t)+V(t))z \tag{18}$$

where $\Delta(t) = -D_n^{-1}(t)D_n'(t)=t^{-1}\mathrm{diag}(0,-1,\dots,1-n)$, $V(t)=e^{-J_0}Ce^{J_0}$, and $C(t)=D_n^{-1}(t)B(t)D_n(t)$.

Using the hypotheses we can see that:

i) $Q(t)=\int_t^\infty V(S)ds$ exists for $t\ge t_0$,

ii) $\Delta(t)Q(t)$, $Q(t)\Delta(t)$ and $Q(t)V(t)\in L_1(t\ge t_0)$, and

iii) $\Delta(t)\in \mathrm{Dic(lev.)}$.

Then, Theorem 2.1. implies that system (18) has a fundamental matrix Z(t) such that

$Z(t)=[I+o(1)]\exp(\int_0^t\Delta(s)ds)=[I+o(1)]\mathrm{diag}(1,t^{-1},\dots,t^{-(n-1)})$ for $t\ \infty$.

Therefore system (16) has a fundamental matrix Y(t) such that

$Y(t)=e^{\lambda_0 t}e^{tJ_0}D_n(t)[I+o(1)]\mathrm{diag}(1,t^{-1},\dots,t^{-(n-1)})=[I+o(1)]e^{tJ(\lambda_0)}$ for

$t\to\infty$.

Finally we verify that i), ii) and iii) hold.

Since $Q(t)=\int_t^\infty V(s)ds=e^{-J_0}(\int_t^\infty C(s)ds)e^{-J_0}$, the condition

$$\int_t^\infty C(s)ds\circ C(t) \in L_1(t\geq t_0) \tag{19}$$

implies $QV\in L_1(t\geq t_0)$and it is not difficult to see that (19) follows from $q_{ij}v_{jk}\in L_1(t\geq t_0)$. Similarly $t^{-1}q_{kj}\in L_1$ implies that ΔQ and $Q\Delta\in L_1$. Moreover, $q_{ij}v_{jk}\in L_1$ implies $QV\in L_1$. Since iii) is evident, the proof is complete.

In [2], Coppel has studied the asymptotic behavior of the solutions of (16) when $\lambda_0=0$ obtaining that if $\int_{t_0}^\infty t^{k-1}|b_{ik}(t)|dt<\infty$, for $i,k\in\{1,2,\ldots,n\}$, then (14) possesses a fundamental matrix $Y(t)$ such that $Y(t)=[I+o(1)\exp(tJ)$ for $t\to\infty$. Thus, Theorem 3.2. is the faithful analogous L_c of the result's Coppel.

We can apply Theorem 3.2. to the differential equations of order n

$$x^{(n)}+(a_1+b_1(t))x^{(n-1)}+\ldots+(a_n+b_n(t))x=0, \tag{20}$$

where the characteristic polynomial of the homogenuous equation (with $b_i=0$) has a root $\lambda_0$with multiplicity n. Thus, we obtain the following L_c-version of **GHIZZETTI'S** theorem [3]:

Theorem 3.3. Assume 1) $v_i(t)=t^{i-1}b_i(t)\in L_c(t\geq t_0)$ and

2) $t^{-1}q_i(t)$, $q_i(t)v_i(t)\in L_1(t\geq t_0)$, $i\in\{1,2,\ldots,n\}$, $(q(t)=\int_t^\infty v_i(s)ds)$.

The equation (20) possesses a fundamental system of solutions $x_0(t), x_1(t),\ldots, x_{n-1}(t)$ such that

$$\lim_{t\to\infty}\frac{x_k^{(i)}(t)}{t^{k-i}}=\begin{cases}\frac{1}{(k-i)!} & \text{si } 0\leq i\leq k\\ 0 & \text{si } k<i<n\end{cases}$$

Proof. Equation (20) is equivalent to the system

$$\left.\begin{aligned} x_1'&=-\sum_{k=1}^{n}(a_k+b_k(t))x_k,\\ x_i'&=x_{i-1}, \qquad i\in\{1,2,\ldots,n\}\end{aligned}\right|$$

whose matricial form is

$$x'=\left\|\begin{pmatrix}\lambda_0 & 0 & \ldots & 0 & 0\\ 1 & \lambda_0 & \ldots & 0 & 0\\ 0 & 1 & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & & 1 & \lambda_0\end{pmatrix}+\begin{pmatrix}-b_1 & -b_2 & \ldots & -b_n\\ 0 & 0 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & 0\end{pmatrix}\right\|x,$$

We apply Theorem 3.2. to this system. Since $b_{ij}=0$ except for $b_{ij}=-b_j$ we must only verify $v_i(t)=t^{i-1}b_i(t)\in L_c(t\ge t_0)$, $i\in\{1,2,\dots,n\}$, which is true by hypothesis. Similarly, condition $t^{-1}q_{ij}(t)\in L_c(t\ge t_0)$ is reduced to $t^{-1}q_i(t)\in L_1(t\ge t_0)$, $i\in\{1,2,\dots,n\}$ because the only elements different from zero in the matrix $Q(t)$ are $q_{11}, q_{12}, \dots, q_{1n}$. Finally, condition $q_{ij}v_{jk}\in L_c(t\ge t_0)$; $i,j,k\in\{1,2,\dots,n\}$, becomes $q_i v_i\in L_1(t\ge t_0)$; $i\in\{1, 2, \dots, n\}$.

4. Non-Simple Eigenvalues. Let A be a $n\times n$ constant matrix with non-simple eigenvalues λ_i; $i=1,2,\dots,s$ with multiplicity n_i respectively. Suppose that A is in the cononical Jordan form

$$A=\bigoplus_{i=1}^{s} J(\lambda_i),\quad J(\lambda_i)=\lambda_i I+J_i \tag{21}$$

where J_i is a $n_i\times n_i$ matrix of the type (17).

The matrix $B(t)$ will be divided in blocks that we will denoted by $B_{ij}(t)$ and elements of each of these blocks will be denoted by $B^{ij}_{\alpha\beta}(t)$. Such elements will be specified in the proof of the following Theorem.

Theorem 4.4. Suppose (21),

1) $v^{ij}_{\alpha\beta}(t)=t^{\alpha-\beta}b^{ij}_{\alpha\beta}(t)\in L_c(t\ge t_0)$; $i,j\in\{1,2,\dots,s\}$ and

2) $q^{ij}_{\alpha\beta}(t)$, $q^{ij}_{\alpha\beta}(t)v^{jk}_{\beta\gamma}(t)\in L_1(t\ge t_0)$; $i,j,k\in\{1,\dots,s\}$; $(q^{ij}_{\alpha\beta}(t)=\int_t^{\infty}v^{ij}_{\alpha\beta}(s)ds)$.

Then

$$y'=(A+B(t))y \tag{22}$$

has a fundamental matrix $Y(t)$ such that

$$Y(t)=[I+o(1)]\exp\left(\bigoplus_{i=1}^{s} t\,J(\lambda_i)\right)\ \text{for}\ t\to\infty.$$

Proof. The unperturbed system $x'=[\bigoplus_{i=1}^{s} J(\lambda_i)]x$ has the fundamental matrix $X(t)=e^{A\cdot t}=[\bigoplus_{i=1}^{s} W_i(t)]$, where

$$W_i(t)=e^{tJ(\lambda_i)}=\begin{vmatrix} 1 & 0 & \dots & 0 \\ t & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{t^{n_i-1}}{(n_i-1)!} & \dots & \dots & 1 \end{vmatrix}\cdot e^{\lambda_i t}\quad \text{for } i\in\{1,2,\dots,s\}.$$

Let $D_n = \mathrm{diag}(D_{n_1}(t), D_{n_2}(t), \ldots, D_{n_s}(t))$, where

$$\begin{cases} D_{n_1}(t) = \mathrm{diag}(1, t, \cdots, t^{n_1-1}) \\ D_{n_k}(t) = \mathrm{diag}\left(t^{\sum_{i=1}^{k-1} n_i}, t^{\sum_{i=1}^{k-1}(n_i+1)}, \cdots, t^{\sum_{i=1}^{k}(n_i-1)}\right), \text{ for } 2 \le k \le s. \end{cases}$$

We make the change of variables

$$y = T(t)z, \qquad T = T_1 \oplus \ldots \oplus T_s, \tag{23}$$

where

$$T_i = e^{tJ(\lambda_i)} D_{n_i}(t) = D_{n_i}(t) e^{J(\lambda_i)}; \; J(\lambda_i) = \lambda_i I + J_i.$$

Then (23) transforms (22) into

$$z' = [\oplus_{i=1}^{s} \lambda_i I + T^{-1}(\oplus_{i=1}^{s} J_i)T - T^{-1}T' + T^{-1}BT]z = [\Delta(t) + V(t)]z, \tag{24}$$

where $\Delta(t) = \Delta_0 + \tilde{\Delta}(t)$, $\Delta_0 = \oplus_{i=1}^{s} \lambda_i I$, $\tilde{\Delta}(t) = \oplus_{i=1}^{s} t^{-1} E_i$ and $V(t) = T^{-1}(t)B(t)T(t)$, with $E_1 = \mathrm{diag}(0, -1, \ldots, -(n_1-1))$, $E_k = \mathrm{diag}\{-(\sum_{i=1}^{k-1} n_i), -(\sum_{i=1}^{k-1} n_i+1), \ldots, -(\sum_{i=1}^{k} n_i - 1)\}$, $2 \le k \le s$.

For computing the elements of $T^{-1}BT$ we need to specify the elements of T^{-1} even more, B and T. As $T_i = D_{n_i} e^{\tilde{J}_i}$ $(\tilde{J}_i = J(\lambda_i))$, $T^{-1}(t) = \oplus_{i=1}^{s} T_i^{-1}(t) = \oplus_{i=1}^{s} e^{-\tilde{J}_i} D_{n_i}^{-1}(t)$, and $B(t) = \left(B_{ik}^{n_i \times n_k}(t)\right)_{\substack{1<i<s \\ 1<k<s}}$ we have that $T^{-1}BT = (T_i^{-1} B_{ik} T_k)$ $1 \le i, k \le s$.

Now, we verify that 1) and 2) imply i) $V \in L_c(t \ge t_0)$ and ii) ΔQ, $Q\Delta$, $QV \in L_1(t \ge t_0)$, where $Q(t) = \int_t^{\infty} V(s)ds$.

Since $T_i^{-1} B_{ik} T_k = e^{-\tilde{J}_i} D_{n_i}^{-1} B_{ik} D_{n_k} e^{\tilde{J}_k}$, we obtain $V = T^{-1}BT \in L_c$, if and only if $D_{n_i}^{-1} B_{ik} D_{n_k} \in L_c$ for $i, k \in \{1, 2, \ldots, s\}$.

For $i=k=1$:

$$D_{n_1}^{-1} B_{11} D_{n_1} = \begin{vmatrix} b_{11} & t\, b_{12} & \ldots & t^{n_1-1} b_{1n_1} \\ t^{-1} b_{21} & b_{22} & \ldots & t^{n_1-2} b_{2n_1} \\ \vdots & \vdots & & \vdots \\ t^{1-n_1} b_{n_1 1} & t^{2-n_1} b_{n_1 2} & \ldots & b_{n_1 n_1} \end{vmatrix} \in L_c, \text{ because}$$

$v_{\alpha\beta}^{11}(t) = t^{\alpha-\beta} b_{\alpha\beta}(t) \in L_c$, for $\alpha, \beta \in \{1, 2, \ldots, n_1\}$.

For $i=k=2$:

$$D^{-1}_{n_2}B_{22}D_{n_2}=\begin{vmatrix} b_{(n_1+1)(n_1+1)} & tb_{(n_1+1)(n_1+2)} & \cdots & t^{n_2-1}b_{(n_1+1)(n_1+n_2)} \\ t^{-1}b_{(n_1+2)(n_1+1)} & b_{(n_1+2)(n_1+2)} & \cdots & t^{n_2-2}b_{(n_1+2)(n_1+n_2)} \\ \vdots & \vdots & \ddots & \vdots \\ t^{-(n_2-1)}b_{(n_1+n_2)(n_1+1)} & t^{-(n_2-2)}b_{(n_1+n_2)(n_1+2)} & \cdots & b_{(n_1+n_2)(n_1+n_2)} \end{vmatrix}$$

belongs to L_c because $v^{22}_{\alpha\beta}(t)=t^{\beta-\alpha}b^{22}_{\alpha\beta}(t)\in L_c$, for $\alpha,\beta\in\{n_1+1,\dots,n_1+n_2\}$.

Analogously, $D^{-1}_{n_s}B_{ss}D_{n_s}\in L_c$ because $v^{ss}_{\alpha\beta}(t)=t^{\beta-\alpha}b^{ss}_{\alpha\beta}(t)\in L_c$, for $\alpha,\beta\in\{\sum_{i=1}^{s-1}n_i+1,\ \dots,\ \sum_{i=1}^{s}n_i\}$.

Thus, we have analysed the diagonal of $V(t)$ in every possible case.

Let us look at the other blocks:

$$D^{-1}_{n_1}B_{12}D_{n_2}=\begin{vmatrix} t^{n_1}b_{1(n_1+1)} & t^{n_1+1}b_{1(n_1+2)} & \cdots & t^{n_1+n_2-1}b_{1(n_1+n_2)} \\ t^{n_1-1}b_{2(n_1+1)} & t^{n_1}b_{2(n_1+2)} & \cdots & t^{n_1+n_2-2}b_{2(n_1+n_2)} \\ \vdots & \vdots & \ddots & \vdots \\ tb_{n_1(n_1+1)} & t^{2}b_{3(n_1+2)} & \cdots & t^{n_2}b_{n_1(n_1+n_2)} \end{vmatrix}\in L_c$$

because $v^{12}_{\alpha\beta}(t)=t^{\beta-\alpha}b^{12}_{\alpha\beta}(t)\in L_c$ for $\alpha\in\{1,\dots,n_1\}$ and $\beta\in\{n_1+1,\dots,n_1+n_2\}$.

Thus, inductively,

$$D^{-1}_{n_1}B_{1s}D_{n_s}=\begin{vmatrix} t^{\sum_{i=1}^{s-1}n_i}\cdot b_{1(\sum_{1}^{s-1}n_i+1)} & t^{\sum_{i=1}^{s-1}n_i+1}\cdot b_{1(\sum_{1}^{s-1}n_i+2)} & \cdots & t^{\sum_{i=1}^{s}n_i-1}\cdot b_{1(\sum_{1}^{s}n_i)} \\ t^{\sum_{i=1}^{s-1}n_i-1}\cdot b_{2(\sum_{1}^{s-1}n_i+1)} & t^{\sum_{i=1}^{s-1}n_i}\cdot b_{2(\sum_{1}^{s-1}n_i+2)} & \cdots & t^{\sum_{i=1}^{s}n_i-2}\cdot b_{2(\sum_{1}^{s}n_i)} \\ \vdots & \vdots & & \vdots \\ t^{\sum_{i=1}^{s-1}n_i-(n_1-1)}\cdot b_{n_1(\sum_{1}^{s-1}n_i+1)} & \cdots\cdots & & t^{\sum_{i=1}^{s}n_i-1(n_1-1)}\cdot b_{n_1(\sum_{1}^{s}n_i)} \end{vmatrix}$$

belongs to $L_c(t\geq t_0)$, because $v_{\alpha\beta}^{1s}(t)=t^{\beta-\alpha}b_{\alpha\beta}^{1s}(t)\in L_c(t\geq t_0)$, for $\alpha\in\{1,\dots,n_1\}$, $\beta\in\{\sum_{i=1}^{s-1}n_i+1, \sum_{i=1}^{s-1}n_i+2, \dots, \sum_{i=1}^{s}n_i\}$. Therefore, in general we have that $V\in L_c$ if $v_{\alpha\beta}^{ij}(t)=t^{\beta-\alpha}b_{\alpha\beta}^{ij}(t)\in L_c(t\geq t_0)$, where $b_{\alpha\beta}^{ij}(t)$ are the entries of the blocks $B_{ij}(t)$.

On other side, $Q(t)=\int_t^{\infty}T^{-1}(s)B(s)T(s)ds=[\int_t^{\infty}T_i^{-1}B_{ik}T_k]_{1\leq i,k\leq s}$

$$=[e^{-J_i}(\int_t^{\infty}D_{n_i}^{-1}(s)B_{ik}(s)D_{n_k}(s)e^{J_k}].$$

Since $\Delta(t)=\Delta_0+\tilde{\Delta}(t)$ then $\Delta Q=\Delta_0 Q+\tilde{\Delta}Q$. Therefore $Q\in L_1(t\geq t_0)$ implies $\Delta Q\in L_1(t\geq t_0)$. Furthermore

$$Q(t)=[e^{-J_i}(\int_t^{\infty}v_{\alpha\beta}^{ik}(s)ds)e^{J_k}]\in L_1(t\geq t_0),$$

since $q_{\alpha\beta}^{ik}(t)=[\int_t^{\infty}v_{\alpha\beta}^{ik}(s)ds\in L_1(t\geq t_0)$, for $i,k\in\{1,2,\dots,s\}$. Then

$$Q(t)V(t)=[e^{-J_i}q_{\alpha\beta}^{ik}(t)v_{\beta l}^{kl}(t)e^{J_l}], \text{ for } 1\leq i,k,l\leq s.$$

Hence $QV\in L_1$ because

$$q_{\alpha\beta}^{ik}(t)v_{\beta l}^{kl}(t)\in L_1(t\geq t_0);\ i,k,l\in\{1,2,\dots,s\}.$$

Thus theorem 2.1, can be applied. Then system (24) has a fundamental matrix $Z(t)$, such that for $t\to\infty$,

$$Z(t)=\exp(\int_0^t\Delta(s)ds)[I+o(1)]=\exp(\Delta_0 t)\exp(\int_{t_0}^t\Delta(s)ds)[I+o(1)]$$

$$=\exp(\Delta_0 t)\cdot\mathrm{diag}(D_{n_1}^{-1}(t),\dots D_{n_s}^{-1}(t))[I+o(1)]=\bigoplus_{i=1}^{s}e^{\lambda_i t}D_{n_i}^{-1}(t)[I+o(1)].$$

Then system (22) has a fundamental matrix $Y(t)$ such that $Y(t)=e^{\bigoplus_{i=1}^{s}tJ(\lambda_i)}[I+o(1)]$, for $t\to\infty$. The proof is now finished.

Remarks. 1. Although this theorem is an extension of Theorem 3.2. because it also determines for the case of non simple eigenvalues the behavior of the solutions of the system (22), it also has to satisfy the stronger condition $q_{\alpha\beta}^{ik}(t)\in L_1(t\geq t_0)$, $i,j\in\{1,2,\dots,s\}$. However, it can be applied exten- sively to several system. In particular, to the linear differential equations of order n of the type

$$x^{(n)}+(a_1+b_1(t))x^{(n-1)}+\dots+(a_n+b_n(t))x=0, \tag{25}$$

for

$$b_j(t)=t^{-\beta_j}\exp(ikt^{\alpha_j}),\quad t\geq t_0>0,$$

where α_j and β_j are positive constants and k is a real number. These functions belong to $L_c(t\geq t_0)$ and its integration improves their integrability properties.

2. If the characteristic polynomial of the matrix associated to the homogeneous equation with constant coefficients a_i is given by $P(t)=(t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}\dots(t-\lambda_s)^{n_s}$ with $n_1+n_2+\dots+n_s=n$, we obtain

Corollary 4.2.. Suppose

1) $v_k(t)=t^{k-1}b_k(t)\in L_c(t\geq t_0)$, $k\in\{1,2,\dots,n\}$ and

2) $q_k(t)$ and $q_1(t)v_k(t)\in L_1(t\geq t_0)$, $k\in\{1,2,\dots,n\}$, $q_k(t)=\int_t^\infty v_k(s)ds$.

Then equation (25) has a fundamental system of solutions $x_0(t)$, $x_1(t)$, ..., $x_{n-1}(t)$, such that for $t\to\infty$ behave as

$$e^{\lambda_1 t},\ te^{\lambda_1 t},\dots,\ t^{n_1-1}e^{\lambda_1 t}$$

$$e^{\lambda_2 t},\ te^{\lambda_2 t},\dots,\ t^{n_2-1}e^{\lambda_2 t}$$

$$\vdots$$

$$e^{\lambda_s t},\ te^{\lambda_s t},\dots,\ t^{n_s-1}e^{\lambda_2 t}$$

respectively.

5. Examples. In this section we present some examples which show the results obtained previously.

Example 1. Consider.

$$x''+(1+b(t))x=0, \tag{26}$$

where i) $b(t)=t^{-\beta}$, $1<\beta<2$ and ii) $b(t)=t^{-\alpha}\sin(t^2)$, $0<\alpha<1$.

In the case i) $b\in L_1(t\geq t_0>0)$ and the equation (26) satisfies **Levinson's** Theorem [5]. On other hand, $b(t)\in L_c(t\geq t_0>0)$, but, since $\beta<2$,

$$Q(t)=\int_t^\infty s^{-\beta}ds=\frac{1}{\beta-1}\,t^{-(\beta-1)}\notin L_1(t\geq t_0).$$

Thus the case i) gives an example of a differential equation which satisfies the hypotheses of **Levinson's** Theorem [5], but we cannot apply Theorem 2.1.

In case ii), we have $b(t)=\sin(t^2)\in L_c(t\geq t_0>0)$, but $b(t)\notin L_1(t\geq t_0>0)$. Furthermore $Q(t)=\int_t^\infty s^{-\alpha}\sin(s^2)ds=\frac{1}{4}\,\frac{\cos(t^2)}{t^{\alpha+1}}+(L_1)$ exists for $t\geq t_0>0$, where (L_1) represents an integrable function. Moreover

$$Q(t)b(t)=\frac{1}{2}\frac{\sin(2t^2)}{t^{2\alpha+1}}+(L_1)\in L_1(t\geq t_0>0).$$

Then, in case ii), the hypotheses of Theorem 2.1. are satisfied but not those of Levinson's Theorem [5]. Thus, this example shows that the mentioned Theorems are different versions of the same problem.

Example 2. Consider the differential system

$$x'=\left[\begin{pmatrix}\lambda_0 & 0\\ 1 & \lambda_0\end{pmatrix}+g(t)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]x, \tag{27}$$

where $g(t)=t^{-\beta}\cos(t^2)$; λ_0 and β are real number with $\beta>0$.

The matrix $B(t)=\begin{pmatrix}0 & t^{-\beta}\cos(t^2)\\ t^{-\beta}\cos(t^2) & 0\end{pmatrix}$ satisfies the hypotheses of Theorem 3.2.

In fact $v_{12}(t)=t^{-(\beta-1)}\cos(t^2)$, $v_{21}(t)=t^{-(\beta+1)}\cos(t^2)\in L_c(t\geq t_0>0)$ and $q_{12}(t)=\int_t^\infty v_{12}=-\frac{\sin(t^2)}{2t^\beta}+(L_1)$, $q_{21}(t)=\int_t^\infty v_{21}=-\frac{\sin(t^2)}{2t^{\beta+2}}+(L_1)$. Hence

$$\left.\begin{aligned}t^{-1}q_{12}(t)&=-\frac{\sin(t^2)}{2t^{\beta+1}}+(L_1),\\ t^{-1}q_{21}(t)&=-\frac{\sin(t^2)}{2t^{\beta+3}}+(L_1),\end{aligned}\right\}\in L_1(t\geq t_0>0).$$

Moreover $q_{21}(t)v_{12}(t)=q_{12}(t)v_{21}(t)=\frac{\sin(2t^2)}{t^{2\beta+1}}+(L_1)\in L_1(t\geq t_0>0)$.

Then system (27) possesses a fundamental matrix $X(t)$ such that

$$X(t)=[I+o(1)]e^{tJ(\lambda_0)}\sim\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}e^{t\lambda_0}\quad\text{for}\quad t\to\infty.$$

Example 3. Consider the differential equation of fourth order

$$x^{(iv)}+(4+b_1(t))x^{(iii)}+(6+b_2(t))x^{(ii)}+(4+b_3(t))x'+(1+b_4(t))x=0, \tag{28}$$

where $b_j(t)=t^{-\beta_j}\sin(t^{\alpha_j})$, $t\geq t_0>0$, and α_j, β_j are positive constant such that $b_j(t)\in L_c(t\geq t_0>0)$, $j=1, 2, 3, 4$. Equation (28) is equivalent to the differential system

$$x' = \left[\begin{pmatrix} -4 & -6 & -4 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} -b_1 & -b_2 & -b_3 & -b_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] x \qquad (29)$$

The matrix A has the cononical Jordan form:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Thus $\lambda_0=-1$ is an eigenvalue with multiplicity n=4 and by Theorem 3.2 system (29) possesses a fundamental matrix X(t) such that $X(t)=[C+o(1)]\exp(tJ(-1))$ for $t\ \infty$, where C is a constant and nonsingular matrix. Then, equation (28) has a fundamental system of solutions $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ such that:

$$x_1(t) \sim e^{-t}$$
$$x_2(t) \sim te^{-t}$$
$$x_3(t) \sim t^2e^{-t}$$
$$x_4(t) \sim t^3e^{-t}$$

for $t\to\infty$.

Example 4. Consider the equation

$$x''+t^{-2}\sin t\ x=0 \qquad (30)$$

The hypotheses of Theorem 3.3. are satisfied because $V_2(t)=t^{-1}\sin t\in L_c(t\geq t_0>0)$ and $t^{-1}q_2(t)=\frac{\cos t}{t^2}+(L_1)\in L_1(t\geq t_0>0)$.

Then solutions of equation (30) behave as straight lines as $t\ \infty$. Since $tb_2(t)\notin L_1(t\geq t_0>0)$, then equation (30) does not satisfy the hypotheses of GHIZZETTI'S Theorem [3].

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