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Kleinian Groups with Common Commutator Subgroup.

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Let us denote by $\mathbb{H}^3$ the hyperbolic three dimensional space. A good model is given by $\mathbb{H}^3 = \{(z, t) \in \mathbb{C} \times \mathbb{R};\ t > 0\}$, with the complete Riemannian metric $ds^2 = (|dz|^2 + dt^2)/t^2$ of constant negative curvature. The group of orientation-preserving isometries of $\mathbb{H}^3$ is isomorphic to the projective linear group $\mathrm{PSL}(2, \mathbb{C})$. A subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is called discrete if it is discrete as subset of the topological space $\mathrm{PSL}(2, \mathbb{C})$. It is well known that a discrete group of $\mathrm{PSL}(2, \mathbb{C})$ acts discontinuously on all $\mathbb{H}^3$. If G is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$, then we say that G is Kleinian if it acts discontinuously somewhere on $\hat{\mathbb{C}}$. We denote by $\Omega(G)$ the region of discontinuity of G on the Riemann sphere $\hat{\mathbb{C}}$.

A Kleinian group G is called a B-group if there is a simply-connected component of its region of discontinuity that is invariant under the action of G. We say that a group G in $\mathrm{PSL}(2, \mathbb{C})$ is a noded Schottky group (respectively, noded π^g group), if it is a free group of finite rank (respectively, isomorphic to the fundamental group of a closed Riemann surface of genus $g \geq 2$) with a finited sided fundamental polyhedron for its action on $\mathbb{H}^3$(it must be necessarily Kleinian as consequence of Marden's isomorphism theorem). A Kleinian group keeping invariant a round disc is called Fuchsian.

To any discrete group G we have associated an 3-orbifold (with boundary a 2-orbifold)) $M(G) = (\mathbb{H}^3 \cup \Omega(G))/G$. The interior $\mathbb{H}^3/G$ is a hyperbolic 3-orbifold and its boundary $\Omega(G)/G$ is a Riemann surface with singularities. If G is torsion-free, then $M(G)$, $\mathbb{H}^3/G$ and $\Omega(G)/G$ are a 3-manifold, a hyperbolic 3-manifold and a Riemann surface, respectively. The commutator of G, denoted by $[G,G]$, is the normal subgroup of G generated by the commutators $aba^{-1}b^{-1}$ with $a, b \in G$. The hyperbolic orbifold $\mathbb{H}^3/[G,G]$ is called the homology cover of $\mathbb{H}^3/G$.

In [4] is proved the following result which can be considered as Torelli's theorem for closed Riemann surfaces.

Theorem 0.1 *(B. Maskit [4]). Let F be a torsion-free, finitely generated Fuchsian group. If F has no parabolic elements and it is of the first kind (that is, the region of discontinuity is exactly the complement of the extended reals), then the commutator subgroup $[F,F]$ determines uniquely F. In particular, the hyperbolic structure of the homology cover of a geometrically finite hyperbolic three manifold N (homeomorphic to $S \times (0,1)$, where S is a closed orientable surface of genus $g \geq 2$) without cusps, determines uniquely (up to isometries) the structure of N.*

We have extended this result for more general Kleinian groups as follows.

Theorem 0.2 *([1], [2] and [3]). Let G be a torsion-free, non-abelian Kleinian group which is either*

(a) B-group; or

(b) geometrically finite and isomorphic to a Fuchsian group,

then G is uniquely determined by its commutator subgroup $[G,G]$. In particular, the hyperbolic structure of a 3-manifold, which either

(1) is geometrically finite and homeomorphic to $S \times (0,1)$, where S is a closed orientable surface of genus greater than one; or

(2) has a incompressible boundary for which the inclution map induces an isomorphism at the level of homotopy,

is uniquely determined (up to isometries) by the hyperbolic structure of its homology cover.

Examples of groups either with torsion or in higher dimensions where the above results are not longer true are given in [2].

Problem. For which class of Kleinian groups holds the above rigidity property?

We prove Theorem 2 in the case that G is a Schottky group.

Theorem 2'. *Let G and H be Schottky groups of genera g and h, respectively. Assume g and h to be greater or equal to 2, that is, G and H are non-cyclic groups. If $[G,G]=[H,H]$, then $G=H$.*

Let us denote by N the common commutator subgroup of G and H. Since N is normal subgroup of both G and H, both have the same region of discontinuity. Denote this region by Ω. The region Ω is nonelementary, that is, Ω is the quotient of the hyperbolic plane by a nonelementary Fuchsian group. In particular, the total group of conformal automorphisms of Ω acts discontinuously on it.

Set $T=G\cap H$ and let R be the group generated by G and H. The fact that the group of conformal automorphisms of Ω acts discontinuously on it implies that the group R is a Kleinian group with Ω as region of discontinuity. Let us consider the following commutative diagram

$$\begin{tikzcd}
 & \Omega \arrow[d, "t"'] & \\
 & \Omega/N \arrow[d, "r"'] & \\
 & \Omega/T \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\Omega/G \arrow[dr, "q_1"'] & & \Omega/H \arrow[dl, "q_2"] \\
 & \Omega/R &
\end{tikzcd}$$

Since Ω/G and Ω/H are closed Riemann surfaces of genus g and h, respectively, they have finite hyperbolic area. In particular, the surface Ω/R must have finite hyperbolic area, and the (branched) coverings q_1 and q_2 are necessarily of finite degree. It follows that the groups G and H have finite index in the group R. Group theory implies that T has finite index in G, H and R. As a consequence, the (branched) coverings p_1 and p_2 have finite degree. We denote by γ the genus of Ω/R.

We must also remark that the coverings p_i are necessarily unbranched. This is consequence of the fact that the coverings $\pi : \Omega \to \Omega/T$, $\pi_1 : \Omega \to \Omega/G$ and $\pi_2 : \Omega \to \Omega/H$ are unbranched.

Now we consider the actions of G and H as isometries of the hyperbolic 3-space $\mathbb{H}^3$. Since R is discrete, it acts discontinuously on $\mathbb{H}^3 \cup \Omega$. Consider the following commutative diagram.

$$\begin{tikzcd}
 & \mathbb{H}^3 \cup \Omega \arrow[d, "T"'] & \\
 & (\mathbb{H}^3 \cup \Omega)/N \arrow[d, "\mathcal{R}"'] & \\
 & (\mathbb{H}^3 \cup \Omega)/T \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & \\
(\mathbb{H}^3 \cup \Omega)/G \arrow[dr, "Q_1"'] & & (\mathbb{H}^3 \cup \Omega)/H \arrow[dl, "Q_2"] \\
 & (\mathbb{H}^3 \cup \Omega)/R &
\end{tikzcd}$$

Since T has finite index in both G and H, the coverings P_k have finite degree, for $k = 1, 2$. Moreover, they are unbranched coverings as we observed above. Consider the induced homomorphisms of the above coverings at the first homology groups (singular homology) with complex coefficients (finite dimensional since our 3-manifolds are compact and homeomorphic to the connect summe of copies of type $D \times S^1$, where S is a closed orientable surface):

$$(P_1)_* : H_1\left(\frac{\mathbb{H}^3 \cup \Omega}{T}, \mathbb{C}\right) \to H_1\left(\frac{\mathbb{H}^3 \cup \Omega}{G}, \mathbb{C}\right)$$

$$(P_2)_* : H_1\left(\frac{\mathbb{H}^3 \cup \Omega}{T}, \mathbb{C}\right) \to H_1\left(\frac{\mathbb{H}^3 \cup \Omega}{H}, \mathbb{C}\right).$$

Since the coverings P_k have finite degree, the above homomorphisms are in fact surjective. The kernel of $(P_k)_*$ corresponds to loops on $\frac{\mathbb{H}^3 \cup \Omega}{T}$ which lift to loops on $\frac{\mathbb{H}^3 \cup \Omega}{N}$. In particular, $(P_1)_*$ and $(P_2)_*$ have the same kernel. It follows that the dimensions of $H_1(\frac{\mathbb{H}^3 \cup \Omega}{G}, \mathbb{C})$ and $H_1(\frac{\mathbb{H}^3 \cup \Omega}{H}, \mathbb{C})$ are the same. Now, the 3-manifold

$\frac{\mathbb{H}^3 \cup \Omega(K)}{K}$, for K a Schottky group of genus g, has fundamental group isomorphic to K, that is, a free group of rank g. In particular, we have the equality

$$g = h.$$

We return to the two dimensional situation. We have a natural embedding of each surface Ω, Ω/N, Ω/T, Ω/G, Ω/H and Ω/R as the natural boundary of the hyperbolic orbifold (a manifold with the only exception of $\frac{\mathbb{H}^3\cup\Omega}{R}$) $\mathbb{H}^3 \cup \Omega$, $\frac{\mathbb{H}^3\cup\Omega}{N}$, $\frac{\mathbb{H}^3\cup\Omega}{T}$, $\frac{\mathbb{H}^3\cup\Omega}{G}$, respectively. We denote the above inclusions by i_Ω, i_N, i_T, i_G, i_H and i_R, respectively. For the particular type of manifolds we are considering, these inclusions define surjective homomorphisms at the level of homology (with complex coefficients). The kernel of each of these homomorphisms is as follows.

(1) $\mathrm{Ker}(i_\Omega)_* = L_\Omega = H_1(\Omega, \mathbb{C})$;

(2) $\mathrm{Ker}(i_N)_* = L_N$;

(3) $\mathrm{Ker}(i_T)_* = L_T$ corresponds to the classes of loops on Ω /T which lift to loops on Ω by $r \circ t$;

(4) $\mathrm{Ker}(i_G)_* = L_G$ corresponds to the classes of loops on Ω /G which lift to loops on Ω by $p_1 \circ r \circ t$;

(5) $\mathrm{Ker}(i_H)_* = L_H$ corresponds to the classes of loops on Ω /H which lift to loops on Ω by $p_2 \circ r \circ t$;

(6) $\mathrm{Ker}(i_R)_* = L_R$.

In particular, the induced homomorphisms $(i_B)_* : \frac{H_1(\Omega/B,\mathbb{C})}{L_B} \to H_1(\frac{\mathbb{H}^3\cup\Omega}{B}, \mathbb{C})$, are isomorphisms for $B \in \{N, T, G, H, R\}$.

Since the coverings p_k and the (branched) coverings q_k have finite degree, for $k = 1, 2$, the induced surjective homomorphisms at homology (with complex coefficients) satisfy the following:

(7) $(p_1)_*(L_T) \subseteq L_G$;

(8) $(p_2)_*(L_T) \subseteq L_H$;

(9) $(q_1)_*(L_G) \subseteq L_R$;

(10) $(q_2)_*(L_H) \subseteq L_R$.

We have natural surjective homomorphisms (induced by the above ones)

$$(\bar{p}_1)_* : \frac{H_1(\Omega/T, \mathbb{C})}{L_T} \to \frac{H_1(\Omega/G, \mathbb{C})}{L_G};$$

$$(\bar{p}_2)_* : \frac{H_1(\Omega/T, \mathbb{C})}{L_T} \to \frac{H_1(\Omega/H, \mathbb{C})}{L_H};$$

$$(\bar{q}_1)_* : \frac{H_1(\Omega/G, \mathbb{C})}{L_G} \to \frac{H_1(\Omega/R, \mathbb{C})}{L_R};$$

$$(\bar{q}_2)_* : \frac{H_1(\Omega/H, \mathbb{C})}{L_H} \to \frac{H_1(\Omega/R, \mathbb{C})}{L_R}.$$

These homomorphisms satisfy the following equalities.

(11) $(P_1)_* \circ (i_T)_* = (i_G)_* \circ (\bar{p}_1)_*$;

(12) $(P_2)_* \circ (i_T)_* = (i_H)_* \circ (\bar{p}_2)_*$;

(13) $(Q_1)_* \circ (i_G)_* = (i_R)_* \circ (\bar{q}_1)_*$;

(14) $(Q_2)_* \circ (i_H)_* = (i_R)_* \circ (\bar{q}_2)_*$.

In particular, the homomorphisms $\bar{p}_1$ and $\bar{p}_2$ have the same kernel.

Now we proceed to construct dual spaces to $\frac{H_1(\Omega/T,\mathbb{C})}{L_T}$, $\frac{H_1(\Omega/G,\mathbb{C})}{L_G}$, $\frac{H_1(\Omega/H,\mathbb{C})}{L_H}$ and $\frac{H_1(\Omega/R,\mathbb{C})}{L_R}$, respectively.

The dual maps of

$$(p_1)_* : H_1(\Omega/T, \mathbb{C}) \to H_1(\Omega/G, \mathbb{C}),$$

$$(p_2)_* : H_1(\Omega/T, \mathbb{C}) \to H_1(\Omega/H, \mathbb{C}),$$

$$(q_1)_* : H_1(\Omega/G, \mathbb{C}) \to H_1(\Omega/R, \mathbb{C}),$$

$$(q_2)_* : H_1(\Omega/H, \mathbb{C}) \to H_1(\Omega/R, \mathbb{C})$$

are giving by the pull-backs

$$(p_1)^* : \mathcal{H}(\Omega/G) \to \mathcal{H}(\Omega/T),$$

$$(p_2)^* : \mathcal{H}(\Omega/H) \to \mathcal{H}(\Omega/T),$$

$$(q_1)^* : \mathcal{H}(\Omega/R) \to \mathcal{H}(\Omega/G),$$

$$(q_2)^* : \mathcal{H}(\Omega/R) \to \mathcal{H}(\Omega/H).$$

Since the homomorphisms at the level of homology are surjective, the above dual maps (by pull-backs) are injective.

Let us consider the orthogonal spaces of L_T in $\mathcal{H}(\Omega/T)$, of L_G in $\mathcal{H}(\Omega/G)$, of L_H in $\mathcal{H}(\Omega/H)$ and of L_R in $\mathcal{H}(\Omega/R)$. Let us denote them by $\mathcal{L}_T$, $\mathcal{L}_G$, $\mathcal{L}_H$ and $\mathcal{L}_R$, respectively. These spaces are dual to $\frac{H_1(\Omega/T,\mathbb{C})}{L_T}$, $\frac{H_1(\Omega/G,\mathbb{C})}{L_G}$, $\frac{H_1(\Omega/H,\mathbb{C})}{L_H}$ and $\frac{H_1(\Omega/R,\mathbb{C})}{L_R}$, respectively.

Observe that the complex dimension of $\mathcal{L}_G$ is g, the dimension of $\mathcal{L}_H$ is h, and the dimension of $\mathcal{L}_R$ is at most γ.

The injective homomorphisms $(p_k)^*$ and $(q_k)^*$ induce injective homomorphisms

$$(\bar{p}_1)^* : \mathcal{L}_G \to \mathcal{L}_T;$$

$$(\bar{p}_2)^* : \mathcal{L}_H \to \mathcal{L}_T;$$

$$(\bar{q}_1)^* : \mathcal{L}_R \to \mathcal{L}_G;$$

$$(\bar{q}_2)^* : \mathcal{L}_R \to \mathcal{L}_H.$$

We can see that the above homomorphisms are the dual maps of $(\bar{p}_1)_*$, $(\bar{p}_2)_*$, $(\bar{q}_1)_*$ and $(\bar{q}_2)_*$, respectively.

The equality of the kernels of $(\bar{p}_1)_*$ and $(\bar{p}_2)_*$ implies the equality of the respective orthogonals spaces in $\mathcal{L}_T$, that is, the equality of the images of $(\bar{p}_1)^*$ and $(\bar{p}_2)^*$. We can see the images of these maps as those 1-forms in $\mathcal{L}_T$ which are invariant under the action of G/T and H/T, respectively. In particular, every 1-form in $\mathcal{L}_T$ is invariant under G/T if and only if it is invariant under H/T. In particular, the image of $\mathcal{L}_R$ under $(\bar{p}_1)^* \circ (\bar{q}_1)^*$ is equal to the image of $(\bar{p}_1)^*$. The injectivity of these maps imply that the dimensions of $\mathcal{L}_G$, $\mathcal{L}_H$ and $\mathcal{L}_R$ are the same, that is, $g = \gamma = h$. It follows that G and H must have index one in R. In particular, $G = H$. ∎

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