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CLIFFORD ANALYSIS AND ITS APLICATIONS IN MATHEMATICAL PHYSICS

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1 On hypercomplex numbers

The really great success of the introduction of complex numbers makes it natural to look for generalizations to higher dimensions. The true reason is founded in the possibility of its geometric interpretation. So the consideration of so-called "geometric algebras" with their elements the "hypercomplex" numbers shifts the geometric-algebraic observation techniques to higher dimensions.

1.1 Complex numbers

We consider in the vector space $\mathbb{R}^2$ pairs of real numbers $z = (x, y)$. Any such pair characterize a free vector in this plane. Two vectors z_1 and z_2 are equal if and only if $x_1 = x_2$ and $y_1 = y_2$ where $z_i = (x_i, y_i)$ $(i = 1, 2)$. Vectors which denote points lying symmetric to the real axis x are called *conjugated*, i.e. $\overline{z} := (x, -y)$ is conjugated to z. The set of all such pairs is denoted by $\mathbb{C}$. Further, we induce in $\mathbb{C}$ the addition and multiplication with real numbers from the vector space R^2. Now any element from $\mathbb{C}$ can be given in the so-called *cartesian form*

$$z = x\mathbf{1} + y\mathbf{i} =: x + \mathbf{i}y,$$

where $\mathbf{1} = (1,0)$ and $\mathbf{i} = (0,1)$. These two special pairs are called *unit vectors* of the corresponding axis x and y. Let $z_i = x_i + \mathbf{i}y_i \in \mathbb{C}$ $(i = 1,2)$. We define a scalar product and a vector product as follows:

$$z_1 \cdot z_2 := x_1x_2 + y_1y_2$$

$$[z_1,\ z_2] := x_1y_2 - x_2y_1$$

By definition we obtain $\mathbf{i} \cdot \mathbf{i} = -1$ and $[\mathbf{i}, \mathbf{i}] = 0$. Each of these products is not suited to generate a commutative field. In the following way we can define a product which transforms $\mathbb{C}$ in a commutative field. We have only to set

$$z_1z_2 = x_1x_2 - y_1y_2 + \mathbf{i}(x_1y_2 + y_1x_2) = \overline{z_1} \cdot z_2 + [\overline{z_1}, z_2].$$

$\mathbb{C}$ is now called *field of complex numbers* and the elements $z = x + \mathbf{i}y$ is called *complex number.* Furthermore, it is called $x = \mathrm{Re}z$ *real part* and $y = \mathrm{Im}z$ *imaginary part* of the complex number z. Complex numbers with zero real part are called *imaginary numbers.* From complex numbers with zero imaginary part we reobtain our traditionally real numbers. Because of multiplication and division of such complex numbers leads to very complicated expressions it is necessary to represent complex numbers in the polar form :

$$z = r(\cos\ \phi + \mathbf{i}\sin\ \phi).$$

Each complex number $z \neq 0$ is uniquely representable by the modulus $r = |z|$ of the complex number z and its argument of $\phi := Arg\ z(mod\ 2\pi)$. It is useful to introduce the abbreviation:

$$\cos\ \phi + \mathbf{i}\sin\ \phi =: e^{\mathbf{i}\phi}.$$

Thus the complex number z has the representation

$$z = re^{\mathbf{i}\phi}.$$

By the help of trigonometric relations and the definition of the complex multiplication we obtain the following rule:

$$r_1e^{\mathbf{i}\phi_1}r_2e^{\mathbf{i}\phi_2} = r_1r_2e^{\mathbf{i}(\phi_1+\phi_2)}.$$

where $z_k = r_k e^{\mathbf{i}\phi_k}$ $(k = 1, 2)$. It is simply to show by induction that

$$(\cos\ \phi + \mathbf{i} \sin\ \phi)^n = \cos\ n\phi + \mathbf{i} \sin\ n\phi \quad (A.\ De\ Moivre).$$

In the history of complex numbers the so-called *circle division equation*

$$z^n = 1$$

plays an important role. We have

$$z^n - 1 = (z - \zeta_0)(z - \zeta_1)...(z - \zeta_{n-1})$$

$$\zeta_k = \mathbf{e}^{\frac{2\pi k}{n}} \quad 0 \le k \le (n-1).$$

The complex numbers ζ_k are just the cornerpoints lying on the unit circle of a equilateral n-angle. Moreover, they are the zeros of the polynomial $p(z) := z^n - 1$, which is a special case of the so-called *fundamental theorem of the algebra*:

Theorem 1 *A non-constant polynomial of degree n posesses exactly n zeros. Here the number of zeros is to compute in accordance with its multiplicity.*

Corollary 2 *The quadratic equation*

$$z^2 + \alpha z + \beta = 0 \quad (\alpha, \beta \in \mathbb{C})$$

has at most two solutions z_1 and z_2 in $\mathbb{C}$.

Remark 3 *An alternative definition can be given in the following way: Let*

$$M_{\mathbb{C}} := \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x, y \in \mathbb{R} \right\} \subset \mathbb{R}(2)$$

furnished with the operation matrix addition and (real) multiplication. $M_{\mathbb{C}}$ has the structure of a field, which is isomorphic to the field of complex numbers $\mathbb{C}$.

1.2 Hamilton's creation – real quaternions

There is a little story on the discovery of quaternions. Briefly we will describe this: *It was on Monday, October 16 in 1843 and it happened that in this morning W.R. Hamilton had to preside at a meeting in the Royal Irish Academy. He was walking with his wife along the Royal Canal in Dublin when the answer of this 10 years old problem came to his mind. With his knife he then and there carved on a stone on Broome Bridge the formulae:*

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{1}$$

By the way in the above mentioned letter of Hamilton named this bridge errounously Brougham Bridge and up to now this bridge bears this name given by W.R. Hamilton.

Hamilton called these numbers

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

quaternions (cf. [18]). In that way the notion of a quaternion was introduced. The first paper on quaternions appeared 14. November 1843 in the Council Books of the Royal Academy at the First General Meeting of the Session (cf. [1]).

Consider the real four-dimensional vector space $\mathbb{R}^4$ with its standard basis $e_0, \mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$ where the element e_k $(k = 0, 1, 2, 3)$ is to identify with that 4-tuple which has at the (k+1)-th component the number one and has zeros otherwise in a cyclic denotation. The element e_0 will be identified in the following by 1 and will mostly be omitted. Notice that $\mathbf{e_k}$ $(k = 1, 2, 3)$ are so-called "axial" vectors in $\mathbb{R}^4$. In case of cancellation of the first component one can identify these "axial" vectors in $\mathbb{R}^4$ with ordinary vectors or "polar" vectors in $\mathbb{R}^3$ (cf. [21]). It is necessary to use sometimes these identifications. An arbitrary element $x \in \mathbb{R}^4$ can now be represented by

$$x = x_0 + \mathbf{x} \quad \text{with} \quad \mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3} \quad (x_k \in \mathbb{R},\ k = 1, 2, 3)$$

The part x_0 is called *scalar part* of x written $x_0e_0 =: \mathrm{Sc}(x)$ and the part $\mathbf{x}$ is called *vector part* and written $\mathbf{x} = \mathrm{Vec}(x)$.

Recalling the computation rules in $\mathbb{R}^4$ we have for $x, y \in \mathbb{R}^4$:

(i) *Identity*: $x = y$ if and only if $x_0 = y_0$ and $\mathbf{x} = \mathbf{y}$

(ii) *Scalar multiplication:* $\mathbb{R} \times \mathbb{R}^4 \ni (\lambda, x) \to \lambda x \in \mathbb{R}^4$ with

$$\lambda x = \lambda x_0 + \lambda x_1 \mathbf{e_1} + \lambda x_2 \mathbf{e_2} + \lambda x_3 \mathbf{e_3} =: \lambda x_0 + (\lambda \mathbf{x}) \quad (\lambda \in \mathbb{R}).$$

(iii) *Addition:* $\mathbb{R}^4 \times \mathbb{R}^4 \ni (x, y) \to x + y \in \mathbb{R}^4$ with

$$\begin{aligned} x + y &= (x_0 + y_0) + (x_1 + y_1)\mathbf{e_1} + (x_2 + y_2)\mathbf{e_2} + (x_3 + y_3)\mathbf{e_3} \\ &=: (x_0 + y_0) + \mathbf{x} + \mathbf{y} \end{aligned}$$

and $x_k, y_k \in \mathbb{R}$.
Now we define a multiplication in the following way:
$\mathbb{R}^4 \times \mathbb{R}^4 \ni (x, y) \to xy \in \mathbb{R}^4$ with

$$xy = x_0 y_0 - \mathbf{x} \cdot \mathbf{y} + x_0 \mathbf{y} + y_0 \mathbf{x} + \mathbf{x} \times \mathbf{y}$$

where

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &:= x_1 y_1 + x_2 y_2 + x_3 y_3 \\ \mathbf{x} \times \mathbf{y} &:= \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &:= (x_2 y_3 - x_3 y_2)\mathbf{e_1} + (x_3 y_1 - x_1 y_3)\mathbf{e_2} + (x_1 y_2 - x_2 y_1)\mathbf{e_3}. \end{aligned}$$

Definition 4 *The space $\mathbb{R}^4$ furnished with the above defined multiplication rule has the structure of an algebra and is called algebra of real quaternions. Its elements are called simply quaternions. In honour of W.R. Hamilton we will denote this algebra by $\mathbb{H}$.*

If $x = \mathbf{x}$ then x is called *pure quaternion* or simply *vector*. The set of all pure quaternions is denoted by **Vec** $\mathbb{H} \subset \mathbb{H}$, while the set of all scalars will be denoted by **Sc** $\mathbb{H} \subset \mathbb{H}$. **Vec** $\mathbb{H}$ and **Sc** $\mathbb{H}$ are real linear subspaces of $\mathbb{H}$ which are not closed relatively to the multiplication introduced above.

Corollary 5 *The map $\rho_y : \mathbb{R}^3 \ni \underline{x} \to -\underline{y}\underline{x}\underline{y}^{-1} \in \mathbb{R}^3$ is a reflection in the plane which lies orthogonal to the vector $\underline{y}$.*

Proof. The map ρ_y is linear and $\rho_y(\underline{y}) = -\underline{y}$, while for any vector $r \perp \underline{y}$ it follows that $\rho_y(r) = -\underline{y}r\underline{y}^{-1} = r\underline{y}\underline{y}^{-1} = r$. ■

Corollary 6 *Each rotation in $\mathbb{R}^3$ has for some non-zero $\underline{y}$ the form $-\rho_y$. Conversely, any such map can be seen as rotation in $\mathbb{R}^3$.*

Proof. We already know that the product of two plane reflections is just a rotation and vice versa. ■

Proposition 7 *Let y be a quaternion. Then there exists a vector $\underline{a} \neq 0$, such that $y\underline{a}$ is also a vector.*

Proof. Let $\underline{a}$ be a vector in $\mathbb{R}^3$ orthogonal to the vector part of y. Then

$$y\underline{a} = y_0\underline{a} + \underline{y} \times \underline{a} \in \mathbb{R}^3. \quad ■$$

Corollary 8 *Each quaternion can be described as a product of two vectors.*

Proposition 9 *An arbitrary unit quaternion can be represented as the product $yxy^{-1}x^{-1}$, where $x = \underline{x}$ and $y = \underline{y}$ are non-zero vectors.*

Proof. We know from proposition 7 that for any unit quaternion e a non-zero vector $\underline{x}$ exists such that $e\underline{x}$ is a vector. Because of $|e| = 1$ we have $|e\underline{x}| = |\underline{x}|$. In this way $e\underline{x}$ has to be a rotation. Then there exists a vector $\underline{y} \neq 0$ with $e\underline{x} = \underline{y}\underline{x}\underline{y}^{-1}$ and so the statement follows. ■

1.2.1 Representation of real quaternions

We will group here some of the most important properties of the representation of real quaternions.

Theorem 10 *An arbitrary quaternion $x \in \mathbb{H}$, $\underline{x} \neq 0$ permits the representation*

$$x = |x|(\cos\phi + \omega(x)\sin\phi),$$

where $\phi = \operatorname{arccot}(x_0/|\underline{x}|)$ and $\omega(x) = \underline{x}/|\underline{x}| \in S^3$.

Proof. It is well known that

$$\sin\phi = \frac{1}{\sqrt{(1+\cot^2\phi)}} \quad \text{and} \quad \cos\phi = \frac{\cot\phi}{\sqrt{(1+\cot^2\phi)}}.$$

We obtain under our assumption

$$\sin\operatorname{arccot}\frac{x_0}{|\underline{x}|} = \frac{1}{\sqrt{1+(\frac{x_0}{|\underline{x}|})^2}} \quad \text{and} \quad \cos\operatorname{arccot}\frac{x_0}{|\underline{x}|} = \frac{\frac{x_0}{|\underline{x}|}}{\sqrt{1+(\frac{x_0}{|\underline{x}|})^2}}.$$

Then we get by a straightforward calculation $\cos\phi + (\underline{x}/|\underline{x}|)\sin\phi = x/|x|$ which verifies our theorem. ■

Example 11 Let $x = 3 + 2e_1 + 2e_2 + e_3$ then $|x| = 3\sqrt{2}$, $|\underline{x}| = 3, \phi = 45°$. Thus we obtain the representation

$$x = 3\sqrt{2}\Big[\cos 45° + \frac{(2e_1 + 2e_2 + e_3)}{3}\sin 45°\Big].$$

Corollary 12 (MOIVRE's formula). *Let $x \in \mathbb{H}, \underline{x} \neq 0, n \in \mathbb{N}$, then the following formula is valid:*

$$(\cos\phi + \omega(x)\sin\phi)^n = \cos\, n\phi + \omega(x)\sin\, n\phi.$$

Proposition 13 *The $\mathbb{R}$-linear hull of the set $\{1,\ x\}$, where x is not real, forms a subalgebra which is isomorphic to $\mathbb{C}$.*

Theorem 14 *The algebra of real quaternions $\mathbb{H}$ can be represented by the matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Therefore each quaternion x permits the representation

$$x = \begin{pmatrix} \overline{z}_1 & -\overline{z}_2 \\ z_2 & z_1 \end{pmatrix}$$

with $z_1 := x_0 + ix_1$ and $z_2 := x_2 + ix_3$.

Corollary 15 *The algebra of real quaternions* $\mathbb{H}$ *can be generated by the real matrices*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

1.3 Pauli algebra – a realization of complex quaternions

More than real quaternions, complex quaternions take an important part in theoretical physics. Let us now discuss the fundamental properties of quaternions with complex-valued coefficients. We will use the so-called PAULI matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

PAULI matrices form an algebra P. Then we get the representation of an arbitrary element $x \in P$ in the form

$$x = x_0\sigma_0 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 + x_4\sigma_2\sigma_3 + x_5\sigma_3\sigma_1 + x_6\sigma_1\sigma_2 + x_7\sigma_1\sigma_2\sigma_3.$$

We have to distinguish four classes of elements, namely complex linear combinations of scalars σ_0, vectors $\sigma_1, \sigma_2, \sigma_3$, bivectors $\sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2$ and pseudoscalars $\sigma_1\sigma_2\sigma_3$. PAULI matrices satisfy the conditions $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$, where δ_{ij} denotes KRONECKER's symbol. On account of $(\sigma_1\sigma_2\sigma_3)^2 = -\sigma_0$ we obtain that the linear space generated by $\{\sigma_0, \sigma_1\sigma_2\sigma_3\}$ is isomorphic to the field of complex numbers $\mathbb{C}$.

Remark 16 The centre of P is $\mathbb{C}$.

Proposition 17 *With* $\varepsilon := i\sigma_0$ *(Hodge star map) we get*

$$-\varepsilon\sigma_0 = \sigma_1\sigma_2\sigma_3, -\varepsilon\sigma_1 = \sigma_2\sigma_3, -\varepsilon\sigma_2 = \sigma_3\sigma_1, -\varepsilon\sigma_3 = \sigma_1\sigma_2, \varepsilon\sigma_1\sigma_2 = +\sigma_3,$$
$$\varepsilon\sigma_2\sigma_3 = +\sigma_1, \varepsilon\sigma_3\sigma_1 = +\sigma_2, \varepsilon\sigma_1\sigma_2\sigma_3 = +\sigma_0,$$

i.e. the multiplication by ε transforms scalars into pseudoscalars, vectors into bivectors, bivectors into vectors, and pseudoscalars into scalars.

2 Clifford algebras

We will consider a class of algebras which form the frame for our further considerations. This structure shall incorporate number systems like complex numbers and quaternions as well as vector and multivector systems. We will be able to realize in this algebra a large number of fundamental geometric and analytic ideas. Let us start with the description of this algebra in expressing the main ideas.

Definition 1 (Real Clifford algebra). *An associative algebra over $\mathbb{R}$ with unit $e_0 := 1$ which is freely generated by n basis elements $e_1, ..., e_n$ together with the defining relations*

$$e_i e_j + e_j e_i = -2\delta_{ij} \quad (i, j = 1, ..., n). \tag{2}$$

is called real Clifford algebra (denoted by: $C\ell_{0,n}$). Here δ_{ij} denotes the Kronecker symbol. This means

$$e_i e_j + e_j e_i = 0 \quad \textit{iff} \quad i \neq j, \quad e_i^2 = -1 \quad (i, j = 1, ..., n).$$

Usually the elements $e_1, e_2, ..., e_n$ belong to the vector space $\mathbb{R}^n$ and form there a basis. After the canonical embedding we have identified the "initial" vectors $e_1, e_2, ..., e_n$ in $\mathbb{R}^n$ with corresponding elements in $C\ell_{0,n}$. Furthermore, we identify the unit element in $C\ell_{0,n}$ with $1 \in \mathbb{R}$.

Definition 2 (Clifford algebra of the (p,q)-type). *Let $e_1, ..., e_q, ..., e_{p+q}$ form a basis of $\mathbb{R}^{p+q}$. We introduce the denotation*

$$\dot{e}_j := \mathrm{i}\ e_{q+j}.$$

The Clifford algebra of the (p,q)-type (denoted by: $C\ell_{p,q}$) is defined following an idea of F. Sommen by the real associative algebra with unit $e_0 := 1$ freely generated by elements of the form

$$\dot{e}_1, ..., \dot{e}_p, e_1, ..., e_q,$$

which fulfil the defining relations

$$\dot{e}_j\,\dot{e}_k + \dot{e}_k\,\dot{e}_j = 2\delta_{jk},$$

$$e_j\,e_k + e_k\,e_j = -2\delta_{jk},$$

$$e_j\,\dot{e}_k + \dot{e}_k\,e_j = 0\quad.$$

Definition 3 *The Clifford algebra $C\ell_{n,0}$ is called complex Clifford algebra can be seen as complexification of the real Clifford algebra $C\ell_{0,n}$ i.e. $C\ell_{n,0} = \mathbb{C} \otimes C\ell_{0,n}$.*

2.1 Involutions and Bott's classification

2.1.1 Involutions

In a CLIFFORD algebra there are at least three involutions. Depending on the degree k of $x \in C\ell_{p,q}$ we define the following:

(i) *grade involution* (or *main involution*) by the formula $\hat{x} := (-1)^k x$,

(ii) *reverse involution* (or *inversion*) by the relation $\tilde{x} := (-1)^{\frac{k(k-1)}{2}} x$,

(iii) *conjugation* by $\bar{x} := (-1)^{\frac{k(k+1)}{2}} x$.

Proposition 4 *These involutions satisfy the following isomorphic and anti-isomorphic conditions with respect to the multiplication of two elements*

$$(\widehat{xy}) = \hat{x}\hat{y},\;\; (\widetilde{xy}) = \tilde{y}\tilde{x},\;\; (\overline{xy}) = \bar{y}\bar{x}.$$

Reversion and conjugation can be extended complex-linear to complexified CLIFFORD algebras. In addition it is usual to introduce a *complex conjugation* by $x^* = x_1 - ix_2$ for $x = x_1 + ix_2$ $(x_k \in C\ell_{p,q})$.

Example 5 Let be $x \in \mathbb{C} \bigotimes C\ell_{3,0}$ the element

$$x = 1 + 3e_1 + 2e_2 + e_{21} + ie_{23} + e_{123},$$

where $e_{kl} := e_k e_l$ and $e_{klm} := e_k e_l e_m$, then we have for the above-mentioned involutions:

$$\begin{aligned}
\hat{x} &= 1 - 3e_1 - 2e_2 + e_{21} + ie_{23} - e_{123},\\
\tilde{x} &= 1 + 3e_1 + 2e_2 - e_{21} - ie_{23} - e_{123},\\
\bar{x} &= 1 - 3e_1 - 2e_2 - e_{21} - ie_{23} + e_{123},\\
x^* &= 1 + 3e_1 + 2e_2 + e_{21} - ie_{23} + e_{123}.
\end{aligned}$$

Definition 6 *Let $x \in C\ell_{n,0} \cup C\ell_{0,n}$ then $\sqrt{Sc(\tilde{x}x)}$ is called absolute value of x. It is denoted by $|x|$.*

Remark 7 For $x \in C\ell_{p,q}$ the product $\tilde{x}x$ is not necessarily real. P. LOUNESTO gave a counterexample: Let $p = 3, q = 1, x = (1 + e_1)(1 + e_{234}) \in C\ell_{3,1}$ and $x\tilde{x} = 0 \in \mathbb{R}$ but $\tilde{x}x = 4e_{234} + 4e_{1234}$ obviously does not belong to $\mathbb{R}$. The so-called *square norm* which can be defined by $\bar{x}x = \|x\|$ is often used.

2.1.2 Table of matrix representations

CLIFFORD algebras are isomorphic as associative algebras with unit to corresponding matrix algebras. Setting now $s := (p + q)/2$ one can show that only eight different rings of matrices occur in the description of the structure of CLIFFORD algebras. The following table gives an overview on the possible structures. For this reason let $M(d, F)$ denote the ring of $d \times d$ matrices over the field F.

$(p\text{-}q)$ mod 8	matrix ring
0	$M(2^{[s]}, \mathbb{R})$
1	$M(2^{[s]}, \mathbb{R}) \oplus M(2^{[s]}, \mathbb{R})$
2	$M(2^{[s]}, \mathbb{R})$
3	$M(2^{[\frac{s-1}{2}]}, \mathbb{C})$
4	$M(2^{[s-1]}, \mathbb{H})$
5	$M(2^{[\frac{s-3}{2}]}, \mathbb{H}) \oplus M(2^{[\frac{s-3}{2}]}, \mathbb{H})$
6	$M(2^{[s-1]}, \mathbb{H})$
7	$M(2^{[\frac{s-1}{2}]}, \mathbb{C})$

We mention that s depends only on the dimension of the algebra and not on its special structure. The denotation $[s]$ means the entire part of s.

2.2 Examples of Clifford algebras

An excellent possibility of representing CLIFFORD algebras, especially those of low dimensions, is to do it with the help of the PAULI *matrices* $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ introduced in Section 1.3. In this way we will interchange the basic elements e_i $(i = 0, 1, 2, 3)$ by the PAULI matrices σ_i $(i = 0, 1, 2, 3)$. It is easy to see that $\sigma_i^2 = \sigma_0$ and that the elements σ_i also fulfil the anti-commutativity property. We will agree upon the use of the symbol " = " for the identification of algebras. Let us now discuss the following examples:

$C\ell_{0,0}$: In this case the basis consists only of the identity element σ_0. Therefore $C\ell_{0,0} = \mathbb{R}$. An isomorphism is explicitly given by $x = x_0\sigma_0$.

$C\ell_{0,1}$: This algebra is generated by σ_0 and $i\sigma_2$ with $\sigma_2^2 = -\sigma_0$, therefore $C\ell_{0,1} = \mathbb{C}$. This is clear by the representation $x = x_0\sigma_0 + x_1 i\sigma_2$.

$C\ell_{0,2}$: The matrices $\sigma_0, -i\sigma_1, -i\sigma_2, \sigma_1\sigma_2$ $(= -i\sigma_3)$ form a basis in $C\ell_{0,2}$. For any element x of this algebra we obtain the representation

$$x = \begin{pmatrix} x_0 - ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & x_0 + ix_1 \end{pmatrix}$$

with $x_i \in \mathbb{R}$ $\quad (i = 0, 1, 2, 3)$.

This describes a matrix realization of the *algebra of real quaternions* $\mathbb{H}$. Hence $C\ell_{0,2} = \mathbb{H}$.

$C\ell_{3,0}$: The matrices $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1, \sigma_1\sigma_2\sigma_3$ form a basis in the algebra $C\ell_{3,0}$. We find that $\sigma_1\sigma_2 = -i\sigma_3, \sigma_2\sigma_3 = -i\sigma_1, \sigma_3\sigma_1 = -i\sigma_2$ and $\sigma_1\sigma_2\sigma_3 = -i\sigma_0$. An arbitrary element x of this algebra can be represented by

$$x = (x_0 - x_7 i)\sigma_0 + (x_1 - x_5 i)\sigma_1 + (x_2 - x_6 i)\sigma_2 + (x_3 - x_4 i)\sigma_3.$$

Then it is easy to get $C\ell_{3,0} = \mathbb{C} \bigotimes \mathbb{H}$. This algebra coincides with the PAULI *algebra*. The denotation by P is made in honour of WOLFGANG PAULI.

$C\ell_{1,3}$: In this algebra a basis can be constructed by the so-called *d-numbers* $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ of DIRAC which are formal 2×2 matrices in which entries are the PAULI matrices. So we have

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

for $i = 1, 2, 3$. Clearly, we have $\gamma_0^2 = \gamma_0$ and $\gamma_i^2 = -\gamma_0$. The basis in $C\ell_{1,3}$ is now given by

$$\begin{aligned}
&1, \\
&\gamma_0, \gamma_1, \gamma_2, \gamma_3, \\
&\gamma_1\gamma_0, \gamma_2\gamma_0, \gamma_3\gamma_0, \gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_3\gamma_1, \\
&\gamma_0\gamma_1\gamma_2, \gamma_0\gamma_2\gamma_3, \gamma_0\gamma_3\gamma_1, \gamma_1\gamma_2\gamma_3, \\
&\gamma_0\gamma_1\gamma_2\gamma_3,
\end{aligned}$$

where $1 : \mathbb{R}^{1,3} \longmapsto \mathbb{R}^{1,3}$ is the unit matrix. This algebra is called *space–time algebra.* One can show that this algebra is isomorphic to $M(4, \mathbb{R})$.

3 Important operators in the theory

3.1 Classes of monogenic functions

Let $G \subset \mathbb{R}^n$ be an open (nonempty) domain and $\Gamma = \partial G$ a piecewise smooth surface. We denote by $\mathcal{B}(\Omega)$ one of the following BANACH spaces of real or complex valued functions on Ω where Ω can be $G, \partial G = \Gamma$, or any suitable subset of $\overline{G}$:

(i) $C^{(k)}(\Omega)$ $(k = 1, 2, ...)$ is the space of k-times continuously $\mathbb{R}$-differentiable functions in Ω.

(ii) $C^{(k,\alpha)}(\Omega)$ $(k = 0, 1, 2, ...; \alpha \in (0, 1])$ is the space of k-times $\mathbb{R}$-differentiable functions, whose k-th derivative is HÖLDER continuous with the exponent α.

(iii) $L_p(\Omega)$, $(1 \leq p < \infty)$, is just the space of all functions, whose p-th power is LEBESGUE integrable in Ω.

(iv) $W_p^k(\Omega)$ is the space of k-times $\mathbb{R}$-differentiable functions in SOBOLEV's sense, whose k-th derivative belongs to $L_p(\Omega)$.

(v) $\overset{\circ}{W_p^k}(G)$ $(1 \leq p < \infty;\ k = 1, 2, ...)$, is the space of k-times $\mathbb{R}$-differentiable functions in SOBOLEV's sense which k-th derivative belongs to $L_p(G)$ and vanish on the boundary Γ.

Definition 1 *Let $\mathcal{E}$ be either $\mathbb{R}$ or $\mathbb{C}$. Then the space $\mathcal{B}(\Omega) \otimes_{\mathcal{E}} \mathcal{A}$, where $\mathcal{A} := C\ell_{p,q}$ is called $\mathcal{E}$-linear space $\mathcal{B}(\Omega, \mathcal{A})$ of all $\mathcal{A}$-valued functions.*

Remark 2 *For both $\mathcal{A} = C\ell_{0,n}$ or $\mathcal{A} = C\ell_{n,0}$ the space $\mathcal{B}(\Omega, \mathcal{A})$ is a* BANACH *space. This means that*

$$\mathcal{B}(\Omega, \mathcal{A}) = \left\{u = \sum_{i=0}^{\dim \mathcal{A}} u_i \varepsilon_i : u_i \in \mathcal{B}\right\}$$

is furnished with the norm $\|u\| = \{\sum_{i=0}^{\dim \mathcal{A}} \|u_i\|^2\}^{\frac{1}{2}}$. Here ε_k is a k-vector such that $\{\varepsilon_0, ..., \varepsilon_{\dim \mathcal{A}}\}$ form a basis in $\mathcal{A}$.

Definition 3 *Let u be given in a basis representation $u = \sum_{i=0}^{\dim \mathcal{A}} u_i(x)\varepsilon_i$, $x \in \Omega \subset \mathbb{R}^n$, where $\{\varepsilon_0, ..., \varepsilon_{\dim \mathcal{A}}\}$ is a basis in $\mathcal{A}$, then we say that $u \in \mathcal{B}(\Omega, \mathcal{A})$ if and only if $u_i \in \mathcal{B}(\Omega)$. Often we will briefly write instead of $\mathcal{B}(\Omega, \mathcal{A})$ $\mathcal{B}(\Omega)$.*

Corollary 4 *Topological properties as continuity, differentiability, and integrability of the coefficients $u_i(x), x \in \Omega$, in a suitable basis representation transfer to the $\mathcal{A}$-valued function $u = \sum_{i=0}^{\dim \mathcal{A}} u_i(x)\varepsilon_i$.*

3.1.1 Clifford regular functions

In general regular functions are those which belong to the kernel of a proper differential operator.

Definition 5 *The operator*

$$D := \sum_{i=1}^{n} e_i \partial_i$$

which acts on the space $C^1(G, C\ell_{p,q})$, *where* $G \subset \mathbb{R}^{p,q}$ *is called* DIRAC *operator and* $\partial_i := \partial/\partial x_i$ *is the i-th partial derivative and* $x = \sum_{i=1}^n x_i e_i \in \mathbb{R}^{p,q}$. *The operator*

$$\partial := \partial_0 + D$$

which acts on $C^1(G, C\ell_{p,q})$ *with* $G \subset \mathbb{R}^{p,q}$ *is called* CAUCHY–FUETER *operator where* $\partial_0 := \partial/\partial x_0$ *and* $x = x_0 + \sum_{i=1}^n x_i e_i \in \mathbb{R} \oplus \mathbb{R}^{p,q}$.

Definition 6 *Let* $G \subset \mathbb{R}^{p,q}$, $p + q = n$, *be an n-dimensional vector space with signature* p, q. *A function* $u \in C^1(G, Cl_{p,q})$ *is called left* CLIFFORD *regular (right* CLIFFORD *regular) if and only if*

$$Du = 0 \ ((uD) = 0).$$

A function which is both left and right CLIFFORD *regular is called two-sided* CLIFFORD *regular. Functions which fulfil the condition*

$$\partial\, u = 0 \quad ((u\partial) = 0)$$

are called left(right) CLIFFORD *holomorphic. We agree that the denotations* $C\ell_{p,q}$*- regular or* $C\ell_{p,q}$*- holomorphic will sometimes be used if it is necessary to emphasize this special algebra.*

Remark 7 *In* $C\ell_{0,n}$ *left(right) Clifford regular functions are often called left (right) monogenic functions. In such algebras the* DIRAC *operator is sometimes called standard* EUCLID*ean* DIRAC *operator. The set of all monogenic functions is denoted by* $\mathcal{M}(G, C\ell_{0,n})$.

Through the whole paper we will view the action of D and ∂ as an action on the left. It is sufficiently known that there are corresponding results for action of these operators on the right. Sometimes we will simply say regular functions instead of left regular functions.

Remark 8 *The* CAUCHY–FUETER *operator* ∂ *which acts on* $C^1(G, C\ell_{p,q})$ *with* $G \in \mathbb{R}^{p,q}$ *can be considered as the* DIRAC *operator on* $C^1(G, C\ell_{p+1,q})$ *with* $G \in \mathbb{R}^{p+1,q}$.

Example 9 *Let* $G \subset \mathbb{R}^3$, $e_0 = (1,0,0,0)^T, e_1 = (0,1,0,0)^T, e_2 = (0,0,1,0)^T$, $e_3 = (0,0,0,1)^T$ *form a basis in* $\mathbb{R}^4$. *Under the governing conditions*

(i) $e_ie_j + e_je_i = -2\delta_{ij}$ $(i,j = 1,2,3;\ i < j)$,

(ii) $e_0^2 = 1, e_ie_0 = e_0e_i = e_i$,

(iii) $e_1e_2 = e_3$,

$\{e_0, e_1, e_2, e_3\}$ *also form a basis in the algebra of real quaternions* $\mathbb{H}$. *The* DIRAC *operator has the form*

$$D := e_1\partial_1 + e_2\partial_2 + e_3\partial_3.$$

Together with $u = u_0e_0 + \underline{u}$, $\underline{u} = \sum_{i=1}^{3} u_ie_i$, *it follows that*

$$Du = -\operatorname{div}\underline{u} + \operatorname{rot}\underline{u} + \operatorname{grad} u_0,$$

where

$$\operatorname{div}\underline{u} := \sum_{i=1}^{3} \partial_i u_i, \quad \operatorname{grad} u_0 := \sum_{i=1}^{3} e_i\partial_i u_0, \quad \operatorname{rot}\underline{u} := \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

For (uD) *we get in an analogous way*

$$(uD) = -\operatorname{div}\underline{u} - \operatorname{rot}\underline{u} + \operatorname{grad} u_0.$$

Proposition 10 *Quaternionic regular functions can be also characterized by the following properties:*

(i) *quaternionic-regular functions are sourceless and in the case of* $u_0 =$ *const. also rotationless vector fields.*

(ii) $Cl_{3,0}$*-regular functions are rotationless and sourceless vector fields with* $u_0 =$ *const.*

Definition 11 *Let be $\overline{\partial}$ defined by*

$$\overline{\partial} := \partial_0 - D.$$

This operator is called adjungated CAUCHY–FUETER *operator.*
In $Cl_{p,q}$, $p+q=n$, the operator $\overline{D}$ defined by

$$\overline{D} = \sum_{i=1}^{n} \overline{e_i}\partial_i$$

is called adjoint DIRAC *operator.*

Assume now that G is a domain $\mathbb{R}^3$, or sometimes in $\mathbb{R}^4$.

Proposition 12 *Let be $u \in C^2(G, C\ell_{0,n})$. Then we have*

$$\overline{D}D\,u = D\overline{D}\,u = \Delta\,u,$$

where $\Delta u = \sum_{i=1}^{n} \partial_i^2\, u$.

Corollary 13 *Let $u \in \ker D \cap C^2(G, C\ell_{0,n}))$ then $u \in \ker\Delta \cap C^2(G, C\ell_{0,n})$, e.g.,* CLIFFORD *regular functions of this type are harmonic.*

Proposition 14 (Generalized LEIBNIZ rule). *Let be $u, v \in C^1(G, \mathbb{H})$ then*

$$D\,(uv) = (D\,u)v + \overline{u}\,D\,v + 2Sc(uD)v.$$

Proof. In [11] is contained the proof for the algebra of real quaternions. ■

Corollary 15 (Product rules in vector analysis). *Let $u, v \in C^1(G, \mathbb{H})$. Then in accordance with the definition of the vector field operations* grad, div, rot *we have the following relations:*

(i) grad $(u_0 v_0) =$ (grad u_0) $v_0 + u_0$ grad v_0,

(ii) div $(u_0\underline{v}) =$ (grad u_0) $\underline{v} + u_0$ div $\underline{v}$,

(iii) rot $(u_0\underline{v}) =$ grad $u_0 \times \underline{v} + u_0$ rot $\underline{v}$,

(iv) $\operatorname{grad}(\underline{u}, \underline{v}) = \underline{u} \times \operatorname{rot} \underline{v} + \underline{v} \times \operatorname{rot} \underline{u} + (\underline{u}, \operatorname{grad}) \underline{v} + (\underline{v}, \operatorname{grad}) \underline{u}$,

(v) $\operatorname{rot}(\underline{u} \times \underline{v}) = \underline{u} \operatorname{div} \underline{v} - \underline{v} \operatorname{div} \underline{u} + (\underline{v}, \operatorname{grad}) \underline{u} - (\underline{u}, \operatorname{grad}) \underline{v}$.

Proposition 16 (Multiplier problem). *Let $u \in C^1(G, \mathbb{H}) \cap \ker D$ be an arbitrarily given function. If for all $v \in \ker D \cap C^1(G, \mathbb{H})$ also $vu \in \ker D \cap C^1(G, \mathbb{H})$ then $u = const$.*

Proof. A straightforward calculation leads to the result. ■

Proposition 17 *The class of $\mathbb{H}$-regular functions does not contain the squares of each of its elements.*

Proof. Let $x = x_1 e_1 - x_2 e_2 \in \ker D \cap C^1(G, Cl_{3,0})$. Then we have

$$x^2 = (x_1 e_1 - x_2 e_2)(x_1 e_1 - x_2 e_2) = x_1^2 + x_2^2.$$

Hence $D\,x^2 = 2(x_1 \sigma_1 + x_2 \sigma_2) \neq 0$. ■

Remark 18 *For a so-called totally analytic variable z with $z(x) \in \ker D \cap C^1(\mathbb{R}^4, \mathbb{H})$ follows with necessity $z^k \in \ker D \cap C^1(R^4, \mathbb{H})$.*
We studied in [11] a generalization of an interpolation formula of LAGRANGE*'s type with quaternionic regular functions. The result is as follows: let $z(x) := \sum_{i=1}^{3} x_i d_i$, $z(a_i^{(j)}) := \sum_{i=1}^{3} a_i^{(j)} d_i$, $a_i^{(j)} \in \mathbb{H}$. We write in abbreviated form:*

$$z_j(x) := z(x) - z(a^{(j)}), \; z_{kj} := z(a^{(k)}) - z(a^{(j)})$$

and demand $z_{kj} \neq 0$ for $k \neq j$. A quaternionic regular interpolation function is then given by the following formula:

$$(L_n\, u)(x) = \sum_{k=1}^{n} \left[\frac{z_1(x)}{z_{k1}} \cdot \ldots \cdot \frac{z_{k-1}(x)}{z_{k(k-1)}} \frac{z_{k+1}(x)}{z_{k(k+1)}} \cdot \ldots \cdot \frac{z_n(x)}{z_{kn}} \right] u_k$$

with

(i) $(L_n u)^k \in \ker D \cap C^1(\mathbb{R}^4, \mathbb{H})$ $(k = 1, \ldots, n)$,

(ii) $(L_n u)(a^{(j)}) = u_j \ (j = 1, ..., n)$.

Proposition 19 *Let* $u \in \ker D \cap C^1(G, \mathbb{H})$, $u(x) \neq 0$ *then* $u^{-1} \in \ker D \cap C^1(G, \mathbb{H})$ *not in general.*

Proof. It is sufficient to give a counterexample. Taking $u(x) = x_1 \mathbf{e}_1 - x_2 e_2 \in \ker D \cap C^1(G, \mathbb{H})$ then for $0 \notin G$ we obtain

$$u^{-1} = \frac{x_2 e_2 - x_1 e_1}{x_1^2 + x_2^2}$$

and one can easily verify that

$$D\, u^{-1} = \frac{2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}\sigma_0 + i\, \frac{4x_1 x_2}{(x_1^2 + x_2^2)^2}\sigma_3 \neq 0. \quad \blacksquare$$

Omission of a LEIBNIZ rule in the classical sense leads to considerable difficulties in the construction of systems of regular functions. We will formulate some easy principles to produce quaternionic regular functions.

3.1.2 Quaternionic regular polynomials

Let $\{e_0, e_1, e_2, e_3\}$ be a basis in $\mathbb{H}$ and be $x = \sum_{i=0}^{3} x_i e_i$. We assume as before that $e_0^2 = e_0$, $e_i e_j + e_j e_i = -2\delta_{ij}$. We consider the new variables

$$z_k := x_0 e_k - x_k e_0 \ (k = 1, 2, 3).$$

For $\partial = \sum_{i=0}^{3} e_i \partial_i$ one has $\partial z_k = 0$. Unfortunately, the product of two of such variables is not quaternionic regular, e.g. $\partial\, z_k z_l \neq 0$ for any k and l. Nevertheless, one can find homogeneous quaternionic regular polynomials of any degree by symmetrization. These polynomials are given by the formula

$$P_m(x) = \frac{1}{m!} \sum_{(\mu_1, ..., \mu_m)} z_{\mu_1} \cdot ... \cdot z_{\mu_m},$$

where $(\mu_1, ..., \mu_m)$ covers all distinguishable permutations of $\{1, ..., m\}$ and $(\mu_1, ..., \mu_m)$ is an arbitrary combination of m elements out of repetitions of the set $\{1, 2, 3\}$. For $m = 0$ we fix $P_0(x) = e_0$. This construction was first made by R. DELANGHE and is also valid in real Clifford algebras. Comprehensive results on CLIFFORD regular polynoms can be found in [6] and also in [7].

3.1.3 Regular singularity functions

Moreover, we will describe an important class of quaternionic regular functions which can be used in collocation methods for boundary value problems of partial differential equations. Let $\underline{x}^{(k)}$ be a system of points outside a given domain $G \subset \mathbb{R}^3$. Then there exists a system of quaternionic regular functions $\{\phi_k\}$ which are defined by

$$\phi_k(\underline{x}) = \frac{\underline{x} - \underline{x}^{(k)}}{|\underline{x} - \underline{x}^{(k)}|^3}.$$

Under some conditions completeness and advantageously numerical properties are possible to obtain. This is a system of singularity functions which are used in calculations.

3.1.4 Regular functions from harmonic functions

Let G be an open set in $\mathbb{R}^n$ and star-shaped with respect to the origin and $u_0 : G \longmapsto \mathbb{R}$ is harmonic in G then

$$u(x) = u_0(x) - \left\{ \int_0^1 t^{n-2}(D\,u_0)(tx)x\,dt - \left(\int_0^1 t^{n-2}(D\,u_0)(tx)x\,dt\right)_0 \right\}$$

is $C\ell_{0,n}$- regular in G.

3.2 On the spherical Dirac operator

In this part we will put together useful properties of operators which result of a suitable decomposition of the DIRAC operator. Let x be the paravector $x = x_0 + \underline{x}$ with $\underline{x} = \sum_{i=1}^{n} x_i e_i$. Setting $\omega_i = x_i/|\underline{x}|$ $(i = 1, ..., n)$ and $\omega = \sum_{i=1}^{n} w_i e_i$. Then $\underline{x} = |\underline{x}|\omega$, where $\omega^2 = -1$. We consider the so-called CAUCHY-FUETER operator $\partial = \partial_0 + D$. Here denotes $\partial_0 := \partial/\partial x_0$ and $D = \sum_{i=1}^{n} e_i \partial_i$ with $\partial_i := \partial/\partial x_i$. Introducing the denotations

$$L := \sum_{i=1}^{n} e_i L_i(x) \qquad \text{with} \qquad L_i(x) = |\underline{x}|\partial_i - x_i \ell_\omega$$

and $\ell_\omega := \sum_{i=1}^{n} \omega_i \partial_i$ we obtain $D = \frac{1}{|\underline{x}|} L + \omega \ell_\omega$.

Proposition 20 *We have for $j = 1, ..., n$*

$$
\begin{array}{ll}
\text{(i)} & \ell_\omega \underline{x} = \omega, \quad \textit{(projection onto the unit sphere)} \\
\text{(ii)} & \ell_\omega \omega = 0 \\
\text{(iii)} & \textit{Let } f = f(|\underline{x}|), \textit{ then } \ \ell_\omega f = d/d_{|\underline{x}|} f =: f' \\
\text{(iv)} & \partial_j \omega_k = \dfrac{1}{|\underline{x}|} \left[\delta_{jk} - \omega_k \omega_j \right], \\
\text{(v)} & L_j \omega_k = \delta_{jk} - \omega_k \omega_j = |\underline{x}| \partial_j \omega_k .
\end{array}
$$

Proof. The following straightforward calculations lead to the result:

$$
\text{(i)} \quad \ell_\omega x_j = \sum_{i=1}^{n} \omega_i \partial_i (x_j) = \sum_{i=1}^{n} \omega_i \delta_{ij} = \omega_j \quad (j = 1, ..., n)
$$

$$
\text{(ii)} \quad \ell_\omega \omega_j = \ell_\omega \left(\frac{x_j}{|\underline{x}|} \right) = \frac{(\ell_\omega x_j)|\underline{x}| - \ell_\omega |\underline{x}| x_j}{|\underline{x}|^2} = \frac{\omega_j |\underline{x}| - x_j}{|\underline{x}|^2} = 0
$$

$(j = 1, ..., n)$.

$$
\text{(iii)} \quad \ell_\omega f = \sum_{i=1}^{n} \omega \partial_i f = \sum_{i=1}^{n} \omega_i f' \partial_i |\underline{x}| = \sum_{i=1}^{n} \omega_i^2 f' = f'
$$

Due to (iv) it is easy to see, that

$$
\partial_i \omega_k = \partial_j \frac{x_k}{|\underline{x}|} = \frac{(\partial_j x_k)|x| - \partial_j |\underline{x}| x_k}{|\underline{x}|^2} = \frac{\delta_{jk}|x| - \omega_j x_k}{|\underline{x}|^2} = \frac{1}{|\underline{x}|} (\delta_{jk} - \omega_j \omega_k) .
$$

Relation (v) we get from

$$
L_j \omega_k = |\underline{x}| \partial_j \omega_k - x_j \ell_\omega \omega_k = \delta_{jk} - \omega_k \omega_j \quad \blacksquare
$$

Corollary 21 *Let $f \in C^1(\mathbb{R})$ and $f = f(|\underline{x}|)$, then $Lf = 0$.*

Proof. By definition we get for $j = 1, ..., n$

$$
L_j(|\underline{x}|) = |\underline{x}| \partial_j |\underline{x}| - \omega_j \ell_x |\underline{x}| = |\underline{x}| \omega_j - \omega_j |\underline{x}| \ell_\omega |\underline{x}| = |\underline{x}| \omega_j - \omega_j |\underline{x}| = 0 .
$$

Hence, it follows

$$L|\underline{x}| = \sum_{i=1}^{n} e_i L_i |\underline{x}| = 0 \,.$$

Furthermore, we obtain

$$\begin{aligned} L_j f(|\underline{x}|) &= |\underline{x}|\partial_j f - \omega_j \sum_{i=1}^{n} x_i \partial_i f = |\underline{x}| \frac{df}{d|\underline{x}|} \partial_j |\underline{x}| - \omega_j \sum_{i=1}^{n} x_i \frac{df}{d|\underline{x}|} \partial_i |\underline{x}| \\ &= \frac{df}{d|\underline{x}|} (|\underline{x}|\partial_j |\underline{x}| - x_j \ell_\omega |\underline{x}|) = \frac{df}{d|\underline{x}|} L_j |\underline{x}| = 0 \end{aligned}$$

and therefore $Lf = 0$. ■

Proposition 22 *It holds*

$$\begin{aligned} &\text{(i)} \quad L\omega = \omega L + (1-n) \\ &\text{(ii)} \quad \omega L = -\underline{x} \wedge D = \sum_{i<j} e_i e_j (x_i \partial_j - x_j \partial_i). \end{aligned}$$

where the operator $\omega L =: \Gamma$ is just the spherical DIRAC *operator.*

Proof. By definition we have

$$\begin{aligned} \Gamma &= \underline{x}D + |\underline{x}|\ell_\omega = -\sum_{i=1}^{n} x_i \partial_i + \sum_{i \neq j} x_i \partial_j e_i e_j + \sum_{i=1}^{n} x_i \partial_i \\ & \sum_{i<j} x_i \partial_j e_i e_j + \sum_{i>j} x_i \partial_j e_i e_j = \sum_{i<j} x_i \partial_j e_i e_j - \sum_{i>j} x_i \partial_j e_j e_i \\ &= \sum_{i<j} x_i \partial_j e_i e_j - \sum_{j>i} x_j \partial_i e_i e_j = \sum_{i<j} (x_i \partial_j - x_j \partial_i) e_i e_j. \end{aligned}$$

■

Theorem 23 *[5] The following formula is valid:*

$$\Gamma\omega = (n-1)\omega.$$

Proof. By definition

$$\begin{aligned} \Gamma\omega &= (\omega L)\omega = \sum_{i,j,k} \omega_i e_i e_j L_j (\omega_k e_k) == \sum_{i,j,k} \omega_i e_i e_j \{\delta_{kj} - \omega_j \omega_k\} e_k \\ &= \sum_{k=j=1}^{n} \sum_{i=1}^{n} \omega_i e_i - \sum_{i,j,k} \omega_i \omega_j \omega_k \, e_i e_j e_k = +n\omega - \sum_{i,j,k} \omega_i \omega_j \omega_k \, e_i e_j e_k \;. \end{aligned}$$

Finally, we find

$$\begin{aligned}\sum_{i,j,k} \omega_i\omega_j\omega_k\, e_i e_j e_k &= \sum_{\substack{k=1\\ i<j}}^{n} (\omega_i\omega_j - \omega_j\omega_i) e_k e_i e_j - \sum_{k=1}^{n}\sum_{i=j=1}^{n} \omega_i^2\omega_k e_i^2 e_k \\ &= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} \omega_i^2\omega_k e_k\right) = \omega\,. \qquad \blacksquare\end{aligned}$$

Corollary 24 *[5] The* DIRAC *operator permits the representation*

$$D = \omega\left(\ell_\omega + \frac{1}{|\underline{x}|}\Gamma\right)$$

Proof. We know that $\omega L = \underline{x}D + |\underline{x}|\ell_\omega$. Then

$$\underline{x}\omega L = -|\underline{x}|^2 D + \underline{x}|\underline{x}|\ell_\omega$$

and

$$\begin{aligned} D &= -\frac{\underline{x}}{|\underline{x}|^2}\omega L + \frac{\underline{x}}{|\underline{x}|}\ell_\omega = \frac{1}{|\underline{x}|}(-L + \underline{x}\ell_\omega) \\ &= \omega\ell_\omega + \frac{w}{|\underline{x}|}\omega L = \omega\left(\ell_\omega + \frac{1}{|\underline{x}|}\Gamma\right) \qquad \blacksquare \end{aligned}$$

3.3 Teodorescu transform

This part is dedicated to the consideration of a very special weakly singular integral operator, which will play a key role in the whole theory.

Let G be a domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial G = \Gamma$. We will use here the following norms:

$$\|u\|_\infty := \|u\|_{L_\infty(G)} = \operatorname{vrai}\max_{x\in G} |u(x)|,$$

where $|\cdot|$ is just the CLIFFORD operator norm . Furthermore, for $1 \le p < \infty$ we define

$$\|u\|_p := \|u\|_{L_p(G)} = \left(\int_G |u|^p dx\right)^{1/p}$$

$$\|u\|_{p,\alpha} := \left(\int_G [|u||x|^\alpha]^p dx\right)^{1/p}$$

We abbreviate

$$\|u\|_{W_p^1(G)} =: \|u\|_{p,1}.$$

By $|G|$ we denote the volume of G.

Definition 25 *Let $u \in C(G))$. Then the linear integral operator*

$$(T_G u)(x) = -\int_G e(x-y)u(y)dy \tag{3}$$

with

$$e(x) = \frac{1}{\sigma_n}\frac{\overline{\omega}(x)}{|x|^{n-1}},\ \omega(x) = \frac{x}{|x|},\ x = \underline{x} = \sum_{i=1}^{n} x_i e_i$$

is called TEODORESCU *transform over G.*

We remark that T_G corresponds to the known T-operator from the one variable complex analysis. In the case $n = 2$ it coincides with the complex T-operator up to a constant factor. First in this section we consider the existence of this integral and some elementary properties.

Proposition 26 *Let $u \in C(\overline{G})$. Then the integral (3) exists for all $x \in \mathbb{R}^n$ and we have*

$$\|T_G u\|_C \le \frac{1}{\sigma_n}\max_{x\in\overline{G}} \int_G \frac{1}{|x-y|^{n-1}}dy\|u\|_C.$$

Proof. Because of the the uniform convergence of $\int_G 1/(|x-y|^{n-1})dy$ it is continuous according to y and, therefore, we can take the maximum over $\overline{G}$. ■

This means that T_G is a continuous operator in $C(\overline{G})$.

Corollary 27 *For $|x| \to \infty$ it follows that $|(T_G u)(x)| \to 0$. Furthermore, we have $(T_G u)(x) \in C^\infty(\mathbb{R}^n \setminus \overline{G})$.*

This property will be essential for the investigation of boundary value problems.

Proposition 28 *Let $u \in L_1(G)$. Then $(T_G u)(x)$ exists almost everywhere on $\mathbb{R}^n$ and belongs to $L_q(G')$ for q with $1 < q < n/(n-1)$ and any domain $G' \subset \mathbb{R}^n$.*

Proof. Let $v \in L_p(G), p > n$. Then we get by HÖLDER's inequality

$$V(x) = \int_G \frac{|v(y)|}{|x-y|^{n-1}} dy \leq \int_G \frac{1}{|x-y|^{(n-1)q}} dy \|v\|_p.$$

Clearly, $(n-1)q < n$. Thus $V(x)$ is defined by a uniformly convergent integral and therefore continuously. Then $V(x)u(x) \in L_1(G)$. Using FUBINI's theorem we obtain the identity

$$\int_G |u(y)| V(y) dy = \int_G |v(y)| U(y) dy \tag{4}$$

with

$$U(x) = \int_G \frac{|u(y)|}{|x-y|^{n-1}} dy.$$

Hence we conclude $U(x) \in L_q(G)$, where $1/p + 1/q = 1$. In this way we get $(T_G u)(x) \in L_q(G)$. ■

Now, we will prove continuity results of T_G in the scale of SOBOLEV spaces.

Proposition 29 *Let $u \in C_0(\mathbb{R}^n)$ then*

$$\partial_k (T_G u)(x) = -\frac{1}{\sigma_n} \int_G \partial_{k,x} e(x-y) u(y) dy + \bar{e}_k \frac{u(x)}{n}.$$

Proof. For the proof we will refer to our book ([11]). ■

Theorem 30 *The operator*

$$\partial_k T : L_p(\mathbb{R}^n) \to L_p(R^n)$$

is continuous and

$$\|\partial_k T_G\|_{L(L_p(R^n))} \leq \left(C\sigma_n^{-\frac{1}{p}} + \frac{1}{n}\right) \quad (1 < p < \infty).$$

and

$$T_G : L_p(G) \to W_p^1(G)$$

is continuous.

By the help of SOBOLEV embedding theorems it is easy to obtain the following results on the continuity of T_G between other important spaces.

Corollary 31
(i) $T_G : [L_p(G), C^{0,\frac{p-n}{p}}(G)]$ *is continuous for* $p > n$.
(ii) *Let* $u \in L_p(G), 1 < p < n$. *Then for all* $r < \frac{np}{n-q}$ *we have that* $T_G : L_p(G) \to L_r(G)$ *is compact. For all* $\varepsilon > 0$ *there exists* $\delta > 0$ *with*

$$\|T_G u(x) - T_G u(x')\|_r \leq \varepsilon \|u\|_p$$

for $|x - x'| < \delta$. *Especially, we have that* T_G *is compact in* $L_2(G)$.

Remark 32 Also in the case of unbounded domains, continuity properties of the operator T_G can be proved. Unfortunately, the proofs require additional expertise. We will only formulate some results and refer to the literature. The TEODORESCU transform T_G is in the case of $p < n$ a continuous mapping from $L_p(G)$ into $L_p^{-1}(G)$. For

$$1 - \frac{n}{p} < \alpha < n\left(1 - \frac{1}{p}\right)$$

the transform T_G is continuous from $L_p^{\alpha}(G)$ into $L_p^{\alpha-1}(G)$. Corresponding considerations for different kinds of unbounded domains can be found in [14].

3.4 Borel-Pompeiu' s formula and its consequences

In this part we will concern ourselves with $C\ell_{0,n}$-versions of classical theorems of the function theory of one complex variable. In the centre of the considerations will be the generalization of BOREL–POMPEIU's formula.

Proposition 33 *Let $u \in C^1(G) \cap C(\overline{G},)$. Then*

$$\int_G (Du)(y)dy = \int_\Gamma n(y)u(y)d\Gamma_y,$$

where $n(y) = \sum_{i=1}^n n_i(y)e_i$ is the unit vector of the outward pointing normal at y.

Proof. Using GAUSS' formula we obtain

$$\begin{aligned} \int_G (Du)(y)dy &= \sum_{i=1}^n \sum_A e_i e_A \int_G \partial_i u_A(y)dy \\ &= \sum_{i=1}^n \sum_A e_i e_A \int_\Gamma n_i(y)u_A(y)d\Gamma_y \\ &= \int_\Gamma n(y)u(y)d\Gamma_y. \quad \blacksquare \end{aligned}$$

Corollary 34 (CAUCHY's integral theorem.)
Let $u \in C^1(G) \cap C(\overline{G})$ and $u \in \ker D$ ($Du = 0$!) then

$$\int_\gamma n(y)u(y)d\Gamma_y = 0,$$

where γ is an arbitrary surface with $\gamma \subset G$.

Let

$$E(x) = \frac{1}{\sigma_n}\frac{1}{(2-n)}|x|^{-(n-2)} \qquad (n > 2) \quad \text{with} \quad \sigma_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

then we obtain $\Delta E(x) = 0$ $(x \neq 0)$ (σ_n is again the surface area of the unit sphere in $\mathbb{R}^n$). It can be easily seen that the DIRAC operator $D = \sum_{i=1}^n e_i \partial_i$ yields

$$e(x) = \overline{D}E(x) = \frac{-x}{\sigma_n |x|^n}.$$

Proposition 35 *Let $u \in C^2(G) \cap C(\overline{G})$. Then*

$$(DT_G u)(x) = \begin{cases} u(x) & \text{in } G, \\ 0 & \text{in } \mathbb{R}^n \setminus \overline{G}. \end{cases}$$

Proof. Firstly, we note that in case of $\partial_{ij} = \partial_{ji}$

$$\begin{aligned} D\overline{D} &= \sum_{i=1}^n \sum_{j=1}^n e_i \overline{e}_j \partial_{ij} = -\sum_{i=1}^n \sum_{j=1}^n e_i e_j \partial_{ij} \\ &= -\sum_{i=j} e_i e_j \partial_{ij} - \sum_{i>j} (e_i e_j - e_j e_i) \partial_{ij} = \Delta. \end{aligned}$$

Furthermore, we have for $x \in G$

$$\begin{aligned} (DT_G u)(x) &= D\Big(-\int_G e(x-y) u(y) dy \Big) = D\Big(-\int_G \overline{D}E(x-y) u(y) dy \Big) \\ &= \Delta\Big(-\int_G \frac{1}{\sigma_n (2-n)} \frac{1}{|x-y|^{n-2}} u(y) dy \Big). \end{aligned}$$

The latter integral is just the volume potential which solves the POISSON equation. Hence, $(DT_G u)(x) = u(x)$.

For $\Delta E(x) = 0$ in $\mathbb{R}^n \setminus \overline{G}$ it follows our assertion. ■

Theorem 36 *Let $G \subset \mathbb{R}^n$ be a domain which is bounded by a piecewise* LIAPUNOV *surface Γ. Then for each $u \in C^1(G) \cap C(\overline{G})$*

$$\int_\Gamma e(x-y) n(y) u(y) d\Gamma_y - \int_G e(x-y)(Du)(y) dy = \left\{ \begin{array}{cc} u(x) & , \; x \in G \\ 0 & , \; x \notin \overline{G} \end{array} \right\}, \quad (5)$$

where $n(y)$ denotes again the unit vector of the outward pointing normal at Γ in point y.

Proof. Let x be an interior point of G, $B_\varepsilon(x)$ a ball centered at x with the radius ε $(\varepsilon > 0)$. Further, let $G_\varepsilon := G \setminus B_\varepsilon(x)$. Applying Proposition 35 and choosing for $u = e(x)$ then it follows that

$$\int\limits_{G_\varepsilon} e(x-y)(Dv)(y)dy =$$

$$= \int\limits_{\Gamma} e(x-y)n(y)v(y)d\Gamma_y - \int\limits_{S_\varepsilon} e(x-y)n(y)v(y)dS_{\varepsilon_y}. \tag{6}$$

On S_ε $(y \in S_\varepsilon)$ holds

$$e(x - yn(y) = \frac{1}{\sigma_n}\frac{\overline{x}-\overline{y}}{|x-y|^n}\frac{y-x}{|x-y|} = \frac{1}{\sigma_n}\frac{1}{|x-y|^{n-1}}.$$

It remains to show that the latter integral converges to $v(x)$ if ε tends to zero. We have

$$\lim_{\varepsilon\to 0}\int\limits_{S_\varepsilon} e(x-y)n(y)v(y)dy = \lim_{\varepsilon\to 0}\left(\frac{1}{\sigma_n\varepsilon^{n-1}}\int\limits_{S_\varepsilon(x)} v(y)dy\right) = v(x). \tag{7}$$

For $\varepsilon \to 0$ the first integral in (8) exists and BOREL-POMPEIU's formula for interior points is shown. For the second part we have to repeat the proof without excluding a neighbourhood of x. ■

Remark 37 Using STOKES' Theorem 36 can be proved on compact n-dimensional, oriented C^∞-manifolds which are lying in a domain of $\mathbb{R}^n$ (cf. [8], [5]).

Definition 38 *Let $u \in C^1(G) \cap C(\overline{G})$. The operator F_Γ defined by*

$$(F_\Gamma u)(x) := \int\limits_{\Gamma} e(x-y)n(y)u(y)d\Gamma_y$$

is called CAUCHY-BIZADSE *operator.*

Remark 39 In this new notation formula (5) has the representation

$$(F_\Gamma u)(x) + (T_G Du)(x) = \left\{ \begin{array}{ll} u(x) & , \ x \in G \\ 0 & , \ x \in \mathbb{R}^n \setminus \overline{G}. \end{array} \right\}.$$

Proof. (Theorem 36) Let x be an interior point of G, $B_\varepsilon(x)$ a ball centered at x with the radius ε $(\varepsilon > 0)$. Further, let $G_\varepsilon := G \setminus B_\varepsilon(x)$. Applying Proposition 35 and choosing for $u = e(x)$ then it follows that

$$\int_{G_\varepsilon} e(x-y)(Dv)(y)dy =$$
$$= \int_\Gamma e(x-y)n(y)v(y)d\Gamma_y - \int_{S_\varepsilon} e(x-y)n(y)v(y)dS_{\varepsilon_y}. \tag{8}$$

On S_ε $(y \in S_\varepsilon)$ holds

$$e(x-y)n(y) = \frac{1}{\sigma_n}\frac{\overline{x}-\overline{y}}{|x-y|^n}\,\frac{y-x}{|x-y|} = \frac{1}{\sigma_n}\,\frac{1}{|x-y|^{n-1}}.$$

It remains to show that the latter integral converges to $v(x)$ if ε tends to zero. We have

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon} e(x-y)n(y)v(y)dy = \lim_{\varepsilon\to 0}\left(\frac{1}{\sigma_n\varepsilon^{n-1}}\int_{S_\varepsilon(x)} v(y)dy\right) = v(x).$$

For $\varepsilon \to 0$ the first integral in (8) exists and BOREL-POMPEIU's formula for interior points is shown. For the second part we have to repeat the proof without excluding a neighbourhood of x. ■

Remark 40 BOREL–POMPEIU's formula is also valid in the case of multiply connected domains. Suppose G_i $(i = 0, 1, ..., n)$ be simply connected bounded domains in $\mathbb{R}^n$ with smooth boundaries Γ_i $(i = 0, 1, ..., n)$ which fulfil the relations

(i) $G_0 \supset \bigcup_{i=1}^{n} \overline{G}_i = K$ and (ii) $\overline{G}_i \cap \overline{G}_j = \emptyset$ $(i \neq j)$. Then

$$(F_{\Gamma_0}u)(x) - \sum_{i=1}^{n}(F_{\Gamma_i}u)(x) + (T_{G_0\setminus K}Du)(x) = \left\{\begin{array}{ll} u(x) & \text{in } G_0 \setminus K \\ 0 & \text{in } \mathbb{R}^n \setminus [G_0 \setminus K] \end{array}\right\}$$

for every function in $C^1(G_0 \setminus K) \cap C(\overline{G \setminus K})$.

Now it is easy to transfer many of the results of the theory of one complex variable which are connected with CAUCHY's integral theorem.

Theorem 41 (CAUCHY's integral formula). *Let $G \subset \mathbb{R}^n$ be a domain with a piecewise* LIAPUNOV *boundary. Furthermore, let $u \in \ker D$, then*

$$(F_\Gamma u)(x) = \left\{ \begin{array}{ll} u(x) & \text{in } G \\ 0 & \text{in } \mathbb{R}^n \setminus \overline{G} \end{array} \right\} \tag{9}$$

holds.

Proof. Because of $Du = 0$ formula (9) holds. ■

Corollary 42 *Let $u \in \ker D$. Then u has partial derivatives $\partial_{i_1 i_2 \ldots i_k}$ $(i_l \in \{1, ..., n\}; l \in \{1, ..., k\})$ of any order.*

Proof. The kernel function $e(x-y)$ has for $x \neq y$ partial derivatives of any order. ■

Corollary 43 (Mean-value theorem). *Let $B_r(x) \subset \mathbb{R}^n$ be a ball centred at x of radius r. For any $u \in \ker D$*

$$u(x) = \frac{n}{\sigma_n r^n} \int\limits_{B_r(x)} u(y)dy \ .$$

Note that $\sigma_n r^n/n$ is just the volume of the ball $B_r(x)$.

Proof. The proof is an immediate consequence of Cauchy's formula. ■

Corollary 44 (Mean-value formula). *Let $B_r(x)$ again be a ball centred at x of radius r. Then*

$$u(x) = \frac{1}{\sigma_n r^{n-1}} \int\limits_{S_r(x)} u(y)dS_{r_y},$$

where $u \in (\ker D)(B_r(x)) \cap C(\overline{B}_r(x))$.

Proof. Applying CAUCHY's integral formula then

$$u(x) = \frac{1}{\sigma_n} \int\limits_{S_r(x)} \frac{\overline{y} - \overline{x}}{|y-x|^n} \frac{y-x}{|y-x|} u(y)dS_{r_y} = \frac{1}{\sigma_n r^{n-1}} \int\limits_{S_r(x)} u(y)dS_{r_y}. \quad ■$$

Theorem 45 (Maximum modulus theorem). *Let $G \subset \mathbb{R}^n$ be a connected bounded domain and $u \in (\ker D)(G) \cap C(\overline{G})$ with $\overline{G} = G \cup \Gamma$, $\Gamma = \partial G$. If there exists some $z \in G$ with*

$$|u(x)| \leq |u(z)| \qquad (\forall\, x \in G)$$

it is necessary and sufficient that u is constant.

Proof. For $z \in G$ there exists a ball $B_\varepsilon(z) \subset G$. Applying the mean value formula we obtain

$$u(z) = \frac{1}{\sigma_n \varepsilon^{n-1}} \int\limits_{S_\varepsilon} u(x) dS_{\varepsilon,x}. \tag{10}$$

Assume now that $|u(x)| \leq |u(z)|$ and u is not constant in $B_\varepsilon(z)$. Then there exists a decomposition of S_ε into two sets

$$S'_\varepsilon = \{x \in S_\varepsilon : |u(x)| = |u(z)|\}, \quad S''_\varepsilon = \{x \in S_\varepsilon : |u(x)| < |u(z)|\}.$$

Formula (10) yields

$$|u(z)| < \frac{1}{\sigma_n \varepsilon^{n-1}} \Big[\int\limits_{S'_\varepsilon} |u(x)| dS'_{\varepsilon,x} + \int\limits_{S''_\varepsilon} |u(z)| dS''_{\varepsilon,x} \Big] = \frac{|S'_\varepsilon| + |S''_\varepsilon|}{\sigma_n \varepsilon^{n-1}} |u(z)| = |u(z)|.$$

This is a contradiction. Hence on S_ε we have $|u(x)| = |u(z)|$.

Choosing now smaller balls $B_{\varepsilon'}(z)$ $(\varepsilon' < \varepsilon)$ we obtain the same. Therefore, $|u(x)| = |u(z)|$ in the whole ball $B_\varepsilon(z)$. Since G is connected we get immediately $|u(x)| = \text{const}$ on G and for the continuity of $|u(x)|$ such also on $\overline{G}$. It remains to show that $u(x)$ is also a constant. From $|u|^2 = \text{const.}$ we conclude

$$0 = D|u|^2 = D\Delta(\sum_A u_A^2) = 2 \sum_A \sum_{j=1}^n (\partial_j u_A) u_A e_j$$

and also

$$0 = \Delta |u|^2 = \overline{D} D |u|^2 = 2 \sum_A \sum_{i=1}^n \sum_{j=1}^n \left[(\partial_{ij} u_A) u_A e_i e_j + (\partial_j u_A)(\partial_i u_A) e_i e_j \right].$$

Since the components of a CLIFFORD left regular function are harmonic one gets

$$0 = \sum_A \sum_{i=1}^{n} (\partial_i u_A)^2$$

and consequently $\partial_i u_A = 0$ $(i = 1, ..., n; A)$. ■

Corollary 46 *Let $G \subset \mathbb{R}^n$ be a connected and bounded domain and $u \in (\ker D)(G) \cap C(\overline{G})$ then*

$$\sup_{x \in \overline{G}} |u(x)| = \sup_{x \in \partial G} |u(x)|.$$

Theorem 47 (LIOUVILLE's theorem). *Let $u \in (\ker D)(\mathbb{R}^n)$. If $|u(x)| \leq M$ $(x \in \mathbb{R}^n)$ then u is a constant.*

Proof. The proof is left to the reader. ■

Theorem 48 (MORERA's theorem). *Let $u \in C^1(G)$, $Du \in L_r(G)$, $r > n$. If for all balls $B_r(x)$ $(r > 0,\ x \in G)$*

$$\int_{B_r(x)} n(y) u(y) d\gamma(y) = 0\,,$$

then $Du = 0$ in G.

Proof. Let $x \in G$ be an arbitrary point (B_{r_k}) $k \in N$ a regular sequence of balls which is contracting to $x \in G$. We have $x \in \bigcap_k B_{r_k}$. For the validity of LEBESGUE's theorem we obtain for each $v \in L_r(G)(r > n)$,

$$\lim_{k \to \infty} \frac{1}{|B_{r_k}|} \int_{B_{r_k}} v(y) dB_{r_k}(y) =: \tilde{v}(x)$$

and $v = \tilde{v}$ in $L_r(G)$. Substituting $v = Du$ it follows by Proposition 33 that

$$\int_{B_{r_k}} (Du)(y) dy = \int_{S_{r_k}} n(y) u(y) dS_{r_{k,y}}$$

and also

$$\frac{1}{|B_{r_k}|}\int_{B_{r_k}}(Du)(y)dy = 0 \qquad (k = 0, 1, 2, \ldots)$$

and, therefore, $(Du)(x) = 0$ almost everywhere on G. ■

Theorem 49 *Let $u \in (\ker D)(\mathbb{R}^n \setminus \overline{G}) \cap C(\mathbb{R}^n \setminus \overline{G})$. Suppose $\lim\limits_{|x|\to\infty} u(x) = u(\infty)$. Then*

$$(F_\Gamma u)(x) = \begin{cases} u(\infty) - u(x) & , \; x \in \mathbb{R}^n \setminus \overline{G} \\ u(\infty) & , \; x \in G \end{cases}.$$

Proof. Let $B_R(x)$ be such a ball that $B_R(x) \supset G$. Write briefly $s = (\omega_x, \omega_y) = (x/|x|, y/|y|)$. On account of

$$\frac{1}{|x-y|^{n-1}} = \Big(\sum_{k=0}^{\infty} C_k^{\frac{n-3}{2}}(s)\left|\frac{x}{y}\right|^k\Big)\frac{1}{|y|^{n-1}} \qquad (|x| < |y|)$$

and CAUCHY's integral formula for multiply connected domains we get

$$\frac{1}{\sigma_n}\int_{S_R}\frac{x-y}{|x-y|^n}n(y)u(y)dS_{R,y} - \frac{1}{\sigma_n}\int_{\Gamma}\frac{x-y}{|x-y|^n}n(y)u(y)d\Gamma_y$$

$$= \begin{cases} 0 & , \; x \in \mathbb{R}^n \setminus \overline{(B_R \setminus \overline{G})} \\ u(x) & , \; x \in B_R \setminus \overline{G} \end{cases}.$$

The first integral transforms into

$$\frac{1}{\sigma_n}\int_{S_R}\frac{x-y}{|x-y|^n}\frac{y-x}{|y-x|}u(y)dS_{R_y} = \frac{1}{\sigma_n}\int_{S_R}\frac{1}{|x-y|^{n-1}}u(y)dS_{R_y} =$$

$$= \sum_{k=0}^{\infty}\frac{1}{\sigma_n}\int_{S^n}\frac{C_k^{\frac{n-3}{2}}(s)}{R^k}u(y)dS_y^n|x|^k.$$

For $R \to \infty$ the right-hand side converges to

$$\frac{C_0^{\frac{n-3}{2}}(s)}{\sigma_n}\int_{S^n}u(\infty)dS_y^n = u(\infty) \quad (C_0^\mu(s) \equiv 1),$$

which provides the result. ■

3.5 Plemelj-Sokhotzkij-type projections

The operator $P_\Gamma := \frac{1}{2}(I + S_\Gamma^\delta)$ denotes the *Plemelj-Sokhotzkij-type projection* onto the space of all $\mathbb{C}\ell_{0,n}$-valued functions which may be $C\ell_{0,n}$-regular extended into the domain G. $Q_\Gamma := \frac{1}{2}(I - S_\Gamma^\delta)$ denotes the *Plemelj-Sokhotzki-type projection* onto the space of all $C\ell_{0,n}$-valued functions which can be $\mathbb{C}\ell_{0,n}$-regular extended into the domain $R^n \setminus \overline{G}$ and vanish at infinity. The operator

$$(S_\Gamma^\delta u) := \frac{4\pi - 2\delta(x)}{4\pi} u(x) + (S_\Gamma u)(x) \qquad 0 < \delta \leq 4\pi)$$

where δ is the space angle taken from outside at the point x. It is easy to see that

$$(S_\Gamma^\delta)^2 = I.$$

Note that Plemelj-Sokhotzkij-type projections coincide in case of the unit ball with the well-known Szegö projections. We have the following statements :

Proposition 50 *[11] Let $u \in C^1(G) \cap C(\overline{G})$. Then we have the formulas*

(i) $(F_\Gamma u)(x) + T_G Du(x) = \left\{ \begin{array}{ll} u(x) & , x \in G \text{ (Borel–Pompeiu formula)} \\ 0 & , x \in R^n \setminus \overline{G} \end{array} \right\}$

(ii) $(DT_G u)(x) = \left\{ \begin{array}{ll} u(x) & , \quad x \in G \\ 0 & , \quad x \in R^n \setminus \overline{G} \end{array} \right\}$

(iii) $(DF_\Gamma)u(x) = 0$ *in* $G \cup (R^n \setminus \overline{G})$

Theorem 51 *[11](*Plemelj–Sokhotzkij's formulas*). Let $u \in C^{0,\alpha}(G)$, $0 < \alpha < 1$. Then we have*

(i) $\lim\limits_{\substack{x \to \xi \in \Gamma \\ x \in G}} F_\Gamma u)(x) = (P_\Gamma u)(\xi)$

(ii) $\lim\limits_{\substack{x \to \xi \in \Gamma \\ x \in R^n \setminus \overline{G}}} F_\Gamma u)(x) = (-Q_\Gamma u)(\xi)$

for any $\xi \in \Gamma$.

Corollary 52 *[11] Let $u \in C^{0,\alpha}(\Gamma)$. Then the relations*

(i) $(S_\Gamma^2 u)(\xi) = u(\xi)$ *(ii)* $(F_\Gamma P_\Gamma u)(\xi) = F_\Gamma u(\xi)$

(iii) $(P_\Gamma^2 u)(\xi) = (P_\Gamma u)(\xi)$ *(iv)* $(Q_\Gamma^2 u)(\xi) = (Q_\Gamma u)(\xi)$

are valid for any $\xi \in \Gamma$.

3.6 Hilbert space decomposition

Let us now consider the Hilbert space $L_2(G)$ with an inner product $(u,v) = \int_G \overline{u}v\,dx \in Cl_{0,n}$.

Theorem 53 *The Hilbert space $L_2(G)$ allows an orthogonal decomposition*

$$L_2(G) = [\ker D \cap L_2(G)] \oplus D[\overset{\circ}{W}{}_2^1(G)]$$

Proof. The right-linear sets $X_1 = L_2(G) \cap \ker D$ and $X_2 = L_2(G) \setminus X_1$ are subspaces of $L_2(G)$. Let $u \in X_2$. Then it follows $v = T_G u \in W_2^1(G)$ and u has a representation $u = Dv, v \in W_2^1(G)$. From $u \in X_2$ we have $\int_G \overline{Dv}\,g\,dx = 0$ for all $g \in X_1$ and in particular $\int_G \overline{Dv}\,e(x-y_l)dx = 0$ for all numbers $l \in N$, where $y_l \in \Gamma_A, \Gamma_A = \partial G_A$, and clos $G \subset G_A$. We assume now that the set $\{u_l\} \in \Gamma_A$, is dense in Γ_A. Integration by parts leads to $\int_\Gamma e(y_l - x)n(x)v(x)d\Gamma_x = (F_\Gamma tr_\Gamma v)(y_l) = 0$ for all $l \in N$, hence, $F_\Gamma tr_\Gamma v = 0$ in G^- and $\lim_{\substack{y\to\eta \\ \eta\in G^-}} (F_\Gamma tr_\Gamma v)(y) = 0$. Consequently, we have $tr_\Gamma v \in$ im $P_\Gamma \cap W_2^{1/2}(G)$. Using theorems about traces we obtain the existence of a function $V \in \ker D \cap W_2^1(G) : tr_\Gamma V = tr_\Gamma v$. If we consider now $v - V \in \overset{\circ}{W}{}_2^1(G)$, then the application of D shows that $D(v-V) = Dv = u$ and we have $u \in D[\overset{\circ}{W}{}_2^1(G)]$. This result means that $((\ker D \cap L_2(G))^\perp \subseteq D[\overset{\circ}{W}{}_2^1(G)]$. If we suppose now that $w \in D[\overset{\circ}{W}{}_2^1(G)]$ we conclude as follows:

$$w \in D[\overset{\circ}{W}{}_2^1(G)] \Rightarrow w = Dz, z \in \overset{\circ}{W}{}_2^1(G) \Rightarrow$$

$$\int_G \overline{u} w dx = \int_G \overline{u}\, Dz dx = \int_G \overline{Du}\, z dx = 0$$

for all $u \in \ker D$.
Then we have $D[\overset{\circ}{W}{}_2^1 (G)] \subseteq [\ker D \cap L_2(G)]^{\perp}$. ∎

Corollary 54 There exist two orthoprojections $I\!P_G$ and Q_G with

$$I\!P_G : L_2(G) \rightarrow \ker D \cap L_2(G)$$
$$Q_G : L_2(G) \rightarrow D[\overset{\circ}{W}{}_2^1 (G)],\ Q_G = I - I\!P_G$$

Remark 55 If we use the operator $Ru = \sum_A r_A u_A e_A$ with positive real numbers r_A and the inner product $[u,v]_R = \int_G \overline{R^{-1}u} R^{-1} v dG$ then Theorem 53 can be generalized
in the following way:

$$L_2(G) = [\ R\ kerD \cap L_2(G)] \oplus_R D[\overset{\circ}{W}{}_2^1 (G)]$$

We can prove this decomposition using the same methods as in the proof of Theorem 53.

Corollary 56 *Let* $f \in L_2(G)$, $(T_G f)(x) = 0$ *for* $x \in I\!R^n \setminus$ *clos* $G \Rightarrow f \in$ *im* Q_G.

Proof. We use the representation $f = Dg$ with $g = T_G f$. From the assumption it follows that $tr_\Gamma g = 0$ and hence, $f = Dg \in D[\overset{\circ}{W}{}_2^1 (G)]$. ∎

The last corollary enables us to formulate a theorem concerning the completeness of $\{e(x - y_l)\}$ and to prove the theorem without using Hahn-Banach's Theorem.

Theorem 57 *Under the above mentioned conditions for* $\{y_l\}_{l \in I\!N} \subset \Gamma_A$ *the system* $\{e(x - y_l)\}_{l \in I\!N}$ *is complete in* $L_2(G) \cap \ker D$.

Proof. As usual, we assume the existence of $u \in \ker D \cap L_2(G) = im\mathbb{P}_G$ with property $(u, e(\cdot - y_l)) = 0 \quad \forall l \in \mathbb{N}$. Note that $im\mathbb{P}_G$ is the image of the operator $\mathbb{P}_G$. Then we have $(T_G u)(x) = 0$ for $\forall x \in \Gamma_A$, because $T_G u \in C^\infty(\mathbb{R}^n \setminus \text{clos } G)$. Furthermore, using $T_G u \in \ker \Delta(\mathbb{R}^n \setminus \text{clos } G)$, $(T_G u)(x) \xrightarrow[|x| \to \infty]{} 0$ we arrive at $(T_G u)(x) = 0$ for $x \in \mathbb{R}^n \setminus \text{clos } G$. The previous corollary implies that $u \in im\ Q_G$. Hence $u \in im\ \mathbb{P}_G \cap im\ Q_G = \{0\}$. This proves the theorem. ■

4 Applications

4.1 A new sort of elementary functions

Let us start with the introduction of an exponential function. For this section we assume that x is a paravector in $C\ell_{0,n}$, e.g. $x \in P\text{Vec}\, C\ell_{0,n} \subset C\ell_{0,n}$. Each paravector permits the representation $x = x_0 + \underline{x}$ with $\underline{x} = \sum_{i=1}^n x_i e_i$. Furthermore, we define a function $\omega : \mathbb{R}^n \to S^n$ given by $\omega(x) := \underline{x}/|\underline{x}|$. We will use for our definition a normally convergent series expansion. For an arbitrary $\varepsilon > 0$ it is always possible to find a sufficiently large number N such that for any $r, s > N$ holds

$$\left| \sum_{k=r}^{s} \frac{x^k}{k!} \right| \le \sum_{k=r}^{s} \frac{K^k |x|^k}{k!} < \varepsilon,$$

where K is a constant which only depends on n and satisfies the inequality $|xy| \le K^2 |x||y|$. Note that $|x|^2 = x\bar{x} = x_0^2 + x_1^2 + \ldots + x_n^2$.

Definition 1 *Let x be a paravector in $C\ell_{0,n}$. The exponential function e^x is defined by the power series*

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{11}$$

Proposition 2 *The exponential function permits the representation*

$$e^x = e^{x_0}(\cos|\underline{x}| + \omega(x)\sin|\underline{x}|).$$

Proof. Using the CAUCHY product of two power series we get

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}=\sum_{k=0}^{\infty}\frac{(x_0+\underline{x})^k}{k!}=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell}x_0^{\ell}\underline{x}^{k-\ell}$$

$$=\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\frac{x_0^{\ell}\underline{x}^{k-\ell}}{\ell!(k-\ell)!}=\sum_{\ell=0}^{\infty}\frac{x_0^{\ell}}{\ell!}\sum_{m=0}^{\infty}\frac{\underline{x}^m}{m!}=e^{x_0}e^{\underline{x}}.$$

It remains to consider $e^{\underline{x}}$. We have

$$\sum_{k=0}^{\infty}\frac{\underline{x}^k}{k!}=\sum_{\ell=0}^{\infty}\frac{\underline{x}^{2\ell}}{(2\ell)!}+\sum_{\ell=0}^{\infty}\frac{\underline{x}^{2\ell+1}}{(2\ell+1)!}$$

$$=\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{|\underline{x}|^{2\ell}}{(2\ell)!}+\omega(x)\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{|\underline{x}|^{2\ell+1}}{(2\ell+1)!}=\cos|\underline{x}|+\omega(x)\sin|\underline{x}|.$$ ■

Corollary 3 *Let x, y be paravectors in $C\ell_{0,n}$. If $xy = yx$ we get*

$$e^{x+y}=e^{x}e^{y}. \tag{12}$$

Corollary 4 *For any paravector $x \in C\ell_{0,n}$ we have*

(i) $e^x \neq 0$, (ii) $e^{-x}e^x = 1$, *(iii)* $e^{\omega(x)\pi} = -1$,

(iv) $e^{k\underline{x}} = (e^{\underline{x}})^k$ $(k \in \mathbb{N})$ *(*MOIVRE's formula*)*, (v) $|e^x| = e^{x_0}$.

Using an idea in [2] we obtain:

Corollary 5 *Let x be a paravector in $C\ell_{0,n}$. Then the exponential function may be described by the limit*

$$e^x=\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m.$$

Proof. Consider the difference

$$e^x-\left(1+\frac{x}{m}\right)^m=\sum_{k=0}^{\infty}\left(\frac{1}{k!}-\frac{\binom{k}{m}}{m^k}\right)x^k.$$

Since $\left(1+\frac{x}{m}\right)^m$ is a polynomial this series is normally convergent. The same is valid for the series e^x. The coefficients in front of x^k are positive. Now we obtain

$$\left|e^x - \left(1+\frac{x}{m}\right)^m\right| \le \sum_{k=0}^{\infty}\left(\frac{1}{k!} - \frac{\binom{k}{m}}{m^k}\right)|Kx|^k = e^{|Kx|} - \left(1+\frac{|Kx|}{m}\right)^m,$$

which tends to zero for m to infinity. ■

Definition 6 *Let x be a paravector in $C\ell_{0,n}$. Then hyperbolic and trigonometric functions are defined by the following formulae:*

$$\sinh x \;:=\; \frac{e^x - e^{-x}}{2}, \quad \cosh x := \frac{e^x + e^{-x}}{2}.$$

For $|\underline{x}| \neq 0$ we define:

$$\sin x \;:=\; \frac{e^{x\omega(x)} - e^{-x\omega(x)}}{2}\omega(x), \quad \cos x := \frac{e^{x\omega(x)} + e^{-x\omega(x)}}{2}.$$

Corollary 7 *It holds*

$$\sin x = -\sin(-x) \quad \textit{and} \quad \cos x = \cos(-x).$$

Proposition 8 *The hyperbolic functions* $\sinh x$ *and* $\cosh x$ *and trigonometric functions* $\sin x$ *and* $\cos x$ *permit the following representations:*

$$\cosh x = \omega(x)\sinh x_0 \sin|\underline{x}| + \cosh x_0 \cos|\underline{x}|,$$
$$\sinh x = \omega(x)\cosh x_0 \sin|\underline{x}| + \sinh x_0 \cos|\underline{x}|.$$

and

$$\sin x = \omega(x)\sinh|\underline{x}|\cos\, x_0 + \cosh|\underline{x}|\sin\, x_0,$$
$$\cos x = \omega(x)\sinh|\underline{x}|\sin\, x_0 + \cosh|\underline{x}|\cos\, x_0.$$

The right-hand sides of these representations can be used for the definition of $\sin$ *and* $\cos$ *in the case of* $|\underline{x}| = 0$.

Definition 9 *Let $x \in C\ell_{0,n}$ be a paravector, e.g. $x = x_0 + \underline{x}$, $x \neq x_0 < 0$. Then a paravector valued logarithm* $\log x$ *is defined by*

$$\log x := \ln |x| + \omega(x) \arccos \frac{x_0}{|x|}, \quad |\underline{x}| \neq 0 \quad \textit{or } |\underline{x}| = 0, x_0 > 0.$$

Corollary 10 *The above defined logarithm* $\log x$ *has the following properties:*

(i) $\log 1 = 0$ *and* $\log e_i = \frac{\pi}{2}$ $(i = 1, ..., n)$,

(ii) $1 - \frac{1}{|x|} - \arctan \frac{|\underline{x}|}{|x_0|} \leq |\log x| \leq |x| - 1 + \arctan \frac{|\underline{x}|}{|x_0|}$.

Theorem 11 *The function* $\log x$ *is the inverse to the exponential function* e^x *introduced above, which means*

$$e^{\log x} = x \quad \textit{and} \quad \log e^x = x.$$

Proof. The proof is to realize by a straightforward calculation. ■

Corollary 12 *Let $xy = yx$ then the well-known logarithm rule*

$$\log (xy) = \log x + \log y$$

is valid.

Definition 13 *Let α be a real number. The general power function x^α is defined by*

$$x^\alpha := e^{\alpha \log x}.$$

One example will confirm this definition.

Example 14 *Let $x = \underline{x}$ and $\alpha = \frac{1}{n}$. We have to line out the following calculation:*

$$\begin{aligned}
\underline{x}^{\frac{1}{n}} &= e^{\frac{1}{n}(\ln |\underline{x}| + \omega(x) \arccos \frac{x_0}{|x|} + \omega(x) 2k\pi)} \\
&= \sqrt[n]{|\underline{x}|} \left[\cos \left(\frac{1}{n} (\arccos 0 + 2k\pi) \right) + \omega(x) \sin \left(\frac{1}{n} (\arcsin 0 + 2k\pi) \right) \right] \\
&= \sqrt[n]{|\underline{x}|} \left[\cos \left(\frac{\pi}{2n} + \frac{2k\pi}{n} \right) + \omega(x) \sin \left(\frac{\pi}{2n} + \frac{2k\pi}{n} \right) \right]
\end{aligned}$$

for $k = 0, 1, ..., n - 1$.

4.1.1 Radially differentiable functions and FUETER's mapping

The elementary functions introduced above are not CLIFFORD regular. This is caused by the operator Γ in the following representation of the CAUCHY-FUETER operator

$$\partial = \partial_0 + \omega(x)\partial_{|\underline{x}|} + \omega(x)\frac{1}{|\underline{x}|}\Gamma.$$

Thus it holds as already proved $\Gamma\omega = (n-1)\omega$. This expression is disturbing the structure of our elementary functions. By the way, we note that scalar and vector fields which are only depending on $|\underline{x}|$ are lying in the kernel of Γ. It seems to be useful to introduce the following operator:

Definition 15 *The differential operator*

$$D_{ra} := \frac{1}{2}(\partial_0 - \omega(x)\partial_{|\underline{x}|})$$

is called radial CAUCHY–FUETER *operator.*

Note the denotation $D_{ra}u := u'$ for a paravector $u \in C\ell_{0,n}$. Let us now investigate in which manner this operator is acting on the elementary functions.

Proposition 16 *We list here the following differential actions:*

$$(i)\ \partial_{|\underline{x}|}\omega(x) = 0, \quad (ii)\ \partial_{|\underline{x}|}|x| = \frac{|\underline{x}|}{|x|}, \quad (iii)\ \partial_{|\underline{x}|}\underline{x} = \omega(x).$$

Proof. For $u \in C^2(\mathbb{R}^n, C\ell_{0,n})$ we have for instance

$$\partial_{|\underline{x}|}\omega_i(x) = \partial_{|\underline{x}|}\partial_i|\underline{x}| = \partial_i\partial_{|\underline{x}|}|\underline{x}| = \partial_i 1 = 0,$$

where $\omega_i(x) = x_i/|\underline{x}|$. We note only for (iii) that $\underline{x} = (\underline{x}/|\underline{x}|)|\underline{x}|$. Property (ii) is easy to verify. ■

Proposition 17 *Let u, v be paravector valued functions. If $\omega(x)u = u\omega(x)$ then for D_{ra} the* LEIBNIZ *rule holds.*

Theorem 18 *The radial differential operator D_{ra} generates in the set of elementary functions the following rules:*

(i) $(e^x)' = e^x$,

(ii) $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$,

(iii) $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$,

(iv) $(\log x)' = \frac{1}{x}$,

(v) $(x^\alpha)' = \alpha x^{\alpha-1}$ *where* $\alpha \in \mathbb{R}$.

Proof. All properties follow straightforward. ■

Definition 19 *Let* $f : \mathrm{PVec}\, C\ell_{0,1} \mapsto \mathrm{PVec}\, C\ell_{0,1}$ *be a paravector valued function in* $C\ell_{0,n}$ *of the type* $f = f_0 + \omega(x) f_1$ *with* $f_0, f_1 \in \mathbb{R}$ *and* $h = h_0 + \omega(x)|\underline{h}|$. *Such a function is called radially differentiable or radially* CLIFFORD *regular if and only if*

$$\lim_{h \to 0} \frac{(f(x+h) - f(x))\bar{h}}{|h|^2}$$

exists. In the case of existence the radial derivative is just $f'(x) = D_{ra}$.

Theorem 20 *Let* $f_i = f_i(x_0, |\underline{x}|)$, $(i = 0, 1)$ *real-valued functions and* $f = f_0 + \omega(x) f_1$. *Further, there exist continuous partial derivatives* $\partial_0 f_i$ *and* $\partial_{|\underline{x}|} f_i$ $(i = 0, 1)$. *Then the function* f *is radially differentiable if and only if the relations*

$$\partial_0 f_0 = \partial_{|\underline{x}|} f_1, \qquad \partial_0 f_1 = -\partial_{|\underline{x}|} f_0$$

are valid.

Corollary 21 *All above defined elementary functions are radially* CLIFFORD *regular.*

Now we will demonstrate how to transform radially regular functions to Clifford regular functions. The basic idea goes back to R. FUETER, who formulated a transformation of holomorphic functions to quaternionic regular functions. Later M. SCE [22] and T. QIAN [20] generalized these results. We will illustrate this mapping in $C\ell_{0,n}$. Let $z := u + iv \in \mathbb{C}$ and $h := u + \omega v \in C\ell_{0,n}$. Introducing by τ the mapping:

$$q = \tau(z) = \Delta^{(n-1)/2} h.$$

Proposition 22 *[20] Let $h = u(x_0, |\underline{x}|) + \omega v(x_0, |\underline{x}|)$ be a function radially regular in an open set of $\mathbb{R}^{n+1}$. Then for any number $k \in \mathbb{N}$ we find*

$$\Delta^k h = u_k(x_0, |\underline{x}|) + \omega v_k(x_0, |\underline{x}|)$$

where

$$u_k = 2k(\partial_{|\underline{x}|})u_{k-1}\frac{1}{|\underline{x}|}, \quad v_k = 2k\left(\partial_{|\underline{x}|}\frac{1}{|\underline{x}|} - \frac{v_{k-1}}{|\underline{x}|^2}\right) = 2k\partial_{|\underline{x}|}\frac{v_{k-1}}{|\underline{x}|}.$$

Proof. The proof given in [20] follows by mathematical induction. ■

Theorem 23 *Let $n = 2$. A radially quaternionic regular function fulfils the following differential equation*

$$|\underline{x}|^2\Delta h - |\underline{x}|\partial_{|\underline{x}|}h + \omega \operatorname{Vec} h = 0$$

Proof. The action of the operator Δ on h.

$$\Delta h = \frac{x}{|\underline{x}|}\partial_{|\underline{x}|}u(|\underline{x}|, x_0) + \omega\partial_{|\underline{x}|}\left(\frac{v(|\underline{x}|, x_0)}{|\underline{x}|}\right).$$

Further we obtain

$$\Delta h = \frac{1}{|\underline{x}|}\partial_{|\underline{x}|}u + \omega\partial_{|\underline{x}|}\left(\frac{v}{|\underline{x}|}\right) = \frac{1}{|\underline{x}|}\left[\partial_{|\underline{x}|}u + \omega\partial_{|\underline{x}|}v\right] - \omega\frac{v}{|\underline{x}|^2}$$

$$\Delta h = \frac{1}{|\underline{x}|}\partial_{|\underline{x}|}h - \omega\frac{\operatorname{Vec} h}{|\underline{x}|^2}.$$

Hence

$$\left[|\underline{x}|^2\Delta - |\underline{x}|\partial_{|\underline{x}|} + \omega \operatorname{Vec}\right]h = 0.$$

Corollary 24 *The Laplacian realizes the mapping*

$$\Delta : \ker \partial_{|\underline{x}|a} \longmapsto \ker \partial,$$

where $\partial_{ra} = \partial_0 + D_{ra}$.

Remark 25 For an arbitrary n we have to substitute the operator Δ by the operator $\Delta^{(n-1)/2}$. In this way we have for odd n that this operator is just a pointwise differential operator and for even n we have to consider the FOURIER multiplier operator induced by the symbol

$$(2\pi i|\xi|)^{n-1}.$$

We will now restrict our considerations to the case of real quaternions.

Theorem 26 FUETER*'s mapping delivers us the following quaternionic regular elementary functions.*

1. $\mathtt{exp}\ x := \left(+\frac{\sin|\underline{x}|}{|\underline{x}|} + \omega\frac{-|\underline{x}|\cos|\underline{x}|+\sin|\underline{x}|}{|\underline{x}|^2}\right)e^{x_0}$
2. $\mathtt{sin}\ x := \sin x_0 \frac{\sin h|\underline{x}|}{|\underline{x}|} + \omega\left(\frac{|\underline{x}|\cos h|\underline{x}| - \sin h|\underline{x}|}{|\underline{x}|^2}\right)\cos x_0$
3. $\mathtt{cos}\ x := \cos x_0 \frac{\sin h|\underline{x}|}{|\underline{x}|} - \omega\left(\frac{-\sin h|\underline{x}| + |\underline{x}|\cos h|\underline{x}|}{|\underline{x}|^2}\right)\sin x_0$
4. $\mathtt{cosh}\ x := +\cosh x_0 \frac{\sin|\underline{x}|}{|\underline{x}|} + \omega\frac{-\sin|\underline{x}| + |\underline{x}|\cos|\underline{x}|}{|\underline{x}|^2}\sinh x_0.$
5. $\mathtt{sinh}\ x := +\sinh x_0 \frac{\sin|\underline{x}|}{|\underline{x}|} + \omega\frac{-\sin|\underline{x}| + |\underline{x}|\cos|\underline{x}|}{|\underline{x}|^2}\cosh x_0$

Proof. Using the paper ([20]) the proof is an exercise. ∎

Remark 27 FUETER*'s method also generates a "logarithm" which we will denote by* $\mathtt{log}\ x$*. After a straightforward computation we find:*

$$\mathtt{log}\ x = \frac{1}{(r^2+x_0^2)} - \omega\left(\frac{1}{r(r^2+x_0^2)} + \frac{1}{r^2}\arccos\frac{x_0}{\sqrt{r^2+x_0^2}}\right).$$

Unfortunately $\mathtt{log}$ *is not the inverse of* $\mathtt{exp}$.

Now we will consider some elementary properties of the regular exponential function $\mathtt{exp}\ x$. Setting

$$\overline{\partial} := \left(\frac{\partial_0 + \overline{D}}{2}\right).$$

Theorem 28 *The exponential function has the property:*

$$\overline{\partial}\mathsf{exp}\lambda x = \lambda\mathsf{exp}\lambda x \quad , \quad \lambda \in \mathbb{C}$$

Proof. We obtain $D = \frac{1}{|\underline{x}|}L + \omega\ell_\omega$. As for definition

$$D\mathsf{exp}\lambda x = \left(\frac{1}{|\underline{x}|}L + \omega\ell_\omega\right)\mathsf{exp}\lambda x$$

using $Lf(|\underline{x}|) = 0$ (for any real function f!) and $L\omega = 2$ then we get.

$$\begin{aligned}
D\mathsf{exp}\lambda x &= \left(\tfrac{1}{|\underline{x}|}L + \omega\ell_\omega\right)\left(\tfrac{\sin\lambda|\underline{x}|}{\lambda|\underline{x}|} + \omega\left(\tfrac{\sin\lambda|\underline{x}| - \lambda|\underline{x}|\cos\lambda|\underline{x}|}{\lambda^2|\underline{x}|^2}\right)\right) \\
&= \tfrac{2}{|\underline{x}|}\left(\tfrac{\sin\lambda|\underline{x}| - \lambda|\underline{x}|\cos\lambda|\underline{x}|}{\lambda^2|\underline{x}|^2}\right) + \omega\left(\tfrac{\lambda\cos\lambda|\underline{x}|(\lambda|\underline{x}|) - \lambda\sin\lambda|\underline{x}|}{(\lambda|\underline{x}|)^2}\right) \\
&\quad \tfrac{[\lambda\cos\lambda|\underline{x}| + \lambda^2|\underline{x}|\sin\lambda|\underline{x}| - \lambda\cos\lambda|\underline{x}|\lambda^2|\underline{x}|^2 - 2\lambda^2|\underline{x}|(\sin\lambda|\underline{x}| - \lambda|\underline{x}|\cos\lambda|\underline{x}|)]}{(\lambda|\underline{x}|)^4} \\
&= 2\left(\tfrac{\sin\lambda|\underline{x}| - \lambda|\underline{x}|\cos\lambda|\underline{x}|}{\lambda^2|\underline{x}|^3}\right) - \tfrac{\lambda\cos\lambda|\underline{x}|}{(\lambda|\underline{x}|)^2} - \tfrac{\lambda^2|\underline{x}|\sin\lambda|\underline{x}|}{(\lambda|\underline{x}|)^2} + \tfrac{\lambda\cos\lambda|\underline{x}|}{(\lambda|\underline{x}|)^2} - \\
&\quad - \tfrac{2\lambda^2|\underline{x}|}{(\lambda|\underline{x}|)^4}[\sin\lambda|\underline{x}| - \lambda|\underline{x}|\cos\lambda|\underline{x}|] + \omega\left(\tfrac{\lambda\cos\lambda|\underline{x}|(\lambda|\underline{x}|) - \lambda\sin\lambda|\underline{x}|}{(\lambda|\underline{x}|)^2}\right) \\
&= 2\left[\tfrac{\sin\lambda|\underline{x}|}{\lambda^2|\underline{x}|^3} - \tfrac{\sin\lambda|\underline{x}|}{\lambda^2|\underline{x}|^3}\right] + 2\left[\tfrac{\lambda^3|\underline{x}|^2}{(\lambda|\underline{x}|)^4}\cos\lambda|\underline{x}| - \tfrac{\lambda|\underline{x}|}{\lambda^2|\underline{x}|^3}\cos\lambda|\underline{x}|\right] + \omega[\ldots] \\
&= -\lambda\left[\tfrac{\sin\lambda|\underline{x}|}{\lambda|\underline{x}|}\right] + \omega\,[\ldots] = -\lambda\mathsf{exp}\lambda x \qquad \blacksquare
\end{aligned}$$

Corollary 29 *It holds*

$$\text{(i)} \quad |\mathsf{exp}x|^2 > 0, \quad \text{(ii)} \quad \lim_{\underline{x}\to 0}|\mathsf{exp}x| = e^{x_0}.$$

4.2 Selected boundary value problems in fluid mechanics

We assume for the moment that the domain G is bounded by a pieceweise smooth bounded Liapunov surface. In the last years these assumptions could be considerable weakened.

4.2.1 Linear equations of Stokes' type

$$\begin{aligned}
-\Delta\underline{u} + \frac{1}{\eta}\nabla p &= \frac{\rho}{\eta}\underline{f} \quad \text{in } G \\
\operatorname{div}\underline{u} &= f_0 \quad \text{in } G \\
\underline{u} &= \underline{g} \quad \text{on } \Gamma.
\end{aligned}$$

Here η is the viscosity and ρ the density of the fluid. We have to look for the velocity u and the hydrostatical pressure p. Between f_0 and $\underline{g}$ we have to fulfil the relation:

$$\int_G f_0 dx = \int_\Gamma n\underline{g} d\Gamma.$$

For $g = 0$ then the measure of the compressibility f_0 satisfies the identity

$$\int_G f_0 dx = 0.$$

For all such real functions f_0 can be represented the unique solution (p is unique up to a real constant) as follows:

Theorem 30 *[12] Let $f := f_0 + \underline{f} \in W_p^k(G, \mathbb{H})$ $(k \geq 0, 1 < p < \infty)$. Then we have*

$$\begin{aligned} \underline{u} &= \frac{\rho}{\eta} T_G \operatorname{Vec} T_G \underline{f} - \frac{\rho}{\eta} T_G \operatorname{Vec} F_\Gamma (tr_\Gamma T_G \operatorname{Vec} F_\Gamma)^{-1} tr_\Gamma T_G \operatorname{Vec} T_G \underline{f} - T_G f_0, \\ p &= \rho \operatorname{Sc} T_G \underline{f} - \rho \operatorname{Sc} F_\Gamma (tr_\Gamma T_G \operatorname{Vec} F_\Gamma)^{-1} tr_\Gamma T_G \operatorname{Vec} T_G \underline{f} + \eta f_0. \end{aligned}$$

In that way we strongly separated velocity and pressure.

4.2.2 Nonlinear equations of Stokes' type

Now we assume that the compressibility depends on the velocity and the nonlinear outer forces. The equations describing this state are the following:

$$\begin{aligned} -\Delta \underline{u} + \frac{1}{\eta} \nabla_p &= \Lambda f(u) \text{ in } G \\ \operatorname{div}(\eta^{-1} \underline{u}) &= 0 \text{ in } G \\ \underline{u} &= 0 \text{ on } \Gamma \end{aligned} \tag{13}$$

The parameter of viscosity η ($\eta > 0$) depends on the position. The main result is given by:

Theorem 31 *[15]*

1. *Let $f \in L_2(G,\mathbb{H}), p \in W_2^1(G), \eta \in C^\infty(G)$. Then every solution of the system (13) permits the integral representation*

$$\begin{aligned} u &= \Lambda RBf - RBDp \\ 0 &= Sc\Lambda QT_GBf - ScQT_GBDp. \end{aligned} \tag{14}$$

Here B is the multiplication operator by η^{-1} and $R := T_GQT_G$.

2. *If the operator function $f(u)$ satisfies the estimates*

$$\begin{aligned} &\text{(i)} \quad \|f(u) - f(v)\|_2 \leq L\|u - v\|_{2,1}, \quad \textit{for} \quad \|u\|_{2,1}, \|v\|_{2,1} \leq 1, \\ &\text{(ii)} \quad \|B\|_2 \leq K \textit{for positive constants} \quad K, L \\ &\text{(iii)} \quad \Lambda < \{\|T\|_{imQ\cap L_2,L_2}\|T\|_{L_2,L_2}KL\}^{-1} \end{aligned}$$

the iteration procedure

$$\begin{aligned} &u_n = \Lambda RBf(u_{n-1}) - RBDp_n \\ &\Lambda ScDBRf(u_{n-1}) = ScDBRDp_n \\ &\|u_0\|_{2,1} \leq 1 \qquad (u_0 \in \overset{\circ}{W}{}_2^1 (G,\mathbb{H})) \end{aligned}$$

is converging to a unique solution $\{u,p\} \in \overset{\circ}{W}{}_2^1 (G,\mathbb{H}) \cap \ker(\mathrm{div}B) \times L_2(G)$ of (14), where p is unique up to a real constant.

4.2.3 Problems of Navier–Stokes type

In the stationary case Navier–Stokes equations are described in the following way:

$$\begin{aligned} -\Delta\underline{u} + \frac{\rho}{\eta}(\underline{u}\cdot\nabla)\underline{u} + \frac{1}{\eta}\nabla p &= \frac{\rho}{\eta}f \quad \text{in} \quad G \\ \mathrm{div}\,\underline{u} &= 0 \quad \text{in} \quad G \\ \underline{u} &= 0 \quad \text{on} \quad \Gamma. \end{aligned} \tag{15}$$

We will abbreviate with $M(u) := M^\bullet(u) - f$, where $M^\bullet(u) = (\underline{u}\cdot\mathrm{grad})\underline{u}$. The main result is now the following:

Theorem 32 *[9][11]*

1. *Let $f \in L_2(G), p \in W_2^1(G)$. Every solution of (15) permits the operator integral representation*

$$u = -\frac{\rho}{\eta} RM(u) - \frac{1}{\eta} T_G Qp \tag{16}$$

$$\frac{\rho}{\eta} ScQT_G M(u) - \frac{1}{\eta} ScQp = 0.$$

2. *The system (16) has a unique solution $\{u, p\} \in \overset{\circ}{W}{}_2^1 (G, \mathbb{H}) \cap \ker(\mathrm{div} B) \times L_2(G)$, where p is unique up to a real constant, if*

$$(i) \quad \|f\|_p \leq (18K^2 C_1)^{-1}$$

$$\text{with} \quad K := \frac{\rho}{\eta} \|T_G\|_{[L_2 \cap \mathrm{im} Q, W_2^1]} \|T\|_{[L_p, L_2]}$$

$$(ii) \quad u_0 \in \overset{\circ}{W}{}_2^1 (G, \mathbb{H}) \cap \ker(\mathrm{div} B)$$

$$\text{with} \quad \|u_0\|_{2,1} \leq min(V, \frac{1}{4KC_1} + W)$$

holds. Here means $V := (2KC_1)^{-1}$, $W := [(4KC_1)^{-2} - \frac{\rho\|f\|_p}{\eta C_1}]^{\frac{1}{2}}$ and $C_1 = 9^{\frac{1}{p}} C$, where C is the embedding constant from W_2^1 in L_2. The iteration procedure (starting with u_0)

$$u_n = \frac{\rho}{\eta} RM(u_{n-1}) - \frac{1}{\eta} RDp_n$$

$$\frac{\rho}{\eta} ScQT_G M(u_{n-1}) = -\frac{1}{\eta} ScQp_n$$

$$(u_0 \in \overset{\circ}{W}{}_2^1 (G, \mathbb{H}) \cap \mathrm{kerdiv})$$

converges in $W_2^1(G, \mathbb{H}) \times L_2(G)$.

4.2.4 Navier–Stokes equations with heat conduction

We will now consider the flow of a viscous fluid under the influence of temperature. The corresponding equations read as follows:

$$-\Delta \underline{u} + \frac{\rho}{\eta}(u \cdot \nabla)\underline{u} + \frac{1}{\eta}\nabla_p + \frac{\gamma}{\eta} \underline{gw} = f \quad \text{in} \quad G$$

$$\begin{aligned}
-\nabla w + \frac{m}{\kappa}(u \cdot \nabla) w &= \frac{1}{\kappa} h && \text{in } G \\
\operatorname{div} \underline{u} &= 0 && \text{in } G \\
\underline{u} &= 0 && \text{on } \Gamma \\
w &= 0 && \text{on } \Gamma
\end{aligned}$$

We denote by ρ the density of the fluid, by η the viscosity, by γ the Grashof number, by κ the temperatur conductivity and by m the Prandl number. As usual $\underline{u}$ stands for the velocity, w for the temperature and p for the hydrostatic pressure. We will here only formulate the main result. It can be shown that the solutions of this system fulfil the following system of operator integral equations, where boundary conditions will be fulfilled automatically. Here we have this system:

$$\begin{aligned}
u &= -R[M(u) - \frac{\gamma}{\eta} e_3 w] - \frac{1}{\eta} T_G \mathcal{Q} p \qquad (17) \\
0 &= ScDR[M(u) - \frac{\gamma}{\eta} e_3 w] - \frac{1}{\eta} \mathcal{Q} p \\
w &= -\frac{m}{\kappa} RSc(uD)w + Rg
\end{aligned}$$

where $M(u) := \frac{\rho}{\eta}(\underline{u} \cdot \operatorname{grad})u + f(u) - F$.

Theorem 33 *[13]*

1. *We consider the following iteration procedure:*

$$\begin{aligned}
u_n &= -R[M(u_{n-1}) - \frac{\gamma}{\eta} e_3 w_{n-1}] - \frac{1}{\eta} T_G \mathcal{Q} p_n \qquad (18) \\
0 &= ScDR[M(u_{n-1}) - \frac{\gamma}{\eta} e_3 w_{n-1}] - \frac{1}{\eta} \mathcal{Q} p_n \\
w_n &= -\frac{m}{\kappa} RSc(u_n D) w_n + Rg.
\end{aligned}$$

The computation of w_n will be done by the inner iteration:

$$w_n^j = \frac{m}{\kappa} RSc(u_n D) w_n^{j-1} + Rg.$$

2. *Let $u_n \in \overset{\circ}{W}{}_2^1$. Further, let $m \neq 4\kappa$ and $u_n \| < \kappa / mKC$. The sequence $\{w_n^{(j)}\}_{j \in N}$ converges in $W_2^1(G)$.*

3. Let $F \in L_2(G,\mathbb{H})$, $g \in L_2(G)$, $f : W_2^1(G,\mathbb{H}) \to L_2(G,\mathbb{H})$ with $\|f(u) - f(v)\|_2 \leq L\|u - v\|_{2,1}$ and $f(0) = 0$. Under the additional smallness conditions

$$(i) \quad \frac{\rho}{\eta}\|F\|_2 + \frac{\gamma}{\eta}K|d|^{-1}\|g\|_2 < \frac{1}{16K^2C} \qquad \left(d := (4 - \frac{m}{\kappa})\kappa\right)$$

$$(ii) \quad \|g\|_2 < \left(1 - \frac{1}{\sqrt{2}}\right)\eta d^2\left(\frac{1}{32K^3Cm}\right)$$

$$(iii) \quad m < 4\kappa$$

the sequence $\{u_n, w_n, p_n\}_{\{n\in\mathbb{N}\}}$ converges in $W_2^1 \times W_2^1 \times L_2$ to the unique solution $(u, w, p) \in \overset{\circ}{W}{}_2^1 (G,\mathbb{H}) \times \overset{\circ}{W}{}_2^1 (G,\mathbb{H}) \times L_2(G)$ of the originally boundary value problem, where p is unique up to a real constant.

Remark 34 *We note that conditions (i) and (ii) can always be realized for fluids with big enough viscosity number.*

4.3 Eigenvalues and Teodorescu transform

4.3.1 On the first eigenvalue of Dirichlet' s problem

We are going to establish connections between suitable norms of the Teodorescu transform and the first eigenvalue of Dirichlet's problem. Assume that G is a bounded domain with a boundary Γ which satisfies weak smoothness conditions for instance the cone property. Then the Dirac operator

$$D : \overset{\circ}{W}{}_2^1 (G) \to im Q$$

is invertible and

$$D^{-1} := T_G : im Q \to \overset{\circ}{W}{}_2^1 (G).$$

We obtain in this way the following estimates:

$$\|T_G v\|_2 \leq c\|v\|_2 \quad (v \in Q)$$

$$\|Du\|_2 \geq c^{-1}\|u\|_2 \quad (u \in \overset{\circ}{W}{}_2^1 (G))$$

For the smallest eigenvalue λ_1 of Dirichlet' s problem we get

$$\lambda_1 = \inf_{\substack{u\in \overset{\circ}{W}{}_2^1(G)\\ u\neq 0}} \frac{(u,-\Delta u)_2}{\|u\|_2^2} = \inf_{\substack{u\in \overset{\circ}{W}{}_2^1(G)\\ u\neq 0}} \frac{(Du,Du)_2}{\|u\|_2^2} = \inf_{\substack{u\in \overset{\circ}{W}{}_2^1(G)\\ u\neq 0}} \frac{\|Du\|_2^2}{\|u\|_2^2}.$$

Hence

$$\|Du\|_2^2 \geq \lambda_1 \|u\|_2^2.$$

With $u := T_G v$ and $v \in imQ$ it follows

$$\|T_G v\|_2^2 \leq \frac{1}{\lambda_1}\|v\|_2^2$$

and $\|T_G\|_{[imQ,W_2^1]} \leq \lambda_1^{-1/2}$. Assume now that u_1 is the corresponding eigenfunction to the eigenvalue λ_1.We obtain

$$\|T_G w\|_2^2 = \|T_G D u_1\|_2^2 = \|u_1\|_2^2 = \|Du_1\|_2^2 = \|w\|_2^2 \frac{1}{\lambda_1}$$

with $w = Du_1$ and $(u_1 \in \overset{\circ}{W}{}_2^1 (G))$. As a consequence, we have

$$\lambda_1 = \|T_G\|_{[imQ,L_2]}^{-2}.$$

In a similar way we find

$$\|T_G\|^2_{[imQ,\overset{\circ}{W}{}_2^1(G)]} = \frac{\lambda_1+1}{\lambda_1}.$$

Because of $\|T_G\|_{[L_2,L_2]} \geq \|T_G\|_{[imQ,L_2]}$ we get, applying Schmidt' s inequality,

$$\|T_G\|_{[L_2,L_2]} \leq \sqrt[n]{\frac{|G|}{|B_1|}}.$$

Here B_1 denotes the unit ball in $\mathbb{R}^n$ and $|B_1|, |G|$ denotes the volume of the unit ball and the domain G, respectively. Thus we have obtained under these very weak conditions the domain G the following lower estimate for λ_1:

$$\lambda_1 \geq \sqrt[n]{\left(\frac{|B_1|}{|G|}\right)^2}$$

Remark 35 *Under additional geometrical and smoothness conditions there exist better lower estimates. Lower bounds for eigenvalues were first established in 1936 by E. Trefftz/F.A. Willers, stimulated by problems in engineering. Essential contributions were given by J. Hersch (1960), W.A. Kondratiev (1967), J. Cheeger (1970), L. Payne/I. Stakgold (1973), W.K. Hayman (1978), M. Taylor (1979), R. Ossermann (1979), C. Bandle (1980), C.B. Croke (1981), M.H. Protter (1981), R. Klötzler (1983), J.V. Jegorov/W.A. Kondratiev (1984) and so on.*

4.3.2 First eigenvalue of Neumann' s problem

By modification of the Teodorescu transform we will get a lower estimation of the first eigenvalue of Neumann' s problem

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in} \quad G, \\ \partial_n u &= 0 \quad \text{on} \quad \Gamma. \end{aligned}$$

For this purpose, we introduce the factor space

$$\widetilde{W}_2^1(G) := \{u \in W_2^1(G) : \int_G u(x)dG = 0\}/\ker \cdot D$$

and the modified Teodorescu transform

$$\widetilde{T}u := Tu - \frac{1}{|G|}\int_G (Tu)(x)dG \quad (u \in L_2(G)).$$

Note that

$$D : \widetilde{W}_2^1(G) \overset{on}{\to} L_2(G).$$

The following properties can be proved:

(i) $\operatorname{im} D(\widetilde{W}_2^1(G)) = L_2(G)$

(ii) $\ker D \cap \widetilde{W}_2^1(G) = \{0\}$

(iii) $\ker \widetilde{T} \cap L_2(G) = \{0\}$

(iv) $\operatorname{im} \widetilde{T}(L_2(G)) = \widetilde{W}_2^1(G)$

Hence, it follows immediately that the inverse of the operator D as a map between this pair of Banach spaces is given by

$$D^{-1} := \widetilde{T}.$$

Using the same technique as before, we prove by consideration of the operator $\widetilde{T}$ (cf. [13]) the estimate

$$\lambda_1^N \geq \frac{1}{4} \sqrt[n]{\left(\frac{|B_1|}{|G|}\right)^2},$$

where λ_1^N denotes the first eigenvalue of Neumann's problem.

4.4 An eigenvalue problem of Stokes type

Let $f : W_2^1(G) \to Vec\, L_2(G)$ be a non-linear operator which fulfils a Lipschitz condition

$$\|f(u) - f(v)\|_2 \leq L\|u - v\|, \quad (u, v) \in \{u \in W_2^1(G) : \|u\| \leq 1\}.$$

We are going to consider the following non-linear Stokes' eigenvalue problem :

$$-\Delta \underline{u} + \eta^{-1}\mathrm{grad}p = \Lambda f(\underline{u}) \quad \text{in} \quad G,$$

$$\mathrm{div}\, \underline{u} = 0 \quad \text{in} \quad G,$$

$$\underline{u} = 0 \quad on \quad \Gamma.$$

The real number Λ is called *eigenvalue parameter*. $\eta = \eta(x)$ describes the viscosity function. After transformation in a corresponding quaternionic language we get the equivalent operator-integral formulation

$$u + \eta^{-1} T_G Q p = \Lambda T_G Q T_G f(u) \tag{19}$$

$$\eta^{-1} Sc Q p = \Lambda Sc Q T_G f(u) \tag{20}$$

where $u = u_0 + \underline{u}$ and $f(\underline{u}) = f(u)$.

Definition 36 *If for problem formulated above the parameter Λ there exist non-zero functions u and p which solve the problem formulated above then Λ is called a point of the spectrum σ of our problem. If u $\|u\|_{2,1} \leq r$ then we denote this part of the spectrum by σ_r.*

Theorem 37 *Let $\Lambda \in \sigma_1, f(0) = 0$. Abbreviate $\|T\|_1 := \|T\|_{[imQ, W_2^1]}$ and $\|T\|_2 := \|T\|_{[imQ, L_2]}$. Then*

$$|\Lambda| \geq (L\|T\|_1\|T\|_2)^{-1}.$$

Proof. The differentiation of the representation formula (19) yields

$$Du + \frac{1}{\eta} Qp = \Lambda QTf(u).$$

As in the case of the linear STOKES system we have

$$\|Du\|_2^2 + \frac{1}{\eta}\|Qp\|_2^2 = |\Lambda|^2\|QTf(u)\|_2^2. \tag{21}$$

It can be shown (cf. [12]) that

$$\sqrt{\frac{\lambda_1}{1+\lambda_1}}\|u\|_{2,1} \leq |\Lambda|\ \|T\|_2\|f(u)\|_2 \leq |\Lambda|\ \|T\|_2 L.$$

Here, we used that $\|f(u)\|_2 \leq L$ for $\|u\|_{1,2} \leq 1$ as a consequence of the local LIPSCHITZ condition. Hence,

$$\|u\|_{2,1} \leq |\lambda|\ \|T\|_1\|T\|_2 L.$$

Now the following iteration procedure can be formulated:

$$\begin{aligned} u_0 &\in \overset{\circ}{W}{}_2^1(G) \cap \ker \operatorname{div}, \quad \|u_0\| \leq 1, \\ u_n &= \lambda TQTf(u_{n-1}) - \frac{1}{\eta} TQp_n \quad (n = 1, 2, \ldots). \end{aligned}$$

At each step a linear STOKES problem has to be solved. Making some operator estimations, we obtain:

$$\|u_n - u_{n-1}\|_{2,1} \leq |\Lambda|\ \|T\|_1\ \|T\|_2 L\|u_{n-1} - u_{n-2}\|_{2,1}.$$

Assuming $|\Lambda|\,\|T\|_1\|T\|_2 L \le 1$ we conclude the contractivity of the non-linear mapping. With the help of (21) it is ensured that $\|u_n\|_{1,2} \le 1$ ($\forall\, u \in$ N). Now, we can apply BANACH's fixed-point theorem and obtain that $\{u_n, p_n\}$ converges to the unique solution. Therefore, Λ such that

$$|\Lambda| \le \frac{1}{L\|T\|_1\|T\|_2}$$

can not belong to σ_1. ■

What happens when we omit the restrictive condition $f(0) = 0$?

Remark 38 *If $f(0) \neq 0$ then, it can not be ensured that $u_n \to 0$, although the proof of convergence goes through. If we assume that $f : \overset{\circ}{W}{}^1_2 \to L_p(G)$ $(\frac{6}{5} < p < \frac{3}{2})$ and*

$$\|f(u) - f(v)\|_p \le L\|u - v\|_{2,1},$$

$$\forall\, u, v \in W_2^1(G):\ \|u\|_{2,1} \le 1,\ \|v\|_{2,1} \le 1.$$

We can prove that

$$\inf_{\Lambda \in \sigma_1} |\Lambda| \ge \left[L\,\|T\|_1\|T\|_{[L_p, L_2]}\right]^{-1}.$$

This is a consequence of

$$\sqrt{\frac{\lambda_1}{1+\lambda_1}}\|u\|_{2,1} \le |\lambda|\,\|T\|_{[L_p, L_2]} L$$

which can be concluded from (21). The assumption $p > \frac{6}{5}$ guarantees that $T : L_p \to L_2$ is bounded.

one has the following theorem:

Theorem 39 *Let $\Lambda \in \sigma_1$, $f(0) = 0$. Then under the assumption*

$$\|f(u) - f(v)\|_2 \le L\|u - v\|_2, \quad \forall\, u, v : \|u\|_2 \le 1, \|v\|_2 \le 1.$$

we have that

$$\inf_{\Lambda \in \sigma_1} |\Lambda| \ge \lambda_1 L^{-1}.$$

Proof. We already know that $\|Du\|_2 \geq \sqrt{\lambda_1}\|u\|_2 \quad (\forall\, u \in \overset{\circ}{W}{}^1_2)$. Therefore, from (21) we get

$$\lambda_1\|u\|_2^2 \leq |\lambda|^2\|T\|_2^2\|f(u)\|_2^2 \leq |\Lambda|^2\|T\|_2^2 L^2\|u\|_2^2.$$

Now $\|T\|_2 \leq 1/\sqrt{\lambda_1}$ makes the proof complete. ■

Remark 40 *The* LIPSCHITZ *condition in L_2 is more useful than the condition between L_2 and W_2^1. But we have to pay for this convenience. Namely, we have another eigenvalue in our final result. Nevertheless, all estimates are constructive, because λ_1 as well as the T-norms can be explicitly estimated by properties of G.*

4.4.1 Lower bound of the spectrum of a non-linear problem in elasticity

Let us consider the following eigenvalue problem (cf. [?][13]):

$$\begin{aligned} -\Delta\underline{u} - \frac{m}{m-2}\text{grad div}\,\underline{u} &= \Lambda f(\underline{u}) \quad \text{in} \quad G \\ \underline{u} &= 0 \quad \text{on} \quad \Gamma \end{aligned}$$

The parameter m denotes Poisson' s contraction number and $\underline{u}$ is the vector of displacements. As in the previous section we complete the vector function $\underline{u}$ to the quaternionic function $u = u_0 + \underline{u}$. We obtain the quaternionic formulation

$$\begin{aligned} DMD\,u &= \Lambda\, f(u) \\ u &= 0, \end{aligned}$$

where

$$Mu := \frac{m-2}{2(m-1)}u_0 + \underline{u}.$$

An iteration procedure is now defined by

$$u_n = T_G Q_M M^{-1}\Lambda T_G f(u_{n-1}$$

It is easy to show that $tr_\Gamma u_n = 0$!). In [12] it is verified that the sequence (u_n) converges to zero if

$$\inf_{\Lambda \in \sigma_1} |\Lambda| \geq \frac{m-2}{2L(m-1)\|T\|_1\|T\|_2}.$$

In that way we get lower bound of σ_1.

Remark 41 *Similarly, one get an estimation of the first eigenvalue of the oscillation problem in linear elasticity:*

$$\begin{aligned} -\Delta\underline{u} - \frac{m}{m-2}\text{grad div }\underline{u} - \theta^2\rho &= \Lambda f(\underline{u}) \quad \textit{in} \quad G \\ \underline{u} &= 0 \quad \textit{on} \quad \Gamma \end{aligned}$$

Here θ denotes the oscillation frequency and ρ the density of the material. The estimation reads :

$$\Lambda_1 \geq \frac{m-2}{2(m-1)}\frac{\lambda_1}{\sqrt{\lambda_1}+1} + \theta^2\rho$$

where λ_1 denotes the first eigenvalue of Dirichlet' s problem.

4.4.2 Comparison of eigenvalues of different eigenvalue problems

In this section we want to deduce relations between the first eigenvalue of Diri-
chlet' s problem, Stokes' problem and Lamé' s problem. The results were obtained by K. Gürlebeck . Confere our book [12]. Lamè's system

$$\begin{aligned} -\Delta\underline{u} - \frac{m}{m-2}\text{grad div }\underline{u} &= \Lambda\underline{u} \quad \text{in} \quad G, \\ \underline{u} &= 0 \quad \text{on} \quad \Gamma, \end{aligned}$$

where $\text{Sc}\, u = 0$ for $\Lambda \in \mathbb{R}$. This system has only the trivial solution if

$$|\Lambda|\|T_G \mathcal{Q}_M M^{-1} T_G\|_{[L_2,L_2]} < 1.$$

Hence

$$\Lambda_1(m,G) \geq \|T_G\|^{-2}_{[L_2,L_2]}\frac{m-2}{2(m-1)}.$$

Now we can estimate, on the one hand,

$$\Lambda_1(m,G) = \inf_{\substack{u\in \overset{\circ}{W}{}^1_2,\ Sc\,u=0\\ u\neq 0}} \frac{(DMD\,u,u)_2}{\|u\|_2^2} \leq \inf_{\substack{u\in \overset{\circ}{W}{}^1_2\cap \ker \operatorname{div},\ Sc\,u=0\\ u\neq 0}} \frac{(Du,Du)}{\|u\|_2^2} = \Lambda_{\bullet\bullet}.$$

On the other hand

$$\Lambda_1(m.G) = \inf\left\{ \frac{(\|Du\|_2^2 + \frac{m}{m-2}\|Sc\ Du\|_2^2)}{\|u\|_2^2},\quad u \in \overset{\circ}{W}{}^1_2,\ \operatorname{Sc} u = 0, u \neq 0 \right\}.$$

Therefore, we have monotonicity relatively to m i.e.

$$\Lambda_1(m,G) \leq \Lambda_1(m',G) \quad \text{for} \quad m < m'.$$

Then there exists the limit

$$\lim_{m\to 2} \Lambda_1 = \Lambda_{\bullet\bullet} \leq \Lambda_{\bullet}$$

We are able to formulate the following result:

Theorem 42 (Gürlebeck[10]) *Let $\Lambda_\bullet$ be the first eigenvalue of Stokes equation under Dirichlet's condition and $m > 2$. Then we have the inclusion*

$$\left(\frac{|B_1|}{|G|}\right)^{2/3} \pi^2 \leq \lambda_1(G) < \Lambda_1(m,G) \leq \Lambda_\bullet(G)$$

and $\Lambda_1(m,G) \to \Lambda_\bullet(G)$. In other words: The first eigenvalue λ_1 of Dirichlet's problem is smaller than the first eigenvalue Λ_1 of Lamé's problem and this again is smaller then first eigenvalue $\Lambda_\bullet$ of Stokes' problem.

Remark 43 *B. Kawohl obtained in 1987 (cf. [16] the estimation*

$$\left(1 + \frac{m}{n(m-2)}\right)\lambda_1(G) \geq \Lambda_1(m,G) \geq \lambda_1(G).$$

Remark 44 *If one has a solution u of Stokes' problem*

$$\begin{aligned} -\Delta u + \eta^{-1}\operatorname{grad} p &= f \quad in \quad G,\\ \operatorname{div} u &= 0 \quad in \quad G,\\ u &= 0 \quad on \quad \Gamma, \end{aligned}$$

then it is possible to get an upper estimate for λ_1. It holds

$$0 < \lambda_1(G) \leq \frac{K^2}{1-K^2}$$

where $K := (\sqrt{2}\|T_G f\|_2 - \eta^{-1}\|Qp\|_2)\|u\|_{2,1}^{-1}$ has to be small enough.

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