Cubo Matemática Educacional
Vol., 4. Nº 2, Junio 2002

LAGRANGE AND HAMILTON SPACES: GEOMETRICAL MODELS IN MECHANICS, NEW THEORETICAL PHYSICS, VARIATIONAL CALCULUS AND OPTIMAL CONTROL.

Radu Miron
University Al. I. Cuza
Faculty of Mathematics
6600 Iasi Romania

In this expository article written for a non-specialist public, I should like to relate about present day picture of the Lagrange and Hamilton geometries applied in Mechanics, Physics etc. Of course, the subject is not a simple one and the attempt to humanize it is a difficult operation. This is the reason because I divided the matter in two parts. One is devoted to an elementary exposition of these geometrical models. The second part addresses to the graduate level.

All problems discussed here can be found in the books [1],[2],[4],[5],[6].

I Considerations on the Lagrange and Hamilton geometries.

1. The Lagrange geometries.

The last 80 years were dominated by the Einsteinian model from General Relativity given by the Riemannian geometries. These were obtained using the manifolds described by test particles endowed with the physical entities, as gravitational or electromagnetic fields etc.

Especially in Relativity, the gravitational field was studied by means of Riemannian model with important results. The attempts of determination of a geometrical model which to describe the both electromagnetic and gravitational fields are very old. In 1941, G. Randers considered an interesting metric, given by

$$ds^2 = [\sqrt{a_{ij}(x)\dot{x}^i\dot{x}^j} + b_i(x)\dot{x}^i]^2 dt^2, \tag{1}$$

where $a_{ij}(x)$ are the gravitational potentials and $b_i(x)$, $(i,j = 1,..,4)$ are the electromagnetic potentials.

Even if it likes as a slight deformation of a Riemannian metric, in fact the metric (1) is a pure Finslerian metric. At present the Finsler geometry of Randers metric is an important chapter of the geometrization of the physical fields.

The Electrodynamics is gouvernated by the known Lagrangian:

$$L(x,\dot{x}) = mca_{ij}(x)\dot{x}^i\dot{x}^j + \frac{2e}{mc}b_i(x)\dot{x}^i + U(x) \tag{2}$$

m, c, e being the known physical constants.

Of course, the geometrical model based only on the Riemannian structure a_{ij} can not characterize all physical properties described by $L(x,\dot{x})$; while the variational problem of the integral of action $I(c) = \int_0^1 L(x,\dot{x})dt$ leads to the Lorentz equations

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt} = F^i_j(x)\frac{dx^j}{dt}. \tag{3}$$

In this case we need a more comprehensive model. Indeed, such a model is provided by the Lagrange space, defined by the Lagrangian (2).

On the ocassion of the Centenary of the birth of J.L.Synge an axiomatic theory of General Relativity by Ehlers, Pirani and Schield, called E.P.S'axioms, was formulated. R.Tavakol and R.Miron proved that the EPS-axioms are satisfied by the metric tensor:

$$g_{ij}(x,\dot{x}) = e^{2\sigma(x,\dot{x})}a_{ij}(x) \tag{4}$$

where $\sigma(x,\dot{x})$ is an arbitrary function. For instance, $\sigma(x,\dot{x}) = 1 - \frac{1}{n^2(x,\dot{x})}$, where $n(x,\dot{x}) > 1$ is the refractive index of a dispersive optic medium, gives a convenient consistence of the previous metric. It is not difficult to prove that, if $\frac{\partial\sigma}{\partial\dot{x}^i}$ nonvanishes, then $g_{ij}(x,\dot{x})$ is not a Riemannian metric tensor.

Finally, another example. In the Relativistic Optics, based on the Synge metric

$$g_{ij}(x,\dot{x}) = mca_{ij} + (1 - \frac{1}{n^2(x,\dot{x})})\dot{x}^i\dot{x}^j \tag{5}$$

where $a_{ij}(x)$ is a Lorentz metric, $n(x,\dot{x}) > 1$ is the refractive index of the considered optic medium and $\dot{x}_i = a_{ij}\dot{x}^j$, we can not consider g_{ij} as a Riemannian structure, because it is not reducible to a such kind of structure.

The previous examples, clearly show that we need some new geometrical models for electrodynamics or relativistic optics etc. In the following we will show that Lagrange and Hamilton geometries can be used in this respect.

A Lagrange space is a pair $L^n = (M, L(x,y))$ in which M is a real, n-dimesional differentiable manifold and $L : (x,y) \in TM \mapsto L(x,y) \in I\!R$ is a regular Lagrangian, for which the fundamental tensor $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i \partial y^j}$ has a constant signature on $\widetilde{TM} = TM \setminus \{0\}$.

Of course, $\pi : TM \longrightarrow M$ is the tangent bundle of M, $\widetilde{TM}$ is TM without the null section and (x^i, y^i), $i = 1, .., n$, are the canonical coordinates of the points $(x,y) \in TM$.

Any Riemann space $\mathcal{R}^n = (M, a_{ij}(x))$, where $a_{ij}(x)$ is the fundamental tensor field of $\mathcal{R}^n$, is a Lagrange space, with $L(x,y) = a_{ij}(x)y^iy^j$. The Finsler space $F^n = (M, F(x,y))$ is the Lagrange space $L^n = (M, F^2(x,y))$.

More general, the pair $GL^n = (M, g_{ij}(x,y))$, where $g_{ij}(x,y)$ is a d-tensor field (d-means distinguished) symmetric, nondegenerate and with constant signature is called a generalized Lagrange space.

Of course, every Lagrange space L^n is a generalized Lagrange space GL^n, but not conversely.

As exemples: The Randers spaces are Finsler spaces; The Lagrange spaces of electrodynamics, $L^n = (M, L(x,y))$ have the function $L(x,\dot{x})$ described in (2).

The metric $g_{ij}(x,y)$ from (4) determines a generalized Lagrange space GL^n. Also the Synge's metric (5) gives us a GL^n space. It is the generalized Lagrange space of Relativistic Optics, [2],[5].

Consequently, the following relations of subordinate between previous spaces, hold:

$$\{\mathcal{R}^n\} \subset \{F^n\} \subset \{L^n\} \subset \{GL^n\}.$$

It is clear that if we are in possesion of the Lagrange geometry of spaces L^n, we can take its restriction to the geometry of Finsler spaces F^n or of Riemann spaces $\mathcal{R}^n$ and we can extend it to that of generalized Lagrange spaces.

In the second part of this article we develop the geometry of Lagrange spaces based on the fundamental principles of Analytical Mechanics: 1° Variational calculus of integral of action $I(c) = \int_0^1 L(x, \dot{x})dt$ determine the Euler-Lagrange equations; 2° The canonical semispray; 3° The law of conservation of Energy; 4° The canonical nonlinear connection; 5° The metrical linear connections (structure equations); 6° Nöther theorem; 7° Applications to electrodynamics; 8° Particularization of this theory to Finsler spaces and 9° Its extension to generalized Lagrange spaces.

The geometrical theory of the spaces from the previous sequence was realized by some Schools from: Romania - R.Miron and his collaborators; Japan - M.Matsumoto et.al.; Germany - D.Langwitz, K.Buchner et.al. Russia - G.S.Asanov et.al.; S.U.A. - S.S.Chern et. al.; Great Britain - M.Crampin et. al.; Canada - P.L.Antonelli et.al.; Italy - Rizza et.al.; Hungary - L.Tamassy et.al. and many others, (see References from the books [2],[3]).

2. The Hamilton Geometries.

The Analytical Mechanics, operates in the same meassure with other fundamental notion: the Hamiltonian function $H(x,p)$ which depends on the particle $x = (x^i)$ and momenta $p = (p_i)$. Therefore an Hamiltonian function is a mapping $H : (x,p) \in T^*M \longrightarrow \mathbb{R}$, where (T^*M, π^*, M) is the cotangent bundle of the manifold M. Of course, the geometrical properties derived from H are based on the canonical geometrical objects on the total space T^*M. These are the Liouville 1-forms $\omega = p_i dx^i$, the symplectic structure $\theta = d\omega = dp_i \wedge dx^i$, and the Poisson structure $\{f,g\} = \dfrac{\partial f}{\partial p_i}\dfrac{\partial g}{\partial x^i} - \dfrac{\partial g}{\partial p_i}\dfrac{\partial f}{\partial x^i}$. Assuming H differentiable on $\widetilde{T^*M} = T^*M \setminus \{0\}$ it is not difficult to see that $\dfrac{1}{2}\dfrac{\partial^2 H}{\partial p_i \partial p_j} = g^{ij}$ is a contravariant tensor field, symmetric.

We define an Hamilton space H^n as a pair $(M, H(x,y))$ for which $det||g^{ij}||$ nonvanishes on $\widetilde{T^*M}$ and the signature of g^{ij} is constant.

It follows that the triple $(\widetilde{T^*M}, \theta, H)$ is an Hamiltonian system. It has the properties:

1°. There exists an unique vector field $X_H \in \chi(T^*M)$ for which

$$i_H\theta = -dH.$$

2°. The integral curves of X_H are given by the Hamilton-Jacobi equations:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}.$$

3°. The following formula holds: $\{f, g\} = \theta(X_f, X_g)$.

For these spaces $H^n = (M, H)$ we can determine, instead of fundamental equations Hamilton-Jacobi, a canonical metrical linear connection (and its structure equations) etc., which give the fundamental geometrical object fields in the geometry of H^n.

The relations between the Lagrange spaces $L^n = (M, L(x, y))$ and Hamilton spaces $H^n = (M, H(x, p))$ are given by the Legendre transformation $\mathcal{L}eg : L^n \longrightarrow H^n$ given by: $x^i = x^i, p_i = \dfrac{1}{2}\dfrac{\partial L}{\partial y^i}$. This is a local diffeomorphism from $\widetilde{TM}$ to $\widetilde{T^*M}$. $\mathcal{L}eg$ transform the regular Lagrangian $L(x, y)$ into the regular Hamiltonian $\mathcal{H}(x, p) = p_i y^i - \mathcal{L}(x, y)$, where y^i is given by $\mathcal{L}eg^{-1}$ and $\mathcal{H} = \dfrac{1}{2}H, \mathcal{L} = \dfrac{1}{2}L$.

By means of the Legendre diffeomorphism $\mathcal{L}eg$ the Euler-Lagrange equations from L^n are applied into the Hamilton-Jacobi equations. The fundamental object fields in L^n are transformed in fundamental object fields in H^n.

But, there are some important subclasses of Hamilton spaces H^n. One of these is that given by the Cartan spaces. Namelly, a Cartan space $C^n = (M, K(x, p))$, where $K(x, p)$ is a positive function, positively 1-homogeneous with respect to the momenta p_i and for which $H^n = (M, K^2(x, p))$ is an Hamilton space.

The Cartan spaces introduced by the author, are totally different of Cartan spaces based on the notion of area. The geometrical importance of these spaces is that they are obtained from Finsler spaces, via Legendre transformation. Therefore, spaces C^n are called dual of Finsler spaces. The geometry of C^n has the same importance, symmetry and beauty as that of Finsler spaces.

Of course any Cartan space $C^n = (M, K(x,p))$ is a Hamilton space $\mathcal{H}^n = (M, K^2(x,p))$. But not conversely.

More general, a generalized Hamilton space is a pair $GH^n = (M, g^{ij}(x,p))$, where $g^{ij}(x,p)$ is a contravariant symmetric tensor on $\widetilde{T^*M}$ having a constant signature and $rank||g^{ij}|| = n = dimM$.

The relations between the previous spaces are given by the inclusions:

$$\{\mathcal{R}^{*n}\} \subset \{C^n\} \subset \{H^n\} \subset \{GH^n\}$$

where $\mathcal{R}^{*n} = (M, g^{ij}(x))$ is a Riemann space.

A good example of Hamilton space is given by the following Hamiltonian [6]:

$$H(x,p) = \frac{1}{mc}a^{ij}(x)p_ip_j - \frac{2e}{mc^2}b^i(x)p_i + \frac{e^2}{mc^3}b^i(x)b_i(x).$$

It is "dual", via Legendre transformation, of the Lagrangian (2) of electrodynamics, with $U(x) = 0$.

The applications in Theoretical Physics, Mechanics, Biology, Relativistic Optics were done by G.Asanov, S.Ikeda, R.G.Beil, R.Tavakol, I.Roxbourgh, V.Balan, P.Stavrinos, P.L.Antonelli, R.Miron, G.Zet and many others. See the References from the book [2].

The details from this geometry can be taken from the book [6], recent published by Kluwer Acad.Publ. (2001).

3. Higher order Lagrange spaces.

The higher order Lagrangians are the functions $L(x, \dot{x}, \ddot{x}, .., \overset{(k)}{x})$ which depend on the particle $x = (x^i)$, velocity $\dot{x} = \dfrac{dx^i}{dt}$ and accelerations $\ddot{x} = \dfrac{1}{2!}\dfrac{d^2x^i}{dt^2}, .., \overset{(k)}{x} = \dfrac{1}{k!}\dfrac{d^kx^i}{dt^k}$. But, on the manifold M the accelerations of order 2,3,..,k have not a covariant meaning. Therefore there are not a *vector bundle* on which to define the higher order Lagrangians. There is a differential bundle, (T^kM, π^k, M), called the higher order accelerations bundle, or k-osculator bundle which give us an adequate geometrical framework for studying the Lagrange space of order k.

This study is important because it is basic for the Analytical Mechanics of higher order.

It is necessary to notice that in classical Lagrangian Mechanics only the acceleration $\frac{d^2x^i}{dt^2}$ is a fundamental notion, since the Newton principle $m\bar{a} = \bar{F}$. The accelerations of higher order, $k \geq 3$, are considered as derivate notions.

In the last twenty years, many scientists, as M.Crampin and collaborators, M.de Leon and collaborators, I.Kollar, D.Krupka, D.Grigore and collaborators, R.Miron and collaborators and many others investigated various aspects of the Lagrangian of higher order. In the first my visit at University of Alberta, in Edmonton, (1995), P.L.Antonelli questioned me: Have you examples of the regular Lagrangians of order $k \geq 3$? I was surprised to constat that in that time there are not of such kind of examples. The difficulty comes from the fact that the old problem, formulated by Bianchi and Bompiani, of the prolongation of the Riemannian structures g, defined on the base manifold M, to the k-osculator bundle (or to T^kM) was not solved yet.

So, I was obligated to solve the following problems:

1°. The prolongations to T^kM of the Riemannian structures g, defined on the base manifold M.

2°. The construction of good examples of nondegenerate (or regular) Lagrangians of order k.

3°. The definition of the notion of Lagrange space of order k.

4°. The study of the geometry of the manifolds T^kM. This is: the notions of k-semisprays, nonlinear connections, N-linear connections, structure equations, etc.

5°. The study of the subspaces in the Lagrange spaces $L^{(k)n}$.

6°. Applications of the previous theory in the study of Lagrange spaces of order k, denoted by $L^{(k)n} = (M, L(x, y^{(1)}, .., y^{(k)}))$.

7°. The introduction of the notion of higher order Finsler spaces $F^{(k)n} = (M, F(x,$
$y^{(1)}, .., y^{(k)}))$.

8°. The introduction of the generalized Lagrange spaces of order k, $GL^{(k)n} = (M, g_{ij}(x, y^{(1)}, .., y^{(k)}))$ such that the following sequence of inclusions hold:

$$\{\mathcal{R}^{(k)n}\} \subset \{F^{(k)n}\} \subset \{L^{(k)n}\} \subset \{GL^{(k)n}\}.$$

For $k = 1$ we have the classical sequence, above mentioned. This sequence is, of course, important in applications. The previous problems have been presented in the books [2],[3].

4. The Hamilton spaces of order $k \geq 1$.

The cotangent bundle T^*M, dual of the tangent bundle TM, carries some canonical object fields as: the Liouville vector field, a symplectic structure and a Poisson structure. They allow to construct a theory of Hamiltonian systems and via Legendre transformation, to transport this theory in that of Lagrangian systems on TM. Therefore the Lagrange spaces $L^n = (M, L(x, y))$ appear as a dual of Hamilton spaces $H^n = (M, H(x, p))$.

In the theory of Lagrange spaces of order k, where the fundamental functions are Lagrangians which depend on particle and higher order accelerations, we do not have a "dual" theory based on a good notion of Hamiltonian of order k, which depends on particle, higher order accelerations and momentum. This is because we have not a convenient differentiable bundle which carries a canonical symplectic (or presymplectic) structure, a canonical Poisson structure and its dimension be the same with dimension of manifold T^kM.

In the book [6] one can see how is possible to eliminate this inconvenient.

Indeed, starting from the k-accelerations bundle (T^kM, π^k, M) we introduce a new bundle $(T^{*k}M, \pi^{*k}, M)$ called "dual"of previous bundle, where the total space $T^{*k}M$ is the fibered product

$$T^{*k}M = T^{k-1}M \times_M T^*M.$$

The canonical coordinate on the manifold $T^{*k}M$ are $(x^i, y^{(1)i}, .., y^{(k-1)i}, p_i)$. So, dimension of $T^{*k}M$ is the same with the dimension of T^kM.

One proves that on $T^{*k}M$, $\omega = p_i dx^i$ is a canonical 1-form and $\theta = d\omega = dp_i \wedge dx^i$ is a presymplectic structure of rank $2n$. The systems of brackets:

$$\{f, g\}_\alpha = \frac{\partial f}{\partial y^{(\alpha)i}}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial y^{(\alpha)i}}\frac{\partial f}{\partial p_i}, \quad (\alpha = 0, 1, .., k-1; y(0) = x)$$

defines a canonical Poisson structure on $T^{*k}M$ for every $\alpha = 0, 1, .., k-1$.

A mapping $H : T^{*k}M \longrightarrow I\!R$ is called a Hamilton function of order k. So, $H(x, y^{(1)}, ..., y^{(k-1)}, p)$ is a function of particle x, of accelerations $y^{(1)}, .., y^{(k-1)}$ and of momenta p. H is called nondegenerate Hamiltonian if *rank* of its Hessian with respect to p_i is nonsingular.

The elements of Hessian are the components of the tensor field $g^{ij} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$.

Thus, we define the notion of Hamilton space of order k, as a pair $H^{(k)n} = (M, H(x, y^{(1)}, .., y^{(k-1)}, p_i))$, where H is a nondegenerate Hamiltonian and the tensor g^{ij} has a constant signature.

One can develop the geometry spaces $H^{(k)n}$ as a natural extension of the classical theory of Hamilton space of order $k = 1$. The relation between the Lagrange spaces of order k, $L^{(k)n} = (M, L(x, y^{(1)}, .., y^{(k-1)}, y^{(k)}))$ and the Hamilton spaces of order k, $H^{(k)n} = (M, H(x, y^{(1)}, .., y^{(k-1)}, p))$ is given by the Legendre mapping, $\mathcal{L}eg : L^{(k)n} \longrightarrow H^{(k)n}$, which is defined by

$$\mathcal{L}eg : (x, y^{(1)}, .., y^{(k-1)}, y^{(k)}) \in T^k M \longrightarrow (x, y^{(1)}, .., y^{(k-1)}, p) \in T^{*k} M$$

where $p_i = \frac{1}{2}\frac{\partial L}{\partial y^{(k)i}}$.

One proves that the Legendre mapping $\mathcal{L}eg$ is a local diffeomorphism.

Consequences one can study the geometry of Hamilton spaces of order k, $H^{(k)n}$ by means of the geometry of Lagrange spaces of order k, $L^{(k)n}$, using the Legendre mapping.

The generalized Hamilton spaces of order k, are the pairs $GH^{(k)n} = (M, g^{ij}(x, y^{(1)}, .., y^{(k-1)}, p))$, where g^{ij} is a tensor field symmetric, nondegenerate and of constant signature on $T^{*k}M$. Of course any space $H^{(k)n}$ is a $GH^{(k)n}$ space, but not conversely.

In particular, if $H(x, y^{(1)}, .., y^{(k-1)}, p)$ is positively 2-homogeneous with respect to the momenta p_i, the spaces $\mathcal{C}^{*(k)n} = (M, H)$ is called the Cartan space of order k. The spaces $\mathcal{R}^{*(k)n} = (M, H)$ with $H = g^{ij}(x)p_i p_j$, $g^{ij}(x)$ being the Riemannian (contravariant) tensor field on the manifold M, are the Riemann spaces. We get the following sequence of inclusions:

$$\{\mathcal{R}^{*(k)n}\} \subset \{\mathcal{C}^{*(k)n}\} \subset \{H^{(k)n}\} \subset \{GH^{(k)n}\}.$$

In the case $k = 1$ we have the sequence studied above.

The applications in higher order Analytical Mechanics, Theoretical Physics or Variational Calculus are remarkable. In this respect one can read the book [6], recent published by Kluwer Acad.Publ. in the Collection "Fundamental Theory of Physics."

II Technical construction of the geometries of spaces $L^n, F^n, L^{(k)n}$.

Without enter in the demonstrations, we briefly describe the construction of the geometry of the most important classes of Lagrange spaces.

1. Lagrange spaces.

The Lagrange spaces are defined over the smooth manifolds M, using the total space TM of the tangent bundle and taking the regular Lagrangians $L : TM \longrightarrow I\!R$.

These spaces were defined twentyfive years ago. They were widly developed by the author of the present article and his group. For the extensive presentation of the geometry of these spaces I refer to the books [1],[5].

Throughout the text we assume that manifolds, mappings etc. are of C^∞-class and the Einstein convention of summarizing is applied.

Let M be a real manifold, of dimension n and (TM, π, M) its tangent bundle. If $u \in TM$ and $x = \pi(u)$, then we denote $u = (x, y)$, y being a tangent vector at $x \in M$. So, the canonical coordinate of the point $u = (x, y)$ are (x^i, y^i), $(i = 1, .., n)$. The transformations of local coordinate on TM are given by

$$(1)\qquad \begin{cases} \widetilde{x^i} = \widetilde{x^i}(x^1, .., x^n), & det(\dfrac{\partial \widetilde{x^i}}{\partial x^j}) \neq 0 \\ \widetilde{y^i} = \dfrac{\partial \widetilde{x^i}}{\partial x^j} y^j. \end{cases}$$

The natural basis $(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^i})$ in $T_u(TM)$ is transformed by (1) as follows

$$(2)\qquad \frac{\partial}{\partial x^j} = \frac{\partial \widetilde{x^i}}{\partial x^j}\frac{\partial}{\partial \widetilde{x^i}} + \frac{\partial \widetilde{y^i}}{\partial x^j}\frac{\partial}{\partial \widetilde{y^i}}, \quad \frac{\partial}{\partial y^j} = \frac{\partial \widetilde{x^i}}{\partial x^j}\frac{\partial}{\partial \widetilde{y^i}}.$$

Therefore the vector fields $(\dfrac{\partial}{\partial y^1}, .., \dfrac{\partial}{\partial y^n})$ spanned a distribution $V : u \in TM \longrightarrow V_u \in T_uTM, \forall u \in TM$. This is the vertical distribution on TM.

The vector field, locally given by

$$(3)\qquad \Gamma = y^i \frac{\partial}{\partial y^i}$$

is globally defined on TM and belongs to the vertical distribution V. It is called the Liouville vector field.

Consider the $\mathcal{F}(TM)$-linear mapping $J : \chi(TM) \longrightarrow \chi(TM)$,

$$(4) \qquad J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}, \quad J(\frac{\partial}{\partial y^i}) = 0.$$

Thus, J is an *almost tangent structure*, globally defined on TM. We have

$$J^2 = 0, Im\ J = Ker\ J = V, rank\ J = n.$$

A vector field $S \in \chi(TM)$ is called a semispray if

$$(5) \qquad JS = \Gamma.$$

Locally, S is uniquelly expressed in the form

$$(6) \qquad S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}.$$

The integral curves of S are

$$(6)' \qquad y^i = \frac{dx^i}{dt}; \ \frac{dy^i}{dt} + 2G^i(x,y) = 0.$$

The system of functions $G^i(x,y)$ gives *the coefficients* of semispray S.

This notion of semispray is fundamental in the construction of the main geometrical object fields on TM.

Now, let $\widetilde{TM} = TM \setminus \{0\}$ be the open submanifold of TM, formed by points (x^i, y^i) with property $rank||y^1, ..., y^n|| = 1$.

Definition 1 *A Lagrange space is a pair $L^n = (M, L(x,y))$ with $L : (x,y) \in TM \longrightarrow L(x,y) \in I\!R$, such that:*

1°. L is differentiable on $\widetilde{TM}$ and continuous on the null section $0 : M \longrightarrow TM$ of the projection π.

2°. The Hessian of L is nonsingular, i.e.

a. $rank|||g_{ij}|| = n$, on $\widetilde{TM}$,

b. $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i \partial y^j}$.

3°. The tensor field $g_{ij}(x,y)$ has a constant signature on $\widetilde{TM}$.

L is called the fundamental function and g_{ij} is called fundamental tensor of the space L^n.

If M is a paracompact manifold, then TM has the same property and there exists pairs $L^n = (M, L)$ which are Lagrange spaces. An example is given by the Lagrange space of electrodynamics, described in the first section.

Starting from the integral of action of the Lagrangian L and based only on the principles of Analytical Mechanics we will determine the fundamental geometrical object fields of the Lagrange space $L^n = (M, L)$.

A curve $c : t \in [0,1] \longrightarrow (x^i(t)) \in U \subset M$, (with a fixed parameterization) has the extension to $\widetilde{TM}$ given by $c^* : t \in [0,1] \longrightarrow (x^i(t), \dfrac{dx^i}{dt}(t)) \in \pi^{-1}(U) \subset \widetilde{TM}$.

The integral of action of the Lagrangian $L(x,y)$ on the curve c is given by the functional:

$$I(c) = \int_0^1 L(x(t), \frac{dx}{dt}(t))dt. \tag{7}$$

Applying the variational principle to the functional $I(c)$ we obtain the Euler-Lagrange equations:

$$E_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} = 0, \quad y^i = \frac{dx^i}{dt}. \tag{8}$$

The curves c solutions of the differential equations (8) are called extremal curves of the space L^n.

The notion of energy of the space L^n can be defined as in Theoretical Mechanics, as follows:

$$E_L = y^i \frac{\partial L}{\partial y^i} - L.$$

It is not difficult to prove the sentences:

The energy E_L of the Lagrange space L^n is constant on every extremal curve.

Also, a Nöther theorem holds.

For the fundamental function $L(x,y)$ of the space L^n, the Euler-Lagrange equations (8) can be written in the equivalent form:

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0, \quad y^i = \frac{dx^i}{dt} \tag{8'}$$

where

$$2G^i(x,y) = \frac{1}{2}\left(\frac{\partial^2 L}{\partial y^j \partial x^h}y^h - \frac{\partial L}{\partial x^i}\right) \tag{9}$$

g^{ij} being the contravariant tensor of fundamental tensor g_{ij}.

But (8)' give the integral curves of the semispray S, (6), with the coefficients (9). Of course, this is a canonical semispray since the coefficients are determined only by the fundamental function $L(x,y)$ of the space.

Throughout in the following considerations we will based on this canonical semispray S.

Taking into account the coefficients G^i, (9), of the canonical semispray S we can introduce the system of functions

$$N^i{}_j = \frac{\partial G^i}{\partial y^j}. \tag{10}$$

So, we obtain a geometrical object field on $\widetilde{TM}$ with the law of transformation

$$\widetilde{N^i}_m \frac{\partial \widetilde{x^m}}{\partial x^j} = N^m{}_j \frac{\partial \widetilde{x^i}}{\partial x^m} - \frac{\partial \widetilde{y^i}}{\partial x^j}. \tag{10'}$$

Therefore $N^i{}_j(x,y)$ are called the coefficients of a canonical nonlinear connection. They depends only on the fundamental function of the space L^n.

It is not difficult to prove that $\dfrac{\delta}{\delta x^i}$:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^j}, \quad (i = 1, .., n), \tag{11}$$

are n-independent vector fields. They spann a new distribution on $\widetilde{TM}$, denoted by N, supplementary to the vertical distribution V, i.e.:

$$T_u(TM) = N_u \oplus V_u, \quad \forall u \in \widetilde{TM}. \tag{12}$$

By the way, we determine a local basis $\left(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right)$, $(i = 1, .., n)$, adapted to distributions N and V or adapted to the direct decomposition (12).

The canonical nonlinear connection N is extemly useful in construction of a canonical linear connection $D\Gamma(N)$. This has the coefficients $(L^i_{jk}(x,y),\ C^i_{jk}(x,y))$ in the adapted basis and leads to two operators of h- and v-covariant derivatives, denoted by "$_|$" and "$|$".

So, for the fundamental tensor field g_{ij}, h- and v-covariant derivatives are given, respectively, by:

$$
g_{ij|h} = \frac{\delta g_{ij}}{\delta x^h} - g_{sj} L^s_{ih} - g_{is} L^s_{jh},
\tag{13}
$$
$$
g_{ij}|_h = \frac{\partial g_{ij}}{\partial y^h} - g_{sj} C^s_{ih} - g_{is} C^s_{jh}.
$$

It is interesting to remark that, with respect to changing of local coordinate (1), the coefficients L^i_{jk} obey the same rule of transformation as the coefficients of Levi-Civita connection on the base manifold M. While C^i_{jk} are the coefficients of a tensor field of type (1,2).

The canonical metrical connection $D\Gamma(N)$ is given by:

Theorem 1 *The following properties hold:*

1°. *There exists an unique N-linear connection D on $\widetilde{TM}$ verifying the axioms:*

$$
g_{ij|h} = 0, \quad g_{ij}|_h = 0,
\tag{14}
$$

$$
L^i_{jh} = L^i_{hj}, \quad C^i_{jh} = C^i_{hj}.
\tag{15}
$$

2°. *This connection has as coefficients the generalized Christoffel symbols*

$$
\begin{aligned}
L^i_{jh} &= \frac{1}{2} g^{is} \left\{ \frac{\delta g_{sh}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^h} - \frac{\delta g_{jh}}{\delta x^s} \right\}, \\
C^i_{jh} &= \frac{1}{2} g^{is} \left\{ \frac{\partial g_{sh}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^h} - \frac{\partial g_{jh}}{\partial y^s} \right\}.
\end{aligned}
\tag{16}
$$

3°. *The previous connection depends only on the fundamental function $L(x,y)$ of the space L^n.*

Let $(dx^i, \delta y^i)$ be the dual basis of the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$. It follows for 1-form fields $\delta y^1, .., \delta y^n$, the expressions:

$$
\delta y^i = dy^i + N^i{}_j dx^j.
\tag{17}
$$

Therefore the geometrical object fields

$$\omega^i_j = L^i_{jh}dx^h + C^i_{jh}\delta y^h \tag{18}$$

are the so called 1-forms connection of $D\Gamma(N)$.

Theorem 2 *The structure equations of the canonical metrical connection $D\Gamma(N)$ are given by*

$$\begin{aligned} d(dx^i) - dx^s \wedge \omega^i_s &= -\overset{(0)}{\Omega^i}, \\ d(\delta y^i) - \delta y^s \wedge \omega^i_s &= -\overset{(1)}{\Omega^i}, \\ d\omega^i_j - \omega^s_j \wedge \omega^i_s &= -\Omega^i_j, \end{aligned} \tag{19}$$

where $\overset{(0)}{\Omega^i}, \overset{(1)}{\Omega^i}$ *are the 2-forms of torsion:*

$$\overset{(0)}{\Omega^i} = C^i_{js}dx^j \wedge \delta y^s; \quad \overset{(0)}{\Omega^i} = \frac{1}{2}R^i_{js}dx^j \wedge dx^s + P^i_{js}dx^j \wedge dx^s, \tag{19$'$}$$

and Ω^i_j *are the 2-forms of curvature:*

$$\Omega^i_j = \frac{1}{2}R_j{}^i{}_{rs}dx^r \wedge dx^s + P_j{}^i{}_{rs}dx^r \wedge \delta y^s + \frac{1}{2}S_j{}^i{}_{rs}\delta y^r \wedge \delta y^s. \tag{19$''$}$$

In the previous equations we set:

$$R^i_{jk} = \frac{\delta N^i{}_j}{\delta x^k} - \frac{\delta N^i{}_k}{\delta x^j}; \quad P^i_{jk} = \frac{\partial N^i{}_j}{\partial y^k} - L^i_{kj}, \tag{20}$$

and

$$\begin{cases} R_j{}^i{}_{kh} = \dfrac{\delta L^i_{jk}}{\delta x^h} - \dfrac{\delta L^i_{jh}}{\delta x^k} + L^s_{jk}L^i_{sh} - L^s_{jh}L^i_{sk} + C^i_{hs}R^s_{kh}, \\ P_j{}^i{}_{kh} = \dfrac{\partial L^i_{jk}}{\partial y^h} - C^i_{jk|h} + C^i_{js}P^s_{kh}, \\ S_j{}^i{}_{kh} = \dfrac{\partial C^i_{jk}}{\partial y^h} - \dfrac{\partial C^i_{jh}}{\partial y^k} + C^s_{jk}C^i_{sh} - C^s_{jh}C^i_{sk}. \end{cases} \tag{21}$$

We call R^i_{jk}, P^i_{jk} the torsion of $D\Gamma(N)$ and $R_j{}^i{}_{kh}, P_j{}^i{}_{kh}, S_j{}^i{}_{kh}$ the curvatures of this connection.

Now we can remark:

Whole geometry of the Lagrange space $L^n = (M, L(x,y))$ can be based on the canonical nonlinear connection N, the canonical metrical connection $D\Gamma(N)$ and the structure equations (19),(19)' and (19)".

The applications in Theoretical Physics, as Einstein equations od $D\Gamma(N)$ or Maxwell equations of $D\Gamma(N)$ can be find in the books [2],[4].

It is important to remark that in the case of Lagrange space of electrodynamics, $L^n = (M, L)$, with the fundamental function

$$L(x,y) = mca_{ij}(x)y^i y^j + \frac{2e}{mc} b_i(x) y^i,$$

the Einstein equations and Maxwell equations, determinate on this geometrical models are the classical one.

2. Finsler spaces.

An important class of Lagrange spaces is provided by the famous Finsler spaces.

Definition 2 *A Finsler space is a pair $F^n = (M, F(x,y))$, formed by a real n-dimensional manifold M and a scalar function $F : TM \longrightarrow I\!R$ having the properties:*

1°. F is differentiable on $\widetilde{TM}$ and continuous on the null section $0 : M \longrightarrow TM$ of the projection $\pi : TM \longrightarrow M$.

2°. F is positive.

3°. F is positively homogeneous of degree 1, with respect to y^i on $\widetilde{TM}$.

4°. The Hessian of F^2, with elements

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{1}$$

is positively defined on $\widetilde{TM}$.

The function F is called fundamental and g_{ij} is called fundamental or metric tensor of F^n.

It follows that g_{ij} is 0-homogeneous, symmetric and nondegenerate. So its contravariant g^{ij} can be considered. Therefore the Finsler spaces can be looked as the particular Lagrange space $L^n = (M, F^2(x,y))$. The previous theory can be applied, taking the regular Lagrangian $L(x,y) = F^2(x,y)$. Thus $L(x,y)$ is 2-homogeneous with respect to y^i.

If we denote by $\gamma^i_{jh}(x,y)$ the Christoffel symbols of the Finsler space F^n, after a straightforward calculus in the formula (9) we find the coefficients of the canonical semispray of the space F^n :

$$G^i = \frac{1}{2}\gamma^i_{jh}(x,y)y^j y^h. \tag{2}$$

Consequently, the canonical nonlinear connection N of the Finsler space F^n is exactly the canonical Cartan nonlinear connection. Its coefficients are in the Cartan form:

$$N^i{}_j = \frac{1}{2}\frac{\partial}{\partial y^j}\left\{\gamma^i_{rs}(x,y)y^r y^s\right\}. \tag{3}$$

Theorem 1 give us the coefficients of the famous Cartan metrical connection of the Finsler space F^n, denoted by $C\Gamma(N)$. In Theorem 2 we determine the structure equations of the Cartan metrical connection $C\Gamma(N)$. The Bianchi identities of $C\Gamma(N)$ are obtained from (19) by calculating the exterior differential of (19), modulo the same system and using the exterior differential of 2-forms of torsion $\overset{(0)}{\Omega^i}, \overset{(1)}{\Omega^i}$ and Ω^i_j. It is not difficult to prove:

Theorem 3 *The Cartan metrical connection $C\Gamma(N)$ of a Finsler space $F^n = (M, F(x,y))$ has the properties:*

1°. $F_{|h} = 0$, $F^2_{|h} = 0$, $F^2|_h = \dfrac{\partial F^2}{\partial y^k}$.

2°. *F is constant on the autoparallel curves of the Cartan nonlinear connection N.*

3°. *The Liouville vector field y^i has the h- and v-covariant derivatives*

$$y^i_{|h} = 0, \quad y^i|_h = \delta^i_h.$$

4°. *$C\Gamma(N)$ is a metrical connection:*

$$g_{ij|h} = 0, \quad g_{ij}|_h = 0.$$

5°. $R^i_{jh} = y^s R_s{}^i{}_{jh}, P^i_{jh} = y^s P_s{}^i{}_{jh}, 0 = y^s S_s{}^i{}_{jh}$.

6°. *The h- and v-torsion* T^i_{jh}, S^i_{jh} *vanish.*

7°. *The tensor* $P_{ijh} = g_{is}P^s_{jh}$ *is complete symmetric.*

8°. $y^sC^i_{sj} = 0,\ y^sC_{sjh} = 0,\ C_{ijh} = \dfrac{1}{2}\dfrac{\partial g_{ij}}{\partial y^h}.$

9°. $R_{ijh} + R_{jhi} + R_{hij} = 0, R_{ijh} = g_{is}R^s_{jh}.$

10°. *The covariant tensors of curvature* $R_{ijhk}, P_{ijhk}, S_{ijhk}$ *are skewsymmetric in the first two indices.*

11°. *In the canonical parameterization, the Euler-Lagrange equations determines the equations of geodesics of space* F^n :

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x, \frac{dx}{ds})\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

Examples.

1°. Let $\mathcal{R}^n = (M, a_{ij}(x))$ be a Riemann space and $\beta = b_i(x)dx^i$ an 1-form on M. The function

$$F(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i,$$

with $b_i(x)y^i > 0$ on an open set from $\widetilde{TM}$, is a fundamental function of a Finsler space. It is used in the geometrical theory of gravitational and electromagnetic fields, in cases when a_{ij} is a semiriemannian structure.

2°. In the same conditions, the function

$$F(x,y) = \frac{a_{ij}(x)y^iy^j}{b_i(x)y^i}$$

is the fundamental function of a Finsler space. It is called Kropina space.

3°. The function

$$F(x,y) = e^{2\alpha_ix^i}\left\{(y^1)^m + \cdots + (y^n)^m\right\}^{\frac{1}{m}},$$

m integer, $m \geq 3$, $\alpha_i = const. \neq 0$ and $F(x, y)$ is expressed in a preferential chart on $\widetilde{TM}$, is a fundamental function of a Finsler space. F is called Antonelli's ecological metric.

3. Almost Kählerian Models of Lagrange and Finsler spaces.

We shall see that the Lagrange spaces $L^n = (M, L)$ or Finsler spaces $F^n = (M, F)$ endowed with the canonical metrical connection $C\Gamma(N)$ can be thought of as the almost Kähler spaces on the manifold $\widetilde{TM}$. We say that such a space is an almost Kählerian model for L^n or F^n.

Moreover, a Lagrangian or Finslerian theory of gravitational and electromagnetic field can be geometrically studied much better on the almost Kählerian model, since the almost symplectic structure of the space is a symplectic one and the nonlinear connection is essential included into this model.

Let be a Lagrange or Finsler space having $g_{ij}(x,y)$ as fundamental tensor and N, with the coefficients $N^i_j(x,y)$ as canonical nonlinear connection.

As usually we consider $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$ the adapted basis to the distribution N and V. Its dual basis is $(dx^i, \delta y^i)$.

Let us define the $\mathcal{F}(\widetilde{TM})$-linear mapping $\mathbb{F} : \chi(\widetilde{\mathrm{TM}}) \longrightarrow \chi(\widetilde{\mathrm{TM}})$:

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i};\ \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}, \quad (i = 1, .., n). \tag{1}$$

It is not difficult to prove that $\mathbb{F}$ is well defined on $\widetilde{TM}$ and it is the following tensor field of type (1,1):

$$\mathbb{F} = -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^i. \tag{1'}$$

Evidently, we have

$$\mathbb{F} \circ \mathbb{F} = -I. \tag{2}$$

So, we obtain:

Theorem 4 1°. *The mapping $\mathbb{F}$ from (1) is an almost complex structure on $\widetilde{TM}$ determined only on the fundamental function of space L^n (or F^n).*

2°. *The structure $\mathbb{F}$ is integrable if and only if the canonical nonlinear connection is integrable.*

Now, let us consider the fundamental tensor $g_{ij}(x,y)$. It determines a pseudo-Riemannian (or Riemannian) structure $\mathbb{G}$ on the manifold $\widetilde{TM}$:

$$\mathbb{G} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j. \tag{3}$$

Theorem 5 1°. $\mathbb{G}$ *is a pseudo-Riemannian (or Riemannian) structure on the manifold* $\widetilde{TM}$, *determined only by the fundamental function of the space.*

2°. *The distributions* N *and* V *are orthogonal with respect to* $\mathbb{G}$.

The pair of structures $(\mathbb{F},\mathbb{G})$ is important in the geometry of the considered space. In this idea we consider the Poincaré-Cartan 1-form ω and 2-form θ :

$$\omega = \frac{1}{2}\frac{\partial L}{\partial y^i}dx^i \tag{4}$$

$$\theta = g_{ij}(x,y)\delta y^i \wedge dx^j. \tag{5}$$

So, we can prove:

Theorem 6 1°. *The following equations hold:*

$$\theta = d\omega, \quad d\theta = 0. \tag{6}$$

2°. θ *is a symplectic structure on* $\widetilde{TM}$, *determined only by the fundamental function* $L(x,y)$ *(or* $F(x,y)$*) of the space* L^n *(or* F^n*).*

Finally, one proves an important result:

Theorem 7 1°. *The pair* $(\mathbb{G},\mathbb{F})$ *is an almost Hermitian structure on* $\widetilde{TM}$ *depending only on the fundamental function* $L(x,y)$ *(or* $F(x,y)$*) of the Lagrange (or Finsler) space* L^n, *(respectively* F^n*).*

2°. *The 2-form associated to the structure* $(\mathbb{G},\mathbb{F})$ *is* θ *given by (5).*

3°. *For any Lagrange space* $L^n = (M,L)$ *or Finsler space* $F^n = (M,F)$ *the space* $K^{2n} = (\widetilde{TM},\mathbb{G},\mathbb{F})$ *is an almost Kählerian space.*

The space K^{2n} is called the almost Kählerian model of the space L^n (or F^n).

We can ude the space K^{2n}, the Einstein equations and Maxwell equations for the gravitational and electromagnetic fields defined by the fundamental function L (or F) of the considered Lagrange (or Finsler) space.

4. Higher order Lagrange spaces.

The existence of Lagrangians which depend on the accelerations of order $k>1$ has been contemplated by mechanicists and physicists for a long time.

Einstein has grasped their presence in connection with the Brownian motion. The variational calculus systematically used the Lagrangians of higher order accelerations: $L(x, \frac{dx}{dt}, .., \frac{1}{k!}\frac{d^kx}{dt^k})$. But, generally such kind of Lagrangians are not regular. This is, the Hessian of L with respect to $y^{(k)i} = \frac{1}{k!}\frac{d^kx}{dt^k}$ is degenerate.

Some good example of higher order regular Lagrangians are given by solving the problem of prolongation to T^kM of the Riemannian structure given on the base manifold M.

Let $\mathcal{R}^n = (M, g_{ij}(x))$ be a Riemann space and (T^kM, π^k, M) its bundle of accelerations of order k. The canonical coordinate of T^kM, of a point $(x, y^{(1)}, .., y^{(k)})$ are $(x^i, y^{(1)i}, .., y^{(k)i})$.

A nonlinear connection N on T^kM is characterized by the dual coefficients $\underset{(1)}{M}{}^i_j, .., \underset{(k)}{M}{}^i_j$ from the adapted cobasis:

$$
\begin{aligned}
&\delta x^i = dx^i, \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M}{}^i_j dx^j, ...,\\
&\delta y^{(k)i} = dy^{(k)i} + \underset{(1)}{M}{}^i_j dy^{(k-1)j} + \cdots + \underset{(k-1)}{M}{}^i_j dy^{(1)j} + \underset{(k)}{M}{}^i_j dx^j.
\end{aligned} \tag{1}
$$

The first important result is given by:

Theorem 8 *For any Riemann space $\mathcal{R}^n = (M, g_{ij}(x))$ there exist nonlinear connections on $\widetilde{T^kM}$ determined only by metric tensor $g_{ij}(x)$. One of them has the dual coefficients*

$$
\begin{aligned}
&\underset{(1)}{M}{}^i_j = \gamma^i_{js}(x) y^{(1)s}, \quad \underset{(2)}{M}{}^i_j = \frac{1}{2}\left\{\Gamma \underset{(1)}{M}{}^i_j + \underset{(1)}{M}{}^i_s \underset{(1)}{M}{}^s_j\right\}, ...,\\
&\underset{(k)}{M}{}^i_j = \frac{1}{k!}\left\{\Gamma \underset{(k-1)}{M}{}^i_j + \underset{(1)}{M}{}^i_s \underset{(k-1)}{M}{}^s_j\right\}
\end{aligned} \tag{2}
$$

where $\gamma^i_{js}(x)$ are the Christoffel symbols of the tensor $g_{ij}(x)$ and Γ is the following nonlinear operator

$$
\Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + \cdots + k y^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}. \tag{3}
$$

Now let us consider the distinguished Liouville vector field:

$$kz^{(k)i} = ky^{(k)i} + (k-1)\underset{(1)}{M}{}^{i}_{j}y^{(k-1)j} + \cdots + \underset{(k-1)}{M}{}^{i}_{j}y^{(1)j}. \tag{4}$$

We obtain the second important result:

Theorem 9 *The following Lagrangian is defined on $\widetilde{T^kM}$, is regular and it is determined only by Riemannian structure $g_{ij}(x)$:*

$$L(x, y^{(1)}, .., y^{(k)}) = g_{ij}(x)z^{(k)i}z^{(k)j}. \tag{5}$$

Also, we have:

Theorem 10 *The tensor field $\mathbb{G}$ on $\widetilde{T^kM}$, given by*

$$\mathbb{G} = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^{(1)i} \otimes \delta y^{(1)j} + \cdots + g_{ij}(x)\delta y^{(k)i} \otimes \delta y^{(k)j}$$

has the property:

$\mathbb{G}$ is a Riemannian structure on $\widetilde{T^kM}$, determined only by the Riemann space $\mathcal{R}^n = (M, g_{ij}(x))$.

The pair $Prol^k\mathcal{R}^n = (\widetilde{T^kM}, \mathbb{G})$ is the prolongation of order k to T^kM of the Riemannian space $\mathcal{R}^n$.

Theorem 9 solves the problem of existence of regular Lagrangians of higher order.

Let us consider the mapping $L : T^kM \longrightarrow I\!R$ differentiable on $\widetilde{T^kM}$ and continuous on the null section. L is called a differentiable Lagrangian of order k. The Hessian of L with respect to variables $y^{(k)i}$ has the elements

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k)j}}. \tag{6}$$

L is called regular if $rank||g_{ij}|| = n$ on $\widetilde{T^kM}$. g_{ij} is called the fundamental tensor of L.

Definition 3 *A Lagrange space of order k is a pair $L^{(k)n} = (M, L(x, y^{(1)}, .., y^{(k)}))$ where L is a regular Lagrangian of order k and its fundamental tensor g_{ij} has a constant signature on $\widetilde{T^kM}$.*

Theorem 9 prove the existence of these spaces on the paracompact manifold M.

The geometry of spaces $L^{(k)n}$ can be constructed by means of the Euler-Lagrange equations.

Indeed, the integral of action of the Lagrangian $L(x, y^{(1)}, .., y^{(k)})$ is the functional

$$I(c) = \int_0^1 L(x(t), \frac{dx}{dt}, ..., \frac{1}{k!}\frac{d^k x}{dt^k})dt.$$

The variational problem of $I(c)$ leads to the Euler-Lagrange equations:

$$(7)\quad \begin{cases} \overset{\circ}{E}_i(L) := \dfrac{\partial L}{\partial x^i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial y^{(1)i}} + \cdots + (-1)^k \dfrac{1}{k!}\dfrac{d^k}{dt^k}(\dfrac{\partial L}{\partial y^{(k)i}}) = 0 \\ y^{(1)i} = \dfrac{dx^i}{dt}, .., y^{(k)i} = \dfrac{1}{k!}\dfrac{d^k x^i}{dt^k}. \end{cases}$$

$\overset{\circ}{E}_i(L)$ is a covector field.

The notions of higher order energies $\varepsilon^k(L), ..., \varepsilon^1(L)$ can be introduced. One proves that the energy of order k, $\varepsilon_c^k(L)$ is conserved along to the integral curves of the equations $\overset{\circ}{E}_i(L) = 0$. A Nöther theorem one demonstrates, too.

There are some covector, which depend by L only, important in the geometry of the spaces $L^{(k)n}$. One of them is as follows

$$\overset{k-1}{E}_i(L) = (-1)^{k-1}\frac{1}{(k-1)!}\frac{\partial L}{\partial y^{(k-1)i}} - \frac{d}{dt}\frac{\partial L}{\partial y^{(k)i}}.$$

The equation $\overset{k-1}{E}_i(L) = 0$ determine a k-semispray:

(8)

$$S = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i(x, y^{(1)}, .., y^{(k)})\frac{\partial}{\partial y^{(k)i}}$$

with the coefficients

$$(8)'\qquad (k+1)G^i = \frac{1}{2}g^{ij}\{\Gamma(\frac{\partial L}{\partial y^{(k)j}}) - \frac{\partial L}{\partial y^{(k-1)j}}\}.$$

This is the canonical k-semispray of the space $L^{(k)n}$.

So, we obtain an important result,[Miron]:

Theorem 11 *For any Lagrange space of order k, $L^{(k)n} = (M, L)$ there exists the nonlinear connection N depending only by Lagrangian L. One of them has the dual coefficients*

$$(9)\qquad \begin{aligned} &\underset{(1)}{M}{}^i_j = \frac{\partial G^i}{\partial y^{(k)j}},\quad \underset{(2)}{M}{}^i_j = \frac{1}{2}\left\{S\,\underset{(1)}{M}{}^i_j + \underset{(1)}{M}{}^i_s\,\underset{(1)}{M}{}^s_j\right\},\ldots,\\ &\underset{(k)}{M}{}^i_j = \frac{1}{k!}\left\{S\,\underset{(k-1)}{M}{}^i_j + \underset{(1)}{M}{}^i_s\,\underset{(k-1)}{M}{}^s_j\right\}\end{aligned}$$

where S is the canonical k-semispray.

Other fundamental notion is that of N-linear connection. It is a natural extension of the N-linear connection $D\Gamma(N)$ of the Lagrange space of order 1. The fundamental tensor g_{ij} of the space $L^{(k)n}$ is covariant constant with respect to $D\Gamma(N)$ if

$$(*)\qquad g_{ij|h} = 0,\quad g_{ij}\overset{(\alpha)}{|}_h = 0,\qquad (\alpha = 1,..,k).$$

One proves that:

There exists an unique $D\Gamma(N)$ which depend only on the regular Lagrangian $L(x, y^{(1)}, .., y^{(k)})$, without some torsion and for which (*) are verified. Its coefficients are given by the generalized Christoffel symbols:

$$(10)\qquad \begin{cases} L^i_{jh} &= \dfrac{1}{2}g^{is}\left\{\dfrac{\delta g_{sh}}{\delta x^j} + \dfrac{\delta g_{js}}{\delta x^h} - \dfrac{\delta g_{jh}}{\delta x^s}\right\},\\ \underset{(\alpha)}{C}{}^i_{jh} &= \dfrac{1}{2}g^{is}\left\{\dfrac{\delta g_{sh}}{\delta y^{(\alpha)j}} + \dfrac{\delta g_{js}}{\delta y^{(\alpha)h}} - \dfrac{\delta g_{jh}}{\delta y^{(\alpha)s}}\right\},\quad (\alpha = 1,..,k).\end{cases}$$

$D\Gamma(N)$ with the coefficients (10) is the canonical metrical connection of the Lagrange space of order k, $L^{(k)n} = (M, L)$.

Whole geometry of spaces $L^{(k)n}$ can be developed only by means of canonical nonlinear connection N and canonical metrical connection $D\Gamma(N)$.

The Finsler spaces of order k are defined by the pair $F^{(k)n} = (M, F(x, y^{(1)}, .., y^{(k)}))$ where $F : T^kM \longrightarrow \mathbb{R}$ is positive, smooth on $\widetilde{T^kM}$, k-homogeneous on the fibres of T^kM. F^2 is the fundamental function of a Lagrange space of order k.

The geometry of these spaces is a special case of that of higher order Lagrange spaces.

I finish with remark that the applications of these theories are find in the books [1],[2],[4],[5].

References

[1] **Antonelli, P.L.** and **Miron, R.**, (Eds.), *Lagrange and Finsler Geometry. Applications to Physics and Biology,* Kluwer Acad. Publ. FTPH no. 76, (1996).

[2] **Miron, R.**, *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics,* Kluwer, Dordrecht, FTPH no. 82, (1997).

[3] **Miron, R.**, *The Geometry of Higher Order Finsler Spaces,* Hadronic Press., Inc. USA, (1998).

[4] **Miron, R.**, and **Anastasiei, M.**, *Vector Bundles and Lagrange Spaces with Applications to Relativity,* Geometry Balkan Press, Bucureşti, no.1, (1997).

[5] **Miron, R.** and **Anastasiei, M.**, *The geometry of Lagrange spaces: Theory and Applications,* Kluwer Acad. Publ., FTPH no. 59, (1994).

[6] **Miron, R.**, **Hrimiuc, D.**, **Shimada, H.**, and **Sabau, V.S.**, *The Geometry of Hamilton and Lagrange spaces,* Kluwer Acad. Publ., FTPH, series Nr. 118, (2001).