Cubo Matemática Educacional
Vol., 4. Nº 2, Junio 2002

RECENT DEVELOPMENTS IN TAXICAB GEOMETRY

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Abstract

Since the time of René Descartes, anlaytic geometry , which is based on the Euclidean metric, has been a popular area of research. In recent years, mathematicians begin to investigate geometry using other metrics such as the taxicab metric. In this paper, results from the research in taxicab metric and related areas are presented. Various ideas and directions for research in taxicab geometry are introduced and discussed to stimulate further research interest in this area.

KEYWORDS AND PHRASES: Bilinear , diamond trigonometric functions, conic sections, eccentricity, iso-taxicab metric, λ-geometry, norm, positive definite, pyramidal sections, taxicab circles, taxicab directed distance, taxicab ellipses, taxicab geometry, taxicab hyperbolas, taxicab inner product, taxicab trigonometry.

AMS(MOS) SUBJ. CLASSIFICATION CODES: 51-02.

1. Introduction. In La gémétrie, René Descartes revealed many of his contributions in analytic geometry, which opened up new avenues to the study of Euclidean geometry. In the late 1800s, Herman Minkowski [17] published a whole family of metrics providing new insight to the study of analytic geometry. Among them, taxicab metric becomes the most popular within the mathematics community, thus it is worthwhile to investigate the importance of taxicab geometry.

In the following sections, many important and interesting results from taxicab geometry and trigonometry, their applications, and their related topics will be presented and discussed. Furthermore, ideas, problems, and directions for future research will be discussed.

Let X be an non-empty set. A *metric* d on X is a function from $X \times X$ to $[0, \infty)$ satisfying the conditions:

(1) $d(P, Q) = 0$ if and only if $P = Q$;
(2) $d(P, Q) = d(Q, P)$;
(3) $d(P, Q) \leq d(P, R) + d(R, Q)$ for all P, Q, R, in X.

In particular, if $X = \Re^n$ for some natural number n, then the function $d_T : \Re^n \times \Re^n \longrightarrow [0, \infty)$, defined by $d_T(P, Q) = \sum_{i=1}^{n} |p_i - q_i|$ where $P = (P_1, ..., P_n)$ and $Q = (q_1, ..., q_n)$, is called the taxicab metric defined on $\Re^n$.

2. Taxicab metric and taxicab geometry on the plane

Although Minkowski [20], Blumenthal [3], and Jacobs [13] have discussed distance goemetry besides the traditional analytic geometry, Krause [17] should be credited for his contribution in popularizing taxicab geometry in recent years. After defining the taxicab metric in $\Re^2$ and comparing it to analytic geometry, Krause discussed, without giving much details, how taxicab geometry can be used in real-life applications and how it can provide simple examples in non-Euclidean geometry. To fill in the missing details, S. So, & Z. Al-Maskari [25] constructed two simple examples showing why taxicab geometry is non-Eulidean.

Inspired by Krause [17], Reynolds [23] defined and studied conics by means ofthe taxicab metric as follow.

Definitions. Let $(h, k) \in \Re^2$, and $r > 0$. Then the set $C = \{(x, y) : d_T((x, y), (h, k)) = r\}$ is called a *taxicab circle* with *center* (h, k) and *radius* r; let $F_1(h_1, k_1), F_2(h_2, k_2) \in \Re^2$ and $c > 0$. Then the set $E = \{P \in \Re^2 : d_T(P, F_1) + d_T(P, F_2) = c\}$ is called a *taxicab ellipse* with *foci* $F_1$and F_2 and *associated constant* c; the set $H = \{P \in \Re^2 : d_T(P, F_1) - d_T(P, F_2) = \pm c\}$ is called a *taxicab hyperbola* with *foci* F_1 and F_2 and *associated constant* c; let $F(h, k) \in \Re^2$ and D be a line on $\Re^2$. Then the set $K = \{P \in \Re^2 : d_T(P, F) = d_T(P, D)\}$ is called a *taxicab parabola* with *focus* F and *directrix* D, where $d(P, D)$ is the distance from P to the point on D which is closest to P.

Reynold [23] then observed that taxicab circles are in "diamond shape";

taxicab ellipses are in "diamond", hexagonal or octagonal in shape. Concerning taxicab hyperbolas, she concluded that their shapes depend on both the associated constant c and the taxicab distance between the foci. In her thesis, Liu [19] determined exactly when a taxicab ellipse is 4–sided, 6-sided, and 8-sided as summarized in the next theorem. She also determined exactly how the shapes of taxicab hyperbolas vary according to the associated constant c and the taxicab distance between the foci. The figure below shows a sample of the graphs of taxicab ellipses and hyperbolas determined by Reynold and Liu.

Theorem 2.1. *Let E be a taxicab ellipse with foci $F_1(h_1, k_1)$ and $F_2(h_2, k_2)$ and associated constant $c > 0$. (1) If $F_1 = F_2$, then E is 4-sided, a taxicab circle. (2) If $h_1 = h_2$ or $k_1 = k_2$, then E is 6-sided. (3) If $h_1 \neq h_2$ and $k_1 \neq k_2$*

then E is 8-sided.

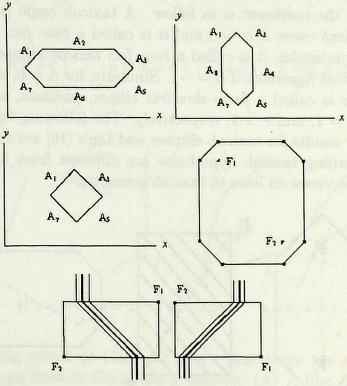


Figure 1.

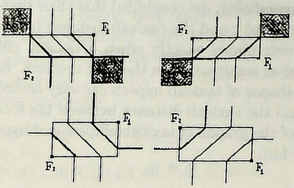
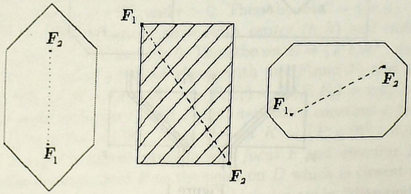
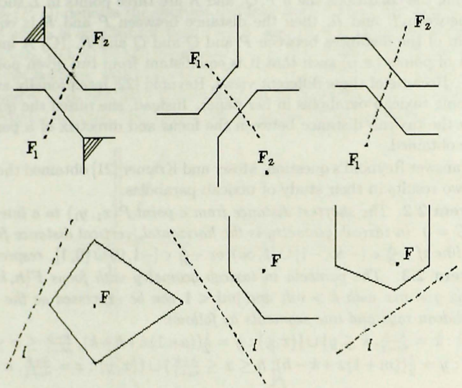


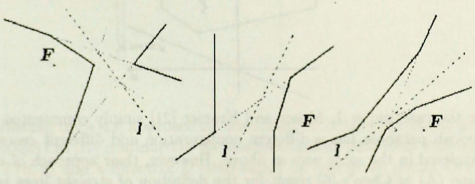
Figure 2.

Through their research in the general equations of taxicab conic sections, Kaya, Akça, Günaltili, and Özcan [14] classified the taxicab conics represented by the equation $|x - h_1| + |y - y_1| + \alpha(|x - h_2| + |y - y_2|) + \beta(|Ax + By + C|) \mp \alpha\gamma = 0$, where $\alpha \in \{-1, 0, 1\}$, $\beta = e(\alpha^2 - 1)(max\,|A|, |B|)^{-1}$, $\gamma \leq 0$, and e is the eccentricity of the related conic into two classes, according to the coefficent α as follow. A taxicab conic is called a *focus-directrix taxicab conic* if $\alpha = 0$ and it is called a *two- foci taxicab conic* if $\alpha = \pm 1$. In particular, it is called a *two- foci taxicab ellipse* if $\alpha = 1$ and a *two- foci taxicab hyperbola* if $\alpha = -1$. Similarly, for $\alpha = 0$, a focus-directrix taxicab conic is called a *focus-directrix ellipse, parabola,* and *hyperbola* if $0 < e < 1$, $e = 1$, and $e > 1$, respectively. The following figures show that most of their results for taxicab ellipses and Liu's [16] are similar, but their results concerning taxicab hyperbolas are different from Liu's because of their different views on lines in taxicab geometry.





$\alpha = 0,\ 0 < e < 1$ $\quad \alpha = 0,\ e = 1, |-\frac{a}{b}| = 1$ $\quad \alpha = 0,\ e = 1, |-\frac{a}{b}| < 1$



$\alpha = 0, |-\frac{a}{b}| > e > 1$ $\quad \alpha = 0,\ |-\frac{a}{b}| = e > 1,$ $\quad \alpha = 0,\ e > |-\frac{a}{b}| > 1$

The following result of Chen [6] shows that there are at least three different views on lines in Euclidean geometry. (A) A line L is the set of points (x, y) satisfying the linear equation $Ax + By + C = 0$ where A, B, C

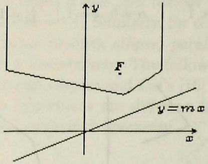
$\epsilon \Re$ with A, B not both zeros. (B) A line L is the set of points (x, y) satisfying the conditions the if P, Q, and R are three points in L such that Q is between P and R, then the distance between P and R is equal to the sum of the distances between P and Q and Q and R. (C) A line L is the set of points (x, y) such that it is equidistant from two given points H and K. Because of these different views, Reynold [22] intentionally avoided describing taxicab parabolas in her paper. Instead, she raised the question of how the taxicab distance between the focus and directrix of a parabola can be obtained.

To answer Reynold's question, Moser and Kramer [21] obtained the following two results in their study of taxicab parabolas.

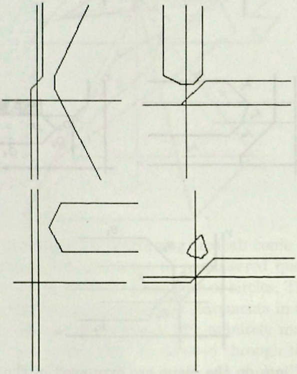
Theorem 2.2. *The shortest distance from a point $P(x_1, y_1)$ to a line $Ax + By + C = 0$ in taxicab geometry is the horizontal, vertical distance from P to the line if $-\frac{A}{B} \epsilon (-\infty, -1] \cup [1, \infty)$ or $-\frac{A}{B} \epsilon [-1, 0) \cup (0, 1]$, respectively.*

Theorem 2.3. *The parabola in taxicab geometry with focus $F(h, k)$ and directrix $y = mx$ with $k > mh$ and $|m| < 1$ can be expressed as the union of Euclidean rays and line segments as follows:*

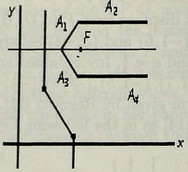
$\{((x, y) : x = \frac{k-h}{m-1}, k \leq y\} \cup \{(x, y) : y = \frac{1}{2}((a-1)x + h + k), \frac{k-h}{m-1} \leq x \leq h\} \cup$
$\{(x, y) : y = \frac{1}{2}((m+1)x + k - h), h \leq x \leq \frac{h+k}{m+1}\} \cup \{(x, y) : x = \frac{h+k}{m+1}, k \leq y\}$.

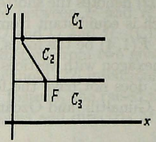


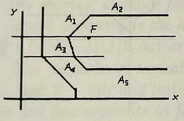
For the case $|m| = 1$, Moser and Kramer [21] simply commented that the taxicab parabola has a different configuration and different cases can be considered in the same way as above. However, their approach of using condition (A) of Chen's [6] result for the definition of straight lines is not entirely satisying because it is unrelated to the underlined metric. Iny [12] rectified the problem by using condition (C) of Chen's result instead, and he defined the distance between a point F and the line D as the minimum of $d_T(F, P)$ where $P \epsilon D$. Furthermore, instead of defining an ellipse by its foci as Reynold [23] did, Iny [12] also defined an "ellipse of the second kind" with respect to a given line D (the directrix), a given point F (the focus), and a given number e (the eccentricity) where $0 < e < 1$ by $\{P \epsilon R^2 : \frac{d_T(P,F)}{d_T(P,D)} = e\}$ where $d_T(P, D)$ denotes the shortest distance from P to D. By letting D be the line which is equidistant from $(-1, 6)$ and $(3, -4)$ satisfying condition (C) of Chen, $F(1, 4)$ be the focus, and $e = \frac{1}{2}$, Iny obtained an ellipse which is a convex hexagon with vertices $(1, 7), (-1, 5), (\frac{-3}{2}, 4), (-1, \frac{11}{3}), (1, 3)$, and $(2, 4)$; and it does not have the form given by Reynold [21], Liu [19], and Kaya, Akça, Günaltili, and Özcan [14] as in the following figure.

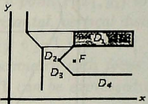


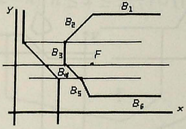
Following Iny's [12] approach on the definition of lines in taxicab geometry, Ho and Liu [10] continued in the study of taxicab parabolas with a given focus $F(h, k)$ and a directrix D determined by two given points $Q(x_1, y_1)$ and $R(x_2, y_2)$. Because of their work together with the work by Kaya, Akça, Günaltili, & Özcan [14], and Moser & Kramer [21], the problem of determining the shapes of taxicab parabolas is essentially solved completely, except for the case $|x_1 - x_2| = |y_1 - y_2|$. The following figure shows a sample of the graphs obtained in these articles.





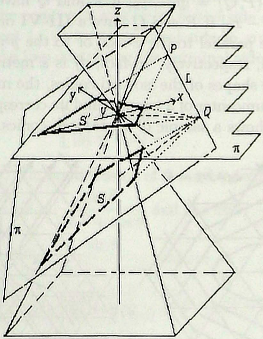






Assume that lines on the plane are expressed in the form of condition (A) of Chen's result [6] and taxicab parabolas, ellipses, and hyperbolas are defined by a given directrix, focus, and eccentricity. Laatsch [18] investigated the intersection of $|x| + |y| = c|z|$ where $c > 0$, a square pyramid (of two nappes) oriented for descriptive purposes with the z-axis vertical, with a plane of the form $z = ax + by + d$ and concluded that taxicab parabolas, ellipses, and hyperbolas can be obtained from the intersection of a plane with the pyramid by perpendicular projection of that intersection onto the x-y plane as shown in the following figure. He also stated that there is an interesting three-dimensional theory of focus and directrix of (unprojected) Euclidean conics, originally described by G. P. Dandelin in 1822 [9], [11],

[24], which has no analogue in taxicab geometry.

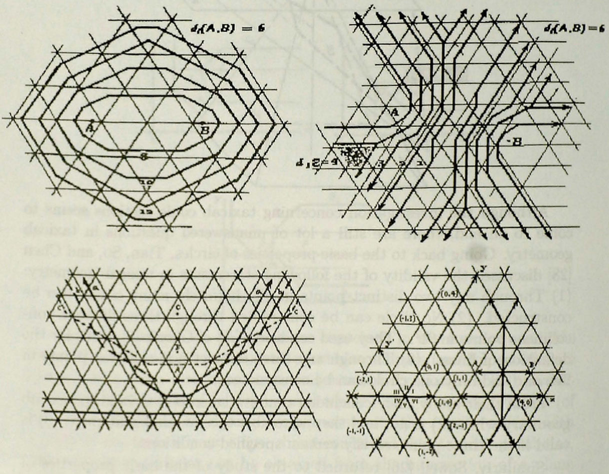


Although the investigation concerning taxicab conic sections seems to come to an end, there are still a lot of unanswered questions in taxicab geometry. Going back to the basic properties of circles, Tian, So, and Chen [28] discussed the validity of the following statements in taxicab geometry: (1) Through any two distinct points in $\Re^2$, infinitely many circles can be constructed. (2) No circle can be constructed through three distinct non-collinear points in $\Re^2$. (They used condition (A) of Chen's [6] result for the definition of lines.) (3) Through any three distinct non-collinear points in $\Re^2$, one and only one circle can be constructed.

From their study, they concluded that statement (1) is valid in taxicab geometry while (2) is not and they were able to show that statement (3) is valid if the three points satisfy certain specified conditions.

Similarly, Sowell [26] returned to the study of the basic properties of the plane $\Re^2$. Based on the fact that there are only three regular polygons which will tessellate the plane: the equilateral triangle, the square, and the regular and the first two of these can be subdivided into smaller polygons (yielding the isometrid grid and the square grid), Sowell decided to study taxicab geometry using the isometric grid.

In iso-taxi geometry, the distance function d_I is defined as follows. For $P(x_1, y_1)$ and $Q(x_2, y_2)$, (1) $d_I(P,Q) = d_T(P,Q)$ if P and Q have a **I-IV** orientation; (2) $d_I(P,Q) = |y_1 - y_2|$ if P and Q have a **II-V** orientation; (3) $d_I(P,Q) = |x_1 - x_2|$ if P and Q have a **III-VI** orientation. If the two points lie on a line parallel to the x-axis or to the y-axis, y'-axis, then (3) or (2) will be used, respectively. In fact, d_I is a metric in $\Re^2$. Sowell [26] then examined the shapes of the iso-taxi conics, the mid-point set, and the location of the circumcenter of a triangle and its corresponding circumcircle. The figure below shows a sample of iso-taxicab conics.



3. Taxicab Trigonometry

In the last chapter of his book, Krause introduced the following research direction: "Define taxi trigonometric functions via wrapping of the unit

taxi circle, and investigate their graphs, trigonometric identities,..." Following Krause's suggestion, Brisbin and Artola [4] defined the Diamond Sine and Diamond Cosine as follow. In the Euclidean approach, trigonometric function can be defined by first considering the unit circle $x^2+y^2=1$ and let θ be the angle with the positive x axis as its initial side and the radial line passing through a point (x,y) as its terminal side so $sin\,\theta = y$, $cos\,\theta = x$, and the equation of the line containing the terminal side is $y = tan\,\theta \cdot x$.
By solving

$$\begin{cases} y = tan\,\theta \cdot x \\ |y| = -|x| + 1, \end{cases}$$

the Diamond Sine and Diamond Cosine of θ can be defined as follows.

In quadrant I , $sin\ d\theta = \dfrac{\sin\theta}{\sin\theta+\cos\theta}$ and $cos\ d\theta = \dfrac{\cos\theta}{\sin\theta+\cos\theta}$.

In quadrant II, $sin\ d\theta = \dfrac{\sin\theta}{\sin\theta-\cos\theta}$ and $cos\ d\theta = \dfrac{\cos\theta}{\sin\theta-\cos\theta}$.

In quadrant III, $sin\ d\theta = \dfrac{-\sin\theta}{\sin\theta+\cos\theta}$ and $cos\ d\theta = \dfrac{-\cos\theta}{\sin\theta+\cos\theta}$.

In quadrant IV, $sin\ d\theta = \dfrac{-\sin\theta}{\sin\theta-\cos\theta}$ and -$cos\ d\theta = \dfrac{\cos\theta}{\sin\theta-\cos\theta}$.

Similar to ordinary trigonometry, they then presented the reference angle θ' of an angle θ and they also discussed the validity of some identities for the Diamond Sine and Diamond Cosine such as $|cos\ d\theta| + |sin\ d\theta| = 1$;

$cos\ d(-\theta) = cos\ d\theta$;
$sin\ d(-\theta) = -sin\ d\theta$.

Akça and Kaya [1] took a different approach in defining taxicab trigonometric functions. First, let $C_T = \{(x,y) : |x| + |y| = r,\ x, y, r \in R,\ r \geqslant 0\}$ be the taxicab circle with center at the origin and radius $r > 0$. To define π_T, the "taxicab π", they used the definition that π_T = the ratio of the circumference to radius of the taxicab circle so $\pi_T = 4$. Given an angle $\theta \in [0, 2\pi_T)$ whose vertex is the origin, initial side is the positive x axis, and is measured in the couterclockwise direction, let $P = (x,y)$ be the intersecting point of the terminal side of θ and C_T.

Then the centra angle θ can be defined by $\theta = \begin{cases} \dfrac{|x-r|+|y|}{r} & \text{if } \theta \ [0, \pi_T/2) \\ \dfrac{2r+|x|+|y-r|}{r} & \text{if } \theta \ [\pi_T/2, \pi_T) \\ \dfrac{4r+|x+r|+|y|}{r} & \text{if } \theta \ [\pi_T, 3\pi_T/2) \\ \dfrac{6r+|x|+|x|+|y+r|}{r} & \text{if } \theta \ [3\pi_T/2, 2\pi_T) \end{cases}$

Let θ be an angle in standard position and let $P(x, y)$ be the intersecting point of the terminal side and the taxicab circle $|x| + |y| = r$. Then the trigonometric functions sin_T, $\cos_T$, and tan_T are defined by $sin_T(\theta) = \frac{y}{r}$, $cos_T(\theta) = \frac{x}{r}$, $tan_T(\theta) = \frac{sin_T(\theta)}{cos_T(\theta)}$.

Thus,

$$sin_T(\theta) = \begin{cases} \frac{\theta}{2} & \text{if } \theta \in \ [0, 2) \\ 2 - \frac{\theta}{2} & \text{if } \theta \in \ [2, 6) \text{ and} \\ -4 + \frac{\theta}{2} & \text{if } \theta \in \ [6, 8), \end{cases}$$

$$\cos_T(\theta) = \begin{cases} 1 - \frac{\theta}{2} & \text{if } \theta \in \ [0, 4] \\ -3 + \frac{\theta}{2} & \text{if } \theta \in \ [4, 8] \end{cases}.$$

From the above discussion, a sample of identities such as

$sin_T(\frac{\pi_T}{2} - \theta) = \cos_T(\theta)$;

$sin_T(\frac{\pi_T}{2} + \theta) = \cos_T(\theta)$;

$sin_T(\pi_T - \theta) = sin_T(\theta)$;

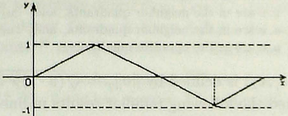
$\cos_T(\frac{\pi_T}{2} - \theta) = sin_T(\theta)$;

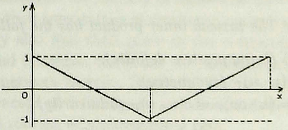
$\cos_T(\frac{\pi_T}{2} + \theta) = -sin_T(\theta)$; and

$sin_T(\pi_T - \theta) = -\cos_T(\theta)$, as well as

$sin_T(2k\pi_T + \theta) = sin_T(\theta)$ and

$cos_T(2k\pi_T + \theta) = \cos_T(\theta)$ for any natural number k can be proved. The following figure shows one-period of the taxicab sine and consine curves defined by Akça and Kaya [1].





Addition and subtraction formulas such as the following can also be established:

If $\alpha, \beta, \alpha+\beta \in [0,2]$, then $cos_T(\alpha+\beta) = -1 + cos_T\alpha + cos_T\beta$; $sin_T(\alpha+\beta) = sin_T\alpha + cos_T\beta$.

If $\alpha, \beta \in [2,4]$, $\alpha+\beta \in [6,8]$, then $cos_T(\alpha+\beta) = -1 - cos_T\alpha - cos_T\beta$; $sin_T(\alpha+\beta) = -sin_T\alpha - cos_T\beta$.

If $\alpha \in [4,6]$, β, $\alpha-\beta \in [2,4]$,then $cos_T(\alpha-\beta) = -1 - cos_T\alpha - cos_T\beta$; $sin_T(\alpha-\beta) = 2 + sin_T\alpha - cos_T\beta$.

If $\alpha \in [2,4]$, β, $\alpha-\beta \in [0,2]$, then $cos_T(\alpha-\beta) = 1 + cos_T\alpha - cos_T\beta$; $sin_T(\alpha-\beta) = 2 - sin_T\alpha - cos_T\beta$.

Comparing the diamond trigonometric functions defined by Brisbin and Artola [4] to the taxicab trigonometric functions defined by Akça and Kaya [1], it is not difficult to see that the latter is better than the former in the sense that the latter is closer to the structure of the taxicab geometry than the former .

By considering $(\Re^2, +, \cdot)$ as a vector space over the field of real numbers with addtion and scalar multiplication defined by $u + v = (u_1 + v_1, u_2 + v_2)$ and $cv = (cv_1, cv_2)$ where $u = (u_1, u_2), v = (v_1, v_2)$, and $c \in \Re$, Ekici, Akça, & Kocayusufoŭlu [8] defined the *taxicab inner product* and the *taxicab norm* of vectors as follow:

$$(u \cdot v)_T = \begin{cases} |u_1v_1| + |u_2v_2|, & u, v \text{ are in the same quadrant} \\ -|u_1v_1| + |u_2v_2|, & u, v \text{ are in the neighbor quadrants, and, } \quad u_1v_1 < 0,\ u_2v_2 > 0 \\ |u_1v_1| - |u_2v_2|, & u, v \text{ are in the neighbor quadrants, and, } \quad u_1v_1 > 0,\ u_2v_2 < 0 \\ -|u_1v_1| - |u_2v_2|, & u, v \text{ are in the opposite quadrants,} \end{cases}$$

The norm of $v = (v_1, v_2)$ is defined by $\|v\|_T = \sqrt{(v \cdot v)_T + 2\,|v_1v_2|}$.

They then proved the following familiar results in linear algebra for the taxicab inner product and taxicab norm.

Theorem 3.1. *The taxicab inner product has the following properties:*

(i) $(v \cdot v)_T \geqslant 0$ *(being positive definite)*;
(ii) $(u \cdot v)_T = (v \cdot u)_T$ *(symmetry)*;
(iii) $(cu \cdot v)_T = (u \cdot cv)_T = c(u \cdot v)_T$ *(bilinearity)*.

Theorem 3.2. *Let u, v, w be vectors in $\Re^2$ and $c \in \Re$. Then*

(i) $\|v\|_T \geqslant 0$;
(ii) $\|cv\|_T = |c|\,\|v\|_T$;
(iii) $\|u + v\|_T \leqslant \|u\|_T + \|v\|_T$;
(iv) $\|u - v\|_T \geqslant \|u\|_T - \|v\|_T$;
(v) $\|u - v\|_T \leqslant \|u\|_T + \|v\|_T$;
(vi) $\|u - v\|_T \leqslant \|u - w\|_T + \|w - v\|_T$.

Finally, in their geometric interpretation of the inner product in taxicab geometry, they built the connection, $(u \cdot v)_T = \|u\|_T\,\|v\|_T \cos_T \theta - R_T$, between the taxicab inner product and norm, where $\cos_T$ is the taxicab cosine function defined by Akça, & Kaya [1] and R_T is a real number depending the vectors u and v, which is called the *taxicab constant* by Ekici, Akça, & Kocayusufoülu [8] .

4. Topics Related to Taxicab Geometry and Trigonometry

In his book, Krause [17] raised the question of the possibility of building a geometry which mimic the movements in Chinese Checker. In response to Krause's proposal, Chen [6] defined a Chinese Checker metric. Following Chen's idea, Bayne [2] examined some properties concerning the Chinese Checker circles. Recently, Tian [27] defined the λ-metric, which is a generalization of both the Euclidean and the taxicab metrics, and Kesh [15] investigated a few properties concerning Tian's λ-circles.

In their recent paper, Özcan & Kaya defined the taxicab directed distance between points P and Q on a directed Euclidean line m as follows:

$$d_T[PQ] = \begin{cases} d_T(P,Q) & \text{if the line segment } \overline{PQ} \text{ and } m \text{ are in the same direction} \\ -d_T(P,Q) & \text{if the line segment } \overline{PQ} \text{ and } m \text{ are opposite direction.} \end{cases}$$

They then established the validity of Menelaus' and Ceva's Theorem.

In addition to their importance in geometry, taxicab geometry and other distance geometry also find their place in the computer age. In their paper, Eisenberg and Khabbaz discussed the applications of taxicab metric in geometry and network theorey. Similarly, Burman, Chen, and Sherwani [5] utilized the concept of λ-Geometry and discussed its applications in the problem of global routing of multiterminal nets.

The discussion in Section 2 and Section 3 concerning taxicab geometry and taxicab trigonometry reveals that ideas in the following list may lead to future research in these areas.

(i) From Chen [6], one can observe that both conditions (B) and (C) are appropriate definitions for lines in taxicab geometry. Since So & Al-Maskari [25] as well as Iny [12] have studied lines defined by condition (C) on the taxicab plane, the characteristics and properties of lines defined by condition (B) will remain to be of great interest for research.

(ii) Using Chen's [6] condition (B) for the definition of lines on the taxicab plane as suggested in (i) above, what would be the shapes and properties of the focus-directrix conic sections?

(iii) Based on the definition of taxicab lines used on the plane, it is quite possible to define taxicab line segments. What would be an appropriate definition for taxicab triangles? Based on the definition of angle measure developed by Akça, & Kaya [1], what are the characteristics and properties of right triangles, isosceles triangles, and equilateral triangles?

(iv) A study of the relationship between taxicab circles and taxicab triangles will definitely be an interesting direction for research.

(v) In his thesis, Bayne [2] has only begun the investigation of Chinese Checker circles. There are still many unanswered questions in Chinese Checker geometry which require further research.

(vi) Tian's [27] study in λ-geometry, which provides the linkage between the Euclidean and taxicab geometry, will definitely attract research interest.

(vii) Based on Özcan & Kaya's [22] research on directed lengths in taxicab geometry, ideas such as the different forms and applications of Menelau's and Ceva's Theorem in taxicab geometry are worthy of pursuing.
(viii) A comparison of the definitions of the diamond trigonometric functions by Brisbin & P. Artola [4] and the taxicab trigonometric functions by Akça, & R. Kaya [1] probably would bring forth some interesting results.
(ix) Assume that taxicab triangles and right triangles are appropriately defined as mentioned in (iii). Can taxicab trigonometric functions be defined by means of taxicab right triangles? How would this definition be different from the ones by Brisbin & P. Artola [4] and Akça, & R. Kaya [1]?
(x) According to Krause [17], Sowell's [26] iso-grids definitely create a different way of solving area problems in taxicab geometry than the approach used by Kocayusufoülu & Ekici's [16].

References

[1] **Z. Akça, & R. Kaya,** On the Taxicab Trigonometry, *J. Inst. Math. Comput. Sci Math. Ser.*, Vol. 10 (1997), No. 3, 151-159.

[2] **D. Bayne,** *Circles in Chinese Checker Geometry and Intersectian of Taxicab Circles*, CMSU Thesis (2001), Warrensburg, MO., USA.

[3] **L. Blumenthal,** *Theory and Applications of Distance Geometry*, Chelsea Publishing Co. (1970), Bronx, New York, 1970.

[4] **R. Brisbin & P. Artola,** Taxicab Trigonometry, *Pi Mu Epsilon J.* Vol 8 (1982), No. 2, 89-95.

[5] **S. Burman, H. Chen, & N. Sherwani,** Improved Global Routing Using λ-Geometry, *Proc. 29th Ann. Allerton Conf. Comm., Control, & Comput.*, (1991), 1083-1092.

[6] **G. Chen,** *Lines and Circles in Taxicab Geometry*, CMSU Thesis (1992), Warrensburg, MO., USA.

[7] **B. Eisenberg &, S. Khabbaz,** Optimal Locations, *The College Math. J.*, Vol. 23 (1992), No. 4, 282-289.

[8] **C. Ekici, Z. Akça, & İ. Kocayusufoǔlu,** The Norm in Taxicab Geometry, *Turkish j. Math.*, Vol. 22 (1998), No. 3, 295-307

[9] **D. Hilbert & S. Cohn-Vossen,** *Geometry and the Imagination*, Chelsea, New York, 1952.

[10] **Y. Ho & Y. Liu,** Parabolas in Taxicab Geometry, *Missouri J. Math. Sci.*, Vol 8 (1996), No. 2, 63-72.

[11] **G. B. Huff,** On Defining Conic Section, *Selected Papers on Geometry*, M.A.A., 1979, 77-78.

[12] **D. Iny,** Taxicab Geometry: Another Look at Conic Sections, *Pi Mu Epsilon J.* Vol 7 (1980), No. 10, 645-647.

[13] **H. Jacobs,** *Mathematics: A Human Endeaver*, W. H. Freeman and Co., San Francisco, 1970.

[14] **R. Kaya, Z. Akça,İ. Günaltili, & M. Özcan,** General Equation for Taxicab Conics and Their Classification, *Mitt. Math. Ges. Hamburg.*, Vol. 19 (2000), 1-14.

[15] **T. Kesh,** *Some Results in λ-Geometry*, CMSU Thesis (2001), Warrensburg, MO., USA.

[16] **İ. Kocayusufoǔlu & C. Ekici,** Some Area Problems in Taxicab Geometry, *J. Inst. Math. Comput. Sci Math. Ser.*, Vol. 12 (1999), No. 2, 95-99.

[17] **E. F. Krause,** Taxicab Geometry, Addison-Wesley, Menlo Park, NJ, 1975.

[18] **R. Laatsch,** Pyramidal Sections in Taxicab Geometry, *Math. Mag.*, Vol. 55 (1982), No. 4, 205-212.

[19] **Y. Liu,** *Conic Sections in Taxicab Geometry*, CMSU Thesis (1995), Warrensburg, MO., USA.

[20] **H. Minkowski,** *Gesammelte Abhandlungen*, Chelsea Publishing Co., New York, 1967.

[21] **J. M. Moser & R. Kramer**, Lines and Parabolas in Taxicab Geometry, *Pi Mu Epsilon J.* Vol 7 (1982), No.7, 441-448.

[22] **M. Özcan & R. Kaya**, On the Ratio of Directed Lengths in the Taxicab Plane and Related Properties, *Missouri. J. Math. Sci.*, to appear.

[23] **B. E. Reynolds**, Taxicab Geometry, *Pi Mu Epsilon J.* Vol 7 (1980), No. 2, 77-88.

[24] **G. Salmon**, *A Treatise on the Conic Sections*, 6th ed., Chelsea, New York, 1954.

[25] **S. So, & Z. Al-Maskari**, Two Simple Examples in Non-Euclidean Geometry, *Kansas Sci. Teachers, J. Math. & Sci. Teaching.*, Vol 11 (1995), 14-18.

[26] **K. O. Sowell**, Taxicab Geometry-A New Slant, *Math. Magazine*, Vol. 62 (1989), No. 4, 238-248.

[27] **S. Tian**, *λ-Geometry*, pre-prints, CMSU, Warrensburg, MO., USA.

[28] **S. Tian, S. So, & G. Chen**, Concerning Circles in Taxicab Geometry, *Int. J. Math. Educ. Sci. Technol.*, Vol 28 (1997), No. 5, 727-733.